

USEFUL INVERSIONS

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Question 3(iii) of the 1973 H.S.C. paper essentially asked for the image, in the complex plane \mathbb{C} , of the set $\{z \in \mathbb{C} : |z| = 1\}$ under the transformation T where

$$\xi = T(z) = \frac{3z - 1}{z - 1}.$$

We will not solve this directly (its solution comes simply, in a quite orthodox way) but rather examine in some detail the related problem obtained by applying the translation $w = z - 1$ to the domain, and the translation followed by a magnification,

$$\eta = \frac{\xi - 3}{2}$$

to the range of T . In which case the pair $(z, T(z))$ becomes

$(w, \mathcal{J}(w))$ where \mathcal{J} is the *inversion*

$$\eta = \mathcal{J}(w) = 1/w.$$

In our discussion of this inversion we introduce extension material which may help students to appreciate the important role played by transformation like T and \mathcal{J} in higher applied mathematics.

Our first main result is

PROPOSITION 1. For any real number k , the inversion \mathcal{J} maps the straight line $\text{Re } w = k$ (parallel to the imaginary axis) onto the circle with centre at $\frac{1}{2}k$ and radius $1/2|k|$, $|\eta - 1/2k| = 1/2|k|$, see Figure 1 (where the image of a point moving down the line, moves round the circle in the direction indicated by the arrows).

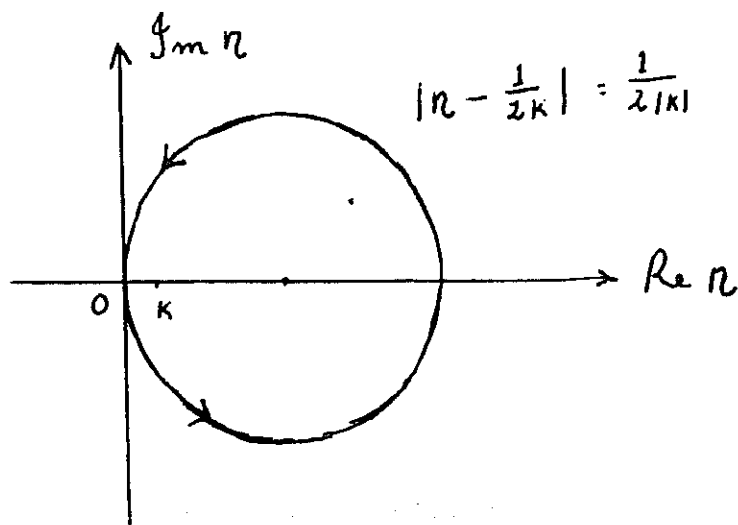


Fig. 1.

Proof:- let $z = u + iv$ (u, v real) then

$$k = \operatorname{Re} z = \operatorname{Re} \frac{1}{z} = \frac{u}{u^2 + v^2} \text{ so}$$

$$u^2 - \frac{1}{k}u + \frac{1}{4k^2} + v^2 = \frac{1}{4k^2}$$

$$\text{or } \left(u - \frac{1}{2k}\right)^2 + v^2 = \frac{1}{4k^2} \text{ the equation of a circle,}$$

centre $\left(\frac{1}{2k}, 0\right)$ radius $\frac{1}{2|k|}$ as required.

Applying this result to a family of lines parallel to the imaginary axis we obtain the double family of co-axial circles depicted in figure 2 (the letters on each curve indicate which circle is the image of which line).

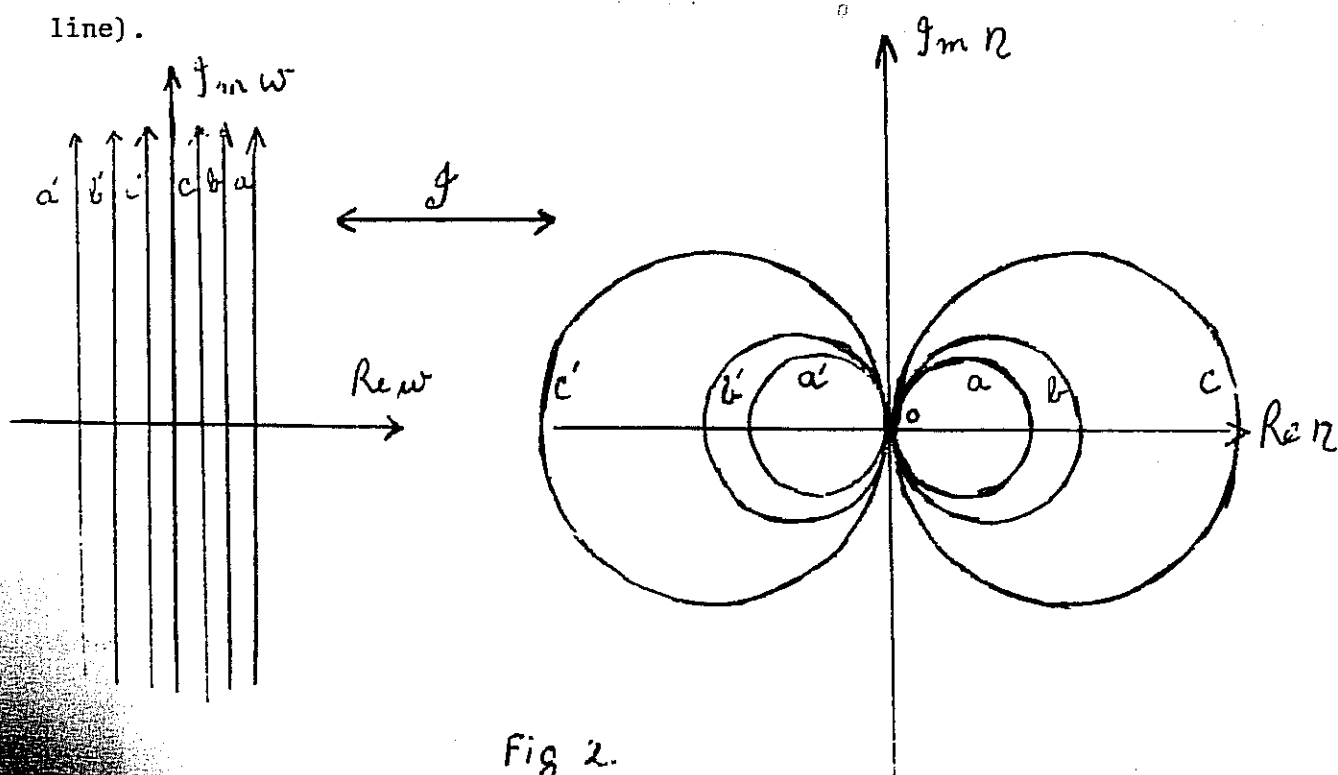


Fig 2.

An almost identical argument to the one above (which is left for the reader to supply) leads to

PROPOSITION 2. For any real number h , the inversion \mathcal{J} maps the straight line $\text{Im} w = h$ (parallel to the real axis) onto the circle

$$\left| w + \frac{i}{2h} \right| = \frac{1}{2|h|}, \text{ with centre at } \frac{-i}{2h} \text{ and radius } \frac{1}{2|h|}.$$

Thus a family of lines parallel to the real axis is mapped under \mathcal{J} to the co-axial family of circles obtained by rotating those of figure 2b through $\frac{\pi}{2}$ about the origin.

An intriguing connection between the family of co-axial circles arising from Proposition 1 and those from Proposition 2, is that any of the circles given by Proposition 1 intersects any circle given by Proposition 2 at right angles, as illustrated in figure 3. (It is a nice exercise for students to prove this).

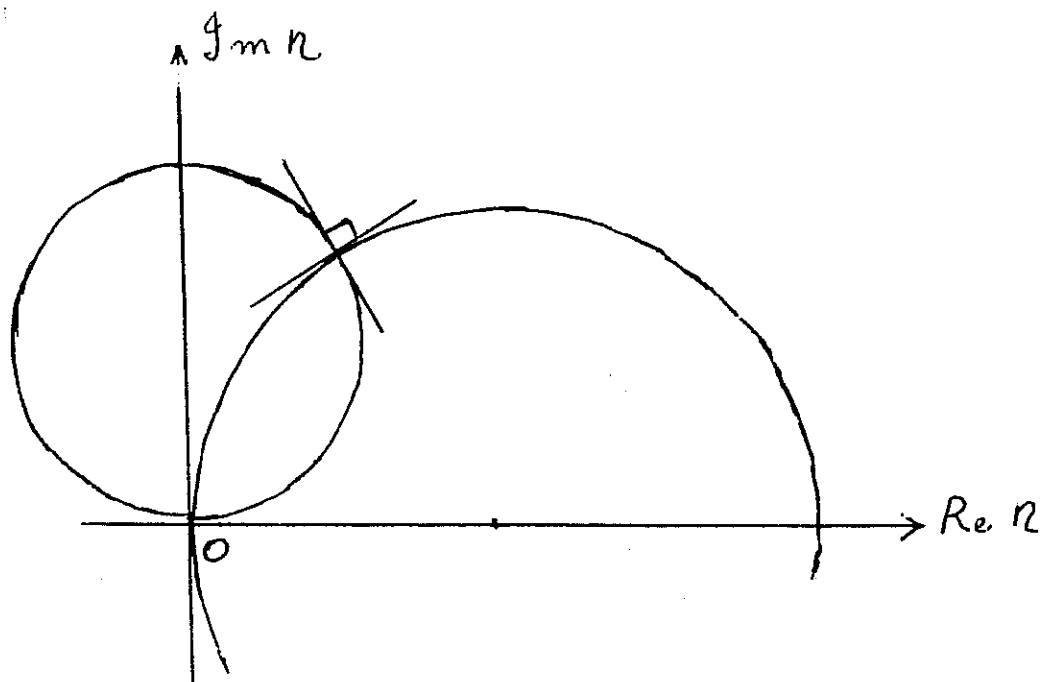


Fig. 3.

The 2 families of circles are mutually orthogonal.

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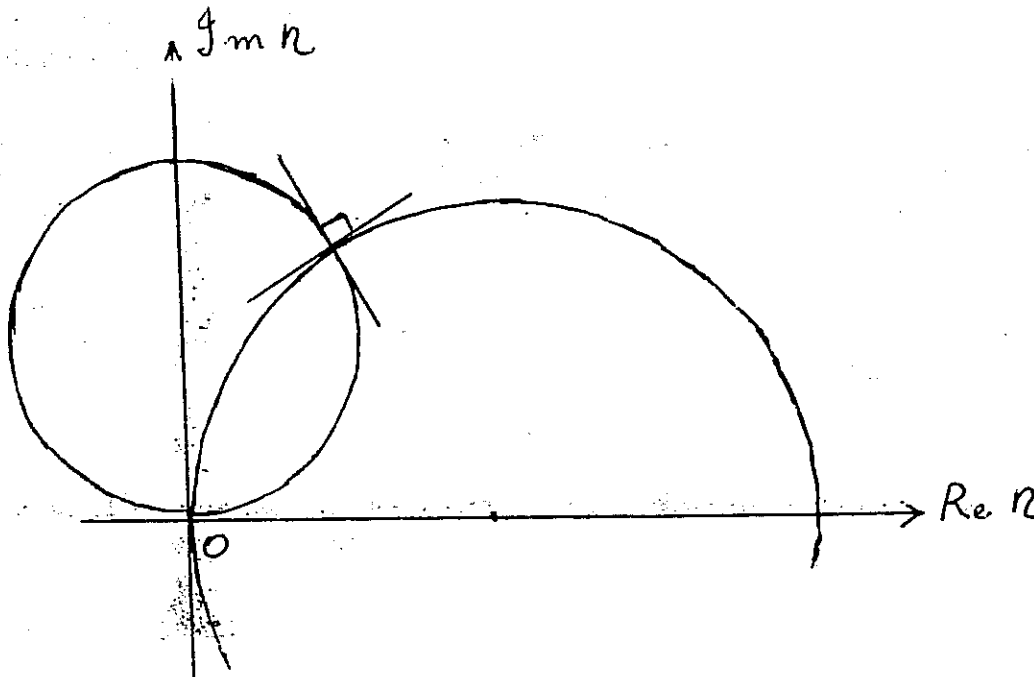


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shown in 4(b).

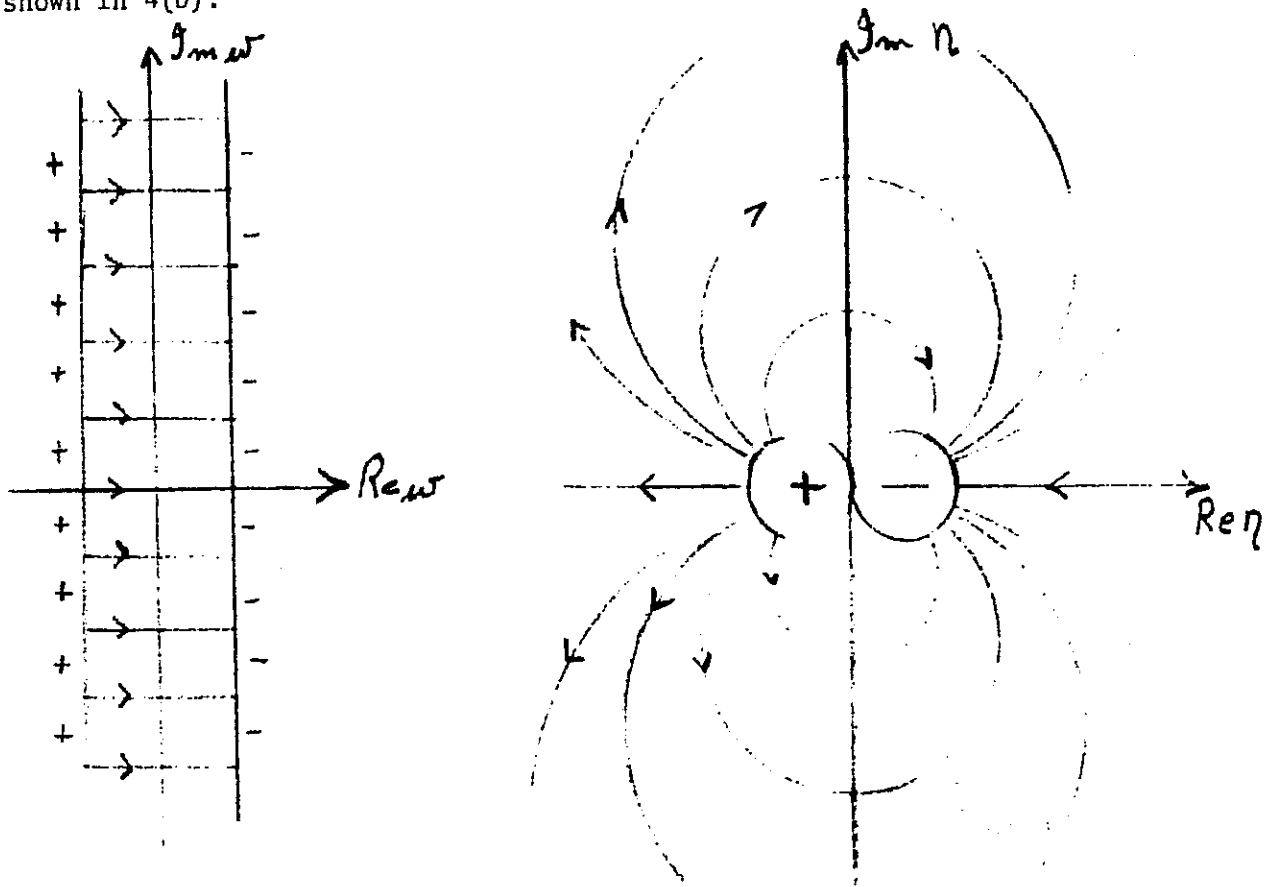


Fig. 4.

Physically we may envisage the two vertical lines in figure 4a as oppositely charged plates of an infinite parallel plate capacitor and the horizontal line segments then represent the electrostatic force field ('lines of force') between the plates.

From our previous discussion it then follows that the circular arcs of figure 4a represent the electrostatic force field which results from two oppositely charged circular rods in contact with one another [2] (as shown in figure 5) whose cross-sectional perimeter is the image under \mathcal{J} of the two condenser planes. Thus using the transformation \mathcal{J} we are "effortlessly" able to demonstrate that the 'lines of force' for such a configuration are circular.

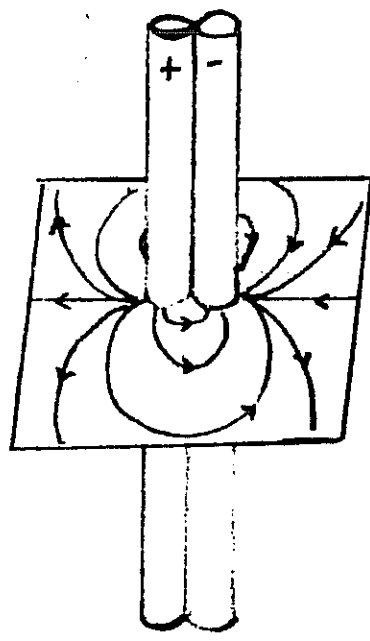


Fig. 5.

- 1) Milne - Thompson, Theoretical Hydrodynamics.
- 2) Bleaney & Bleaney, Electricity and Magnetism.

MATHEMATICAL LIMERICK

A cheerful old fellow is Ben,
 He is three times as old as young Len;
 While three years ago,
 Their age ratio
 Was thirty-three units to ten.

Find the ages of Ben and Len.

* * * * *

ANSWERS

PALINDROMIC NUMBERS

$$1 \times 1 = 1$$

$$11 \times 11 = 121$$

$$111 \times 111 = 12321$$

$$1111 \times 1111 = 1234321.$$