Renorming to gain the fixed point property

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Preliminaries

$X \equiv (X, \|\cdot\|)$ is Banach space with norm $\|\cdot\| : X \to \mathbb{R}^+$.

X^* is the dual space of continuous linear functionals $f : X \to \mathbb{R}$, with dual norm $\|f\|^* = \sup\{|f(x)| : \|x\| = 1\}$.

X^{**} is the second dual of X, the set of all continuous linear functionals $F : X^* \to \mathbb{R}$.

$J : X \to X^{**}$ is the natural embedding given by $J(x)(f) = f(x)$ for each $x \in X$.
X is *reflexive* if J is a bijection, that is, if X is isometric to X^{**} under the natural embedding.

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Two norms, $\|\cdot\|$ and $|||\cdot|||$ on X are *equivalent* if there exists $m, M \in \mathbb{R}$ such that $0 < m \leq M$ and $m\|x\| \leq |||x||| \leq M\|x\|$ for all $x \in X$.

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(e_n) is a *(Schauder) basis* for X if for each $x \in X$ there exists a unique sequence of real numbers $(x(n))$ such that

$$x = \sum_{n=1}^{\infty} x(n) e_n.$$

(e_n) is a *basic sequence* if it forms a basis for its closed linear span. The sequence (e_n) is *normalised* if $\|e_n\| = 1$ for every $n \in \mathbb{N}$.

(u_n) is a *block basic sequence* if

$$u_n = \sum_{n=p_j+1}^{p_{j+1}} a_n e_n$$

for $\{e_n\}$ a basic sequence, $\{a_n\}$ a sequence of real numbers and $p_1 < p_2 < \ldots$ a sequence of integers.

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For (e_n) a basis and $k \in \mathbb{N}$, we define the *k-th natural projection* by,

$$P_k(\Sigma_{i=1}^{\infty} x(i) e_i) = \Sigma_{i=1}^{k} x(i) e_i.$$

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A space X is said to be *asymptotically isometric* to ℓ_1 if there is a real sequence (ρ_n) decreasing to 0 and a basis $\{e_n\}$ such that,

$$\sum_{n=1}^{\infty} |a_n|(1-\rho_n) \leq \| \sum_{n=1}^{\infty} a_n e_n \| \leq \sum_{n=1}^{\infty} |a_n|,$$

for all $a = (a_n) \in \ell_1$.

For C and D two nonempty bounded subsets of X, the *Hausdorff distance* between them is,

$$H(C,D) := \max\left\{\sup_{x\in C}\inf_{y\in D}\|x-y\|, \sup_{y\in D}\inf_{x\in C}\|y-x\|\right\}.$$

For X and Y two Banach spaces, their *Banach-Mazur distance* is given by,

$d(X,Y) = \inf\left\{\|S\|\|S^{-1}\| : S \text{ is an isomorphism from } X \text{ onto } Y\right\}.$

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Background

For C a nonempty subset of a Banach space X, a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$.

A nonempty subset C of a Banach space X has the *fixed point property* (**FPP)** if every nonexpansive self mapping T of C has a fixed point, that is, an $x \in C$ such that $T(x) = x$.

C has the *hereditary fixed point property* if every closed convex nonempty subset of C has the FPP.

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A Banach space X has the *fixed point property (FPP)* if every **closed bounded convex nonempty** subset has the FPP.

X has the *weak fixed point property (***w-FPP***)* if every **weak - compact convex nonempty** subset of X has the FPP.
Note in reflexive spaces, the w-FPP and the FPP are equivalent.

Similarly, the dual of X, X^*, has the *weak* fixed point property* (**w*-FPP** if every **weak* (that is $\sigma(X^*, X)$) - compact convex nonempty** subset of X^* has the FPP.

A sequence (x_n) is an *approximate fixed point sequence (***afps***)* for T if $\lim_n \|x_n - Tx_n\| = 0$.
It follows from the Banach contraction mapping principle that every nonexpansive self mapping of a closed bounded convex subset of a Banach space has an afps.

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Examples

c_0 (and hence c and ℓ_∞) fails to have the FPP: the mapping $T(x) = (1, x(1), x(2), ...)$ is a fixed point free nonexpansive self-mapping of the unit ball of c_0 [Kakutani, 1943].

However, Maurey [1981] showed that both c_0 (and c) have the w-FPP.

Indeed Dowling, Lennard, and Turret [2004] proved a nonempty closed convex subset of c_0 has the FPP if and only if it is weak compact.

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Whether a dual space has the w*-FPP depends on the choice of pre-dual and hence of w* topology.

For instance:

ℓ_1 as c_0^* has the w*-FPP [It has the w*-Opial property and hence w*-normal structure],

On the other hand,

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Lennard's example

Take the dual action of $(f(n)) \in l_1$ on $(x(n)) \in c$ to be

$$(f(n))(x(n)) = f(1)x(1) + f(2)\lim_n x(n) + f(3)x(2) + \cdots).$$

For $\delta \in (0,1]$ and $(\epsilon_k) \subset [0,1)$ a sequence such that $\sum_{k=1}^{\infty} \epsilon_k < \infty$ and $\Pi_{k=1}^{\infty}(1-\epsilon_k) > 0$, the mapping

$$T(x) := (\delta(1-x(1)) + \sum_{k=1}^{\infty}(1-\epsilon_k)x(k+1), \delta(1-x(1)), (1-\epsilon_1)x(2), (1-\epsilon_2)x$$

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Since W.A. Kirk [1965] effectively proved Banach spaces with normal structure have the w-FPP, an important aspect of metric fixed point theory has been to **find easily verified widely applicable sufficient conditions for a space to have the w-FPP**.

In 1981, Alspach gave the first and essentially only known example of a space failing the w-FPP. He showed that in $L_1[0,1]$ a modification of the baker transform is a fixed point free isometry on the order interval $[\mathbf{0},\mathbf{1}]$.

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Note that the closure of the span of the order interval $[\mathbf{0}, \mathbf{1}]$, the smallest Banach space into which Alspach's example embeds, is a non-reflexive separable subspace of $L_1[0,1]$.

Since every separable space embeds isometrically into ℓ_∞ and $C[0,1]$, this shows that both these spaces fail the w-FPP. However, there is no known intrinsic example showing this in either space **?**.

Complementary to Lin's result, Maurey [1980] used ultrapower techniques to prove that every reflexive subspace of $L_1[0,1]$ has the FPP.

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Many sufficient conditions for a Banach spaces to have the w-FPP have been discovered. Leaving as the major unresolved question in this direction: **Do all reflexive spaces have the FPP?**

The less ambitious question of whether all super-reflexive spaces have the FPP also remains open.

In 1997, Dowling and Lennard exploited the earlier example of $\ell_1 = c^*$ failing the FPP to show that **every Banach space containing an asymptotically isometric copy of ℓ_1 (or c_0) fails the FPP**.

And, gave several examples of such spaces including all non-reflexive subspaces of $L_1[0,1]$.

This combined with Maurey's 1980 result shows that a subspace of $L_1[0,1]$ has the FPP if and only if it is reflexive.

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Dowling Lennard and Turett [1996] also showed that not all non-reflexive Banach spaces can be equivalently renormed to have the FPP;

They showed that under any equivalent norm, ℓ_∞ or $\ell_1(\Gamma)$, where Γ is uncountable, admits an asymptotically isometric copy of ℓ_1 and any equivalent norm on $c_0(\Gamma)$ admits an asymptotically isometric copy of c_0.
Hence, ℓ_∞, $\ell_1(\Gamma)$ and $c_0(\Gamma)$ all fail to have the FPP in any equivalent norm.

All of these results led to the **conjecture**: A Banach space with the FPP is necessarily reflexive.

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All of these results led to the **conjecture**: A Banach space with the FPP is necessarily reflexive.

In the other direction, there are non-reflexive spaces that fail to contain an asymptotically isometric copy of ℓ_1 or c_0.

Indeed, Dowling, Johnson, Lennard and Turett [1967] showed that for $(\gamma_k) \subset (0, 1)$ a non-decreasing sequence converging to 1,

$$|||x||| = \sup_k \gamma_k \sum_{n=k}^{\infty} |x(n)|$$

is an equivalent norm on ℓ_1 and $(ell_1, ||| \cdot |||)$ does not contain an asymptotically isometric copy of ℓ_1.

Similarly, the pre-dual, $(c_0, ||| \cdot |||)$ does not contain an asymptotically isometric copy of c_0, where $|||x||| = \sup_k \gamma_k \sup_{n=k}^{\infty} |x(n)|$.

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Then surprisingly, Pei-Kee Lin [2008] showed that when $\gamma_k = \frac{8^k}{1+8^k}$, $(\ell_1, |||\cdot|||)$ **has the FPP**.

This was the first example of a non-reflexive space with the FPP, showing the previous conjecture to be false, and so initiating a new line of research to determine which (non-reflexive) Banach spaces can be equivalently renormed to have the FPP.

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Almost all proofs that a space has the w-FPP rely on w-compactness ensuring nonexpansive maps have minimal invariant sets and the existence and properties of diameterizing sequences for such sets.

This technique is not available for the FPP in non-reflexive spaces. In its place Lin used the following **lemma, which should should have wider applications**.

Lemma

Let $(X, \|\cdot\|)$ be a Banach space with a linear topology τ such that every bounded sequence has a τ -convergent subsequence. Let C be a closed convex bounded subset of X and $T : C \to C$ a nonexpansive mapping. If T is fixed point free, then there exists a closed convex T-invariant subset D of C such that

$$\inf\left\{\limsup_n \|x_n - x\| : (x_n) \subset D, (x_n) \ an\ afpt,\ \&\ x_n \xrightarrow{\tau} x\right\} > 0.$$

The proof relies on Cantor's intersection theorem.

Since Lin's initial example, a number of extensions and generalizations have been made. We list some of them.

In what follows $|||\cdot|||$ denotes the equivalent norm on ℓ_1 defined by,

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |x(n)|,$$

where (γ_k) is a given non-decreasing sequence in $(0,1)$ which converges to 1.

One of the most general results on renorming ℓ_1 to have the FPP is due to Linares, Japn and Llorens-Fuster [2012].

Theorem

Let $p(\cdot)$ be an equivalent norm on ℓ_1 such that

$$\limsup_n p(x_n + x) = \limsup_n p(x_n) + p(x),$$

for every w-null sequence (x_n) and for all $x \in \ell_1$.*
Then for every $\lambda > 0$ the norm

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In 2010, Dowling Lin and Turret proved:

Theorem

For $m \in \mathbb{N}$,

$$\oplus_{i=1}^{m}(\ell_1, |||\cdot|||)$$

has the FPP.

A basis $\{e_n\}$ is *premonotone* if for every $n \leq k \leq m$ and $(a_i) \subset \mathbb{R}$

$$\|\sum_{i=k}^{m} a_i e_i\| \leq \|\sum_{i=n}^{m} a_i e_i\|$$

A space Y is said to be *asymptotically isometric* to ℓ_1 if it has a basis $\{e_n\}$ such for some sequence (ρ_n) decreasing to 0 and every $x = (x(n)) \in \ell_1$,

$$\sum_{n=1}^{\infty} |x(n)|(1-\rho_n) \leq \|\sum_{n=1}^{\infty} x(n)e_n\| \leq \sum_{n=1}^{\infty} |x(n)|,$$

In 2009, Fetter and Buen proved:

Theorem

Suppose $(X, \|\cdot\|)$ is a Banach space with a premonotone basis $\{e_n\}$, and every infinite dimensional subspace of X has a subspace that is asymptotically isometric to ℓ_1. Then provided $\gamma_1 > \frac{4}{5}$, every subspace of $(X, |||\cdot|||)$ contains an infinite dimensional subspace that has the FPP.

We mention that extensions of Lin's result to the *non-commutative L_1-spaces* associated with infinite dimensional finite von Neumann algebras have been made by Linares and Japon [2011].

Density of norms with the fixed point property

Thus far we have considered the existence of renormings that have have the FPP. We now consider whether such renormings are common amongst the set of equivalent renormings, $\mathcal{P}(X)$.

An equivalent norm $|||\cdot|||$ on a Space $(X, \|\cdot\|)$ is said to be *distorted* if $(X, |||\cdot|||)$ does not contain an almost isometric copy of $(X, \|\cdot\|)$.
X is emphnon-distortable if no equivalent norm is distorted.

James' Distortion Theorem [1964] states that ℓ_1 and c_0 are non-distortable. In 1981, Partington proved that ℓ_∞ is also non-distortable.

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In 2012, Benavides used these results to prove

Theorem

Let X be a Banach space which contains an isomorphic copy of a non-distortable space Y which fails the (w-)FPP. Then, the subset of $\mathcal{P}(X)$ failing the (w-)FPP is dense in $\mathcal{P}(X)$.

In particular, the set of norms on c_0 or ℓ_1 failing the FPP are dense in the set of all equivalent norms.

What about norms with the (w-) FPP?

In 2009 Benavides also proved the following result.

Theorem

Let X be a Banach space such that for some set Γ there is a one-to-one linear continuous mapping $J : X \to c_0(\Gamma)$. Define the equivalent norm $|||x|||^2 = \|x\|^2 + \|Jx\|_0^2$, the space $(X, |||\cdot|||)$ has the w-FPP.

Since J can be replaced with λJ for any $\lambda \in (0,1)$, we see that the equivalent norm $|||x|||$ can be made arbitrarily close to the original norm.

Since every reflexive space admits such a mapping, we deduce the following, which solves a long standing question in metric fixed point theory.

Corollary

For every reflexive space X the set of norms with the w-FPP is dense in $P(X)$.

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Stability of the (w-) FPP

A space X has the *stable (w-) fixed point property (stable (w-) FPP)* if every Banach space isomorphic to X with a Banach-Mazur distance to X less than some constant M has the (w-) FPP.

For example, the Hilbert space ℓ_2 has the stable w-FPP, with the best known stability constant $M = \sqrt{\frac{5+\sqrt{13}}{2}}$ being obtained by Lin [1999].

In 2012, Benavides proved the following result.

Theorem

Let X be a Banach space with a monotone basis , $\{e_n\}$. For $a \in (-1, 0)$ such that $1 - a^8 < 1/28$, consider the equivalent norm
$\|x\|_a =$
$\sup\left\{\|\sum_{n=1}^{\infty} a^{p(n)} t_n e_n : (p_n)\right.$ *is a nondecreasing bounded sequence* $\subset \mathbb{Z}$
where $x = \sum_{n=1}^{\infty} t_n e_n$. If Y is a Banach space isomporphic to X with Banach Mazur distance to $(X, \|\cdot\|_a)$ less than $2/\sqrt{3 + 28(1 - a^8)}$ then Y has the w-FPP.

This theorem resolves a major question in metric fixed point theory:

Corollary

Every separable Banach space X can be renormed to have the stable w-FPP.

Proof.

Since X is separable, it can be embedded isometrically in $C[0, 1]$. $C[0, 1]$ has a monotone basis so there is a renorming $|||\cdot|||$ on

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Some important open questions:

Is the set of renormings of ℓ_1 with the FPP dense in $\mathcal{P}(\ell_1)$? There is currently no known non-reflexive space X such that equivalent renormings with the FPP are dense in $\mathcal{P}(X)$.

It is unknown whether there are any equivalent renormings of ℓ_1 which do not contain an asymptotically isometric copy of ℓ_1 but fail to have the FPP. (Discovering that there are no such renormings would lead to a characterisation of the renormings of ℓ_1 with the FPP.)

Can c_0 be renormed to have the FPP?

Benavides proved that every reflexive space has an arbitrarily close renorming with the w-FPP, but it remains unclear whether every reflexive space or even every super-reflexive space has the FPP.

What is the upper bound for the stability constant of a Hilbert space? (If all superreflexive spaces had the FPP the upper bound would be infinite.)