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A DYADIC APPROACH TO THE RIEMANN INTEGRAL

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ABSTRACT

We propose a treatment of the definite integral which is substantially simpler than the approach adopted by the standard texts.

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Introduction. We propose a treatment of the definite integral which is substantially simpler than the approach adopted by the standard texts.

A student's first rigorous (or perhaps not so rigorous) encounter with the definite integral should be through a development which is, as far as possible,

(i) *intuitive*, and

(ii) *elementary and technically uncluttered.*

At the same time, it should meet other, possibly conflicting, objectives. It should

(iii) *yield a class of integrable functions which is adequate for most applications*, and

(iv) *allow an easy passage to the basic properties of the integral* . Here, we have in mind the "obvious" properties of an area, and those properties necessary for the development of the integral calculus.

Whilst most working analysts use the Lebesgue integral, an overwhelming majority of elementary textbooks chooses to develop the Riemann integral. This choice is presumably based on the questionable view that the Riemann theory, though often technically inadequate, is at least more elementary and accessible than the Lesbesgue theory. We will not debate this point here; rather we will present an integration theory which is entirely equivalent to the Riemann theory (see the appendix) but which is unencumbered by many of the intricacies inherent in standard treatments.

The real analysis we require is kept to a bare minimum. We do assume familiarity with the basic facts about sequence limits, and in particular we rely heavily on *the convergence of bounded monotonic sequences.* In addition, we need to be able to manipulate suprema and infima but only of real-valued functions of a real variable. If we wish to show that continuous functions on compact intervals are integrable, we must, as in all other approaches, know that they are ***uniformly continuous.***

Our method is the simplest formalization of the idea of estimating the area under a curve drawn on graph paper by

(a) counting the number of squares contained in the area (a lower estimate) and

(b) counting the number of squares intersecting the area (an upper estimate),

together with the observation that

(c) the error is reduced by further subdividing the rulings.

We find it convenient to use dyadic (rather than decimal) graph paper.

Notation. The n^{th} *dyadic partition* of the real line $\mathbf{R}$ is the set of points $x_n^k := k/2^n$ for $k = 0, \pm 1, \pm 2, \ldots$. We set $I_n^k := [x_n^{k-1}, x_n^k]$.

Let $f : [a,b] \to \mathbf{R}$ be a *bounded* function defined on a *compact* interval. We shall identify f with the extension $f : \mathbf{R} \to \mathbf{R}$ obtained by setting $f(x) = 0$ for x outside $[a,b]$.

For such a function we write

$$M_n^k(f) = \sup\{f(x) : x \in I_n^k\} \quad \text{and} \quad m_n^k(f) = \inf\{f(x) : x \in I_n^k\}.$$

These quantities enable us to define the n^{th} ***upper sum of f*** as

$$U_n(f) = \frac{1}{2^n} \sum_k M_n^k(f)$$

and the n^{th} *lower sum of f* as

$$L_n(f) = \frac{1}{2^n} \sum_k m_n^k(f).$$

It is important to note that ***in both cases only a finite number of terms in the sum is non-zero.***

When f or n is clear from the context, we may drop one or both from the notations above and simply write $x^k, M^k(f), L_n$, and so on.

Fundamental facts. The sequences $(U_n(f))$ and $(L_n(f))$ are respectively decreasing and increasing. Since each is bounded, they must both converge.

Definitions. The *upper D-integral of f* is $U(f) := \lim_{n\to\infty} U_n(f)$ and the *lower D-integral of f* is $L(f) := \lim_{n\to\infty} L_n(f)$.

Note that $L(f) \leq U(f)$. We say that f *is D-integrable* if $L(f) = U(f)$. The common value, which we denote by $\int_a^b f$,is the *D-integral of* $f : [a,b] \to \mathbf{R}$.

The following is obvious, but very useful.

Basic Criterion for D-integrability.

f is D-integrable if and only if $\lim_{n\to\infty} \left[U_n(f) - L_n(f)\right] = 0$.

This should be compared with the usual criterion for Riemann integrability [1,pp.242-3]. The fundamental advantage of our approach is that we need only consider *one sequence* of partitions, which is quite independent of the function f.

Fundamental properties of the D-integral. Simplified versions of well-worn arguments establish the basic properties of D-integrals. The following is a reasonably complete list of properties, and the order reflects an appropriate sequence when supplying proofs.

We assume that $f, g, h : [a,b] \to \mathbf{R}$ are D-integrable.

(1) *Rectangles have the correct area.* If $[c,d] \subseteq [a,b]$ then the characteristic function $\chi_{[c,d]}$ is D-integrable and $\int_a^b \chi_{[c,d]} = d - c$.

(2) *Linearity.* If $\lambda \in \mathbf{R}$ then $f + \lambda g$ is D-integrable and $\int_a^b (f + \lambda g) = \int_a^b f + \lambda \int_a^b g$.

(3) *Products.* fg is D-integrable.

(4) *Quotients.* f/g is D-integrable provided that g is bounded away from zero.

(5) *Positivity.* If f has non-negative values then $\int_a^b f \geq 0$.

(6) *Sandwich property.* If $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g \leq \int_a^b h$.

(7) *Absolute values and the triangle inequality.* $|f|$ is D-integrable and $|\int_a^b f| \leq \int_a^b |f|$.

(8) *Restrictions.* If $[c,d] \subseteq [a,b]$ then $f\chi_{[c,d]}$ is D-integrable over [c,d] and $\int_c^d f := \int_c^d f\chi_{[c,d]} = \int_a^b f\chi_{[c,d]}$.

(9) *Additivity over intervals.* If $c \in [a,b]$, then $\int_a^b f = \int_a^c f + \int_c^b f$.

The proofs present few problems but we would like to give a flavour of what is required. First, we prove an easy result.

Proof of (1). There are at most two I_n^k's on which M_n^k and m_n^k differ.

It follows that

$$d - c - 2^{1-n} \le L_n \le U_n \le d - c + 2^{1-n}.$$

Take limits to obtain the desired conclusion.

We now show how to establish one of the more "difficult" results.

Proof of (8). By linearity and the boundedness of the functions, it is no loss to assume that f and g have non-negative values. We examine the n^{th} upper and lower sums. Observe that

$$M^k(fg) \le M^k(f)M^k(g) \quad \text{and} \quad m^k(fg) \ge m^k(f)m^k(g).$$

So if $f(x) \le F$ and $g(x) \le G$ for all $x \in [a,b]$, we obtain

$$\begin{aligned} 0 &\le M^k(fg) - m^k(fg) \le M^k(f)M^k(g) - m^k(f)m^k(g) \\ &= [M^k(f) - m^k(f)]M^k(g) + [M^k(g) - m^k(g)]m^k(f) \\ &\le [M^k(f) - m^k(f)]G + [M^k(g) - m^k(g)]F. \end{aligned}$$

Hence

$$0 \le U_n(fg) - L_n(fg) \le G\big(U_n(f) - L_n(f)\big) + F\big(U_n(g) - L_n(g)\big).$$

Our basic criterion now yields the result.

Classes of D-integrable functions. Step-functions, monotonic functions and continuous functions may be shown to be D-integrable by using streamlined versions of the standard proofs of the Riemann integrability of such fuctions.

Further developments. The integral calculus may now be developed, via the Fundamental Theorem of Calculus [an easy consequence of (1),(6),(8) and (9)] without further reference to partitioning. Certain esoteric results, like the integrability of $f \circ g$ when g is D-integrable and f is continuous, and the Fundamental Theorem that $\int_a^b f' = f(b) - f(a)$, even when f' is not piecewise continuous, may be proved with the aid of dyadic partitions, but we feel that these are inappropriate in a first treatment.

APPENDIX. The equivalence of the D-integral and the Riemann Integral.

We recall a standard criterion for Riemann integrability [1,p.242]. *The function* $f : [a,b] \to \mathbf{R}$ *is Riemann integrable if and only if for every* $\epsilon > 0$ *there is a partition* π *of* $[a,b]$ *such that* $U_\pi(f) - L_\pi(f) < \epsilon$. [Here $U_\pi(f)$ and $L_\pi(f)$ denote the upper and lower sums of f relative to the partition π.]

Since dyadic partitions of the line yield partitions of $[a,b]$, it is clear that D-integrability implies Riemann integrability.

Conversely, suppose $f : [a,b] \to \mathbf{R}$ is Riemann integrable. Fix $\epsilon > 0$ and let $\pi := \{a = p_0 < p_1 < \ldots < p_m = b\}$ be a partition of $[a,b]$ for which $U_\pi(f) - L_\pi(f) < \epsilon$.

Let $\delta = \min\{p_r - p_{r-1} : 1 \le r \le m\}$ and choose an integer n with $\frac{1}{2^n} \le \delta\epsilon$. If $I_n^k \subseteq [p_{r-1}, p_r]$, it is clear that the oscillation of f on I_n^k cannot exceed the oscillation of f on $[p_{r-1}, p_r]$. Now, at most $m + 1 \le 2(b-a)/\delta$ of the I_n^k's fail to fall into this category. So if $M = \sup\{|f(x)| : x \in [a,b]\}$ we obtain

$$U_n(f) - L_n(f) \le U_\pi(f) - L_\pi(f) + \left[2(b-a)/\delta\right] 2M\,\delta\,\epsilon < K\,\epsilon,$$

where K is a constant independent of n.

This shows that $U_n(f) - L_n(f) \to 0$ and we conclude that f is D-integrable.

It is clear that the $D\text{-}\int_a^b f$ and the $R\text{-}\int_a^b f$ have the same value.

Reference.

[1] R.G.Bartle and D.R.Sherbert, *Introduction to real analysis,* Wiley (1982).