The Algebra of Polynomials

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Polynomials are among the most elementary functions studied. Despite this (or perhaps because of it) they play a prominent and fundamental role in mathematics. A study of polynomials is therefore important; it also provides, along with Euclidean geometry, an opportunity to illustrate the logical development of mathematical theory, typical of more advanced modern mathematics, but all too often missing from school treatments of the subject. This article provides a summary for such a development which is not far from that suggested for 4-Unit Mathematics students in New South Wales.

A *polynomial of degree n* 'over F' is an expression of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $a_0, a_1, a_2, \ldots, a_n$ are elements of F with $a_n \neq 0$.

For us, F will be one of

$\mathbb{Z}$	$\subset$	$\mathbb{Q}$	$\subset$	$\mathbb{R}$	$\subset$	$\mathbb{C}$
ring of integers		field of rational numbers		field of real numbers		field of complex numbers

However, if one so wished, the thrust of the study could also be illustrated by taking F to be a finite ring, e.g. the integers modulo 4, or a finite field, e.g. the integers modulo 7.

If $p = p(x)$ is a polynomial over F_1 and $F_1 \subseteq F_2$, then by allowing values from F_2 to be substituted for x, we obtain a function from F_2 to F_2. This is how we will regard polynomials, and as we shall see, the choice of domain F_2 determines much of what can be said about p.

We can divide polynomials over F_1 by the usual process of *long division.* In particular, for $a \in F_2$, we may divide $p(x)$ by $x - a$ to obtain

$$p(x) = (x - a)q(x) + R,$$

where the *quotient* $q(x)$ is a polynomial over F_2 of degree one less than that of $p(x)$, and the *remainder* R is an element of F_2.

Putting $x = a$, we see that

$$p(a) = 0 + R = R.$$

Thus we have arrived at the following

Remainder Theorem: The remainder from dividing $p(x)$ by $x - a$ is $p(a)$.

For example, taking $p(x) = x^4 - 2x^3 + 2x - 3$ and $a = 1$, the division of $p(x)$ by $x - a$ may be set out as follows:

$$
\begin{array}{r|rrrrr l}
 & x^3 & - x^2 & - x & + 1 & & \leftarrow \text{quotient} \\
\hline
x - 1 & x^4 & - 2x^3 & + 0x^2 & + 2x & - 3 & \leftarrow p(x) \\
 & x^4 & - x^3 & & & & \\
\cline{2-6}
 & & - x^3 & + 0x^2 & + 2x & - 3 & \\
 & & - x^3 & + x^2 & & & \\
\cline{3-6}
 & & & - x^2 & + 2x & - 3 & \\
 & & & - x^2 & + x & & \\
\cline{4-6}
 & & & & x & - 3 & \\
 & & & & x & - 1 & \\
\cline{5-6}
 & & & & & - 2 & \leftarrow \text{remainder} = p(1)
\end{array}
$$

So $x^4 - 2x^3 + 2x - 3 = (x-1)(x^3 - x^2 - x + 1) - 2$.

From the remainder theorem it follows that r is a root of p, that is $p(r) = 0$, if and only if p can be *factored* as:

$$p(x) = (x - r)q(x),$$

a result known as the **Factor Theorem.**

By repeated application of this we see that **a polynomial of degree n can have at most n roots.**

This naturally raises questions concerning the actual number and types of roots a given polynomial has. To discuss these, let us begin by recalling the following theorem.

The Fundamental Theorem of Algebra*: When $F_2 = \mathbb{C}$ every polynomial has a root, and consequently, by repeated application of the factor theorem, **every polynomial of degree n has precisely n roots** (repeated roots being counted according to their multiplicity).

In one sense, this provides a complete answer to the last question; however, it leaves open questions such as: When does a polynomial over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ have integer, rational, or real roots, respectively. The fact that the complex numbers $\mathbb{C}$ cannot be replaced by $\mathbb{R}$, for example, in the Fundamental Theorem of Algebra is amply demonstrated by the polynomial $p(x) = x^2 + 1$ which has no real roots.

Note: For a polynomial over $\mathbb{Q}$, so far as roots are concerned, by multiplying the polynomial by the least common multiple of the denominators of the coefficients, we may, without loss of generality, assume the polynomial to be over $\mathbb{Z}$. For example, $\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x$ has the same roots as $3x^4 - 4x^3 - 24x^2 + 48x$.

Polynomials over $\mathbb{R}$

Let $p(x)$ be a polynomial over $\mathbb{R}$. If z is a complex root of $p(x)$, then p(z)=0, so

$$0 = \overline{0} = \overline{p(z)} = \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}$$
$$= a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^n$$

(as the coefficients are real). Thus $\overline{z}$ is also a root of $p(x)$.

That is, **for polynomials with real coefficients, unreal complex roots always occur in conjugate pairs.** It follows that $p(x)$ may be factorized as

$$p(x) = (x - r_1) \cdots (x - r_m)(x - z_1)(x - \overline{z_1}) \cdots (x - z_k)(x - \overline{z_k}),$$

where $r_1, \ldots, r_m$ are real roots and there are an even number of unreal complex roots $z_1, \overline{z_1}, \ldots, z_k, \overline{z_k}$, occuring as conjugate pairs.

Hence, **every such polynomial of odd degree must have at least one real root.****

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* This is the only result we will assume rather than prove. It was first given a satisfactory proof by the German mathematician Carl Friedrich Gauss in 1799. (At least this was the claim made by the then 22-year-old Gauss. In fact the 1746 proof by d'Alembert is in retrospect equally acceptable and perhaps more elementary.) While a proof need not entail any mathematics more advanced than that assumed in this note, it does require sophisticated and somewhat lengthy computations with complex-valued functions of a complex variable.

** More delicate consideration leads to Descartes' rule of signs for the numbers of positive and negative roots of a polynomial over $\mathbb{R}$. A study of this would provide excellent extension material.

Noting that

$$(x-z)(x-\overline{z}) = x^2 - (z+\overline{z})x + z\overline{z}$$
$$= x^2 - (2\,\mathrm{Re}\,z)x + |z|^2$$

is a quadratic over $\mathbb{R}$, we see that every polynomial over $\mathbb{R}$ can be factorized as:

$$p(x) = \begin{pmatrix}\text{a product of}\\ \text{linear factors}\end{pmatrix} \times \begin{pmatrix}\text{a product of}\\ \text{irreducible}\\ \text{(no real roots)}\\ \text{quadratics over } \mathbb{R}\end{pmatrix}.$$

Polynomials over $\mathbb{Z}$ or $\mathbb{Q}$

As noted previously, it is enough to consider polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, a_2, \ldots, a_n$ are in $\mathbb{Z}$, and $a_n \neq 0$.

Suppose $\frac{p}{q}$, where p,q are relatively prime integers, and $q > 0$, is a root of $p(x)$, i.e.

$$a_0 + a_1 \frac{p}{q} + a_2 \frac{p^2}{q^2} + \cdots a_n \frac{p^n}{q^n} = 0.$$

So

$$a_0 q^n + a_1 p q^{n-1} + a_2 p^2 a^{n-2} + \cdots + a_n p^n = 0.$$

Since p is a factor of all but the first term it follows that p divides $a_0 q^n$, and so, since p and q are relatively prime, p must divide a_0. That is, when expressed with lowest denominator, **the numerator of any rational root of $p(x)$ is a divisor of the constant term a_0**. Similarly, a_n must be divisible by q, i.e. **the denominator of such a root divides the highest order coefficient a_n**.

In particular, we have

(1) The only possible integer roots must be $\pm$ factors of the constant term a_0.

Example: Show that $p(x) = x^3 - x^2 - 3x + 1$ has no integer roots.

Solution: The only possible integer roots are $\pm$ factors of 1, and so ae $+1$ or -1. By direct checking, $p(1) = -2$, $p(-1) = 2$. Neither of these are roots, hence no integer roots exist.

(2) If a_n and a_0 are primes, the only possible rational roots are ± 1, $\pm a_0$, $\pm 1/a_n$, $\pm a_0/a_n$. So,

(3) When a_n, $a_0 = \pm 1$, the only possible rational roots are ± 1.

E.g. Not only does $p(x) = x^3 - x^2 - 3x + 1$ have no integer roots (see above) it has no **rational roots.**

Note: If $a_0 = 0$, p factorizes as $p(x) = xq(x)$, in which case 0 is a root and we can apply the above considerations to the $(n-1)$-degree polynomial q.

These results are not only of theoretical interest; they also provide useful limits on the search for integer or rational roots of a polynomial.

Relationships among the roots

Let $p(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$ (regarded as a function on $\mathbb{C}$) be factorized as

$$p(x) = (x - r_1)(x - r_2)\cdots(x - r_n).$$

Then, multiplying out, we have

$$x^n + \cdots + a_0 = x^n - (r_1 + \cdots + r_n)x^{n-1} + \cdots + (-1)^n r_1 \cdots r_n.$$

Equating coefficients leads to the *symmetric functions* of the roots and their relationships with the coefficients; in particular, the product of the roots is given by $r_1 \cdots r_n = (-1)^n a_0$ and the sum of the roots is given by $r_1 + \cdots + r_n = -a_{n-1}$.

These relationships are a source of endless problems.

E.g. (NSW, 1987 HSC) Let the polynomial $p(x)$ be given by $p(x) = x^7 - 1$. Let $\rho \neq 1$ be that complex root of $p(x) = 0$ which has the smallest positive argument. Let $\theta = \rho + \rho^2 + \rho^4$ and $\phi = \rho^3 + \rho^5 + \rho^6$.

Show that $\theta + \phi = -1$ and $\theta\phi = 2$.

We first observe that $\rho = cos(2\pi/7) + i\sin(2\pi/7)$ and that the seven roots of $x^7 - 1$ are $1, \rho, \rho^2, \ldots, \rho^6$. Then

$$\theta + \phi = \rho + \rho^2 + \cdots + \rho^6 = \text{sum of roots} - 1 = 0 - 1 = -1,$$

and

$$\begin{aligned}\theta\phi &= (\rho + \rho^2 + \rho^4)(\rho^3 + \rho^5 + \rho^6) \\ &= \rho^4 + \rho^5 + \rho^6 + 3\rho^7 + \rho^8 + \rho^9 + \rho^{10} \\ &= 3 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 \quad (\text{using } \rho^7 = 1) \\ &= 2 + (1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6) \\ &= 2 \quad (\text{as sum of the roots is } 0),\end{aligned}$$

as required.

E.g. (NSW 1983 HSC) If α, β, γ are the roots of $x^3 + qx + r \quad (r \neq 0)$, find a polynomial with coefficients expressed in terms of q and r whose roots are α^2, β^2, γ^2.

Now, $r = \alpha\beta\gamma$, $q = \alpha\beta + \alpha\gamma + \beta\gamma$, and $\alpha + \beta + \gamma = 0$, and a polynomial with roots α^2, β^2, γ^2 is:

$$\begin{aligned}(x - \alpha^2)(x - \beta^2)(x - \gamma^2) \\ &= x^3 - (\alpha^2 + \beta^2 + \gamma^2)x^2 + (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)x - \alpha^2\beta^2\gamma^2 .\end{aligned}$$

Further,

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) .$$

So,

$$0 = \alpha^2 + \beta^2 + \gamma^2 + 2q .$$

Hence,

$$\alpha^2 + \beta^2 + \gamma^2 = -2q .$$

Also

$$\begin{aligned} q^2 &= (\alpha\beta + \alpha\gamma + \beta\gamma)^2 \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2) \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 ,\end{aligned}$$

and

$$r^2 = (\alpha\beta\gamma)^2 = \alpha^2\beta^2\gamma^2 .$$

Thus a polynomial of the required form is

$$x^3 + 2qx^2 + q^2x - r^2 .$$

The Calculus as a Tool for Studying Polynomials

The possibilities here are too numerous to do more than give a couple of typical examples.

E.g. (NSW 1986 HSC) If $(x - 1)^2$ is a factor of $p(x) = \alpha x^{n+1} + \beta x^n + 1$, find α and β.

From the product rule for differentiation, we see that $(x - 1)$ is also a factor of $p'(x)$, and so -1 is a root of both $p(x)$ and $p'(x)$. Hence

$$\begin{aligned} \alpha + \beta + 1 &= 0 \qquad (p(1) = 0) \\ \text{and } (n + 1)\alpha + n\beta &= 0 \qquad (p'(1) = 0) .\end{aligned}$$

Solving these equations gives $\alpha = n$ and $\beta = -(1 + n)$.

E.g. For the polynomial

$$p(x) = x^3 - 3ux + v\,,$$

investigate the number of real roots for various values of u and v.

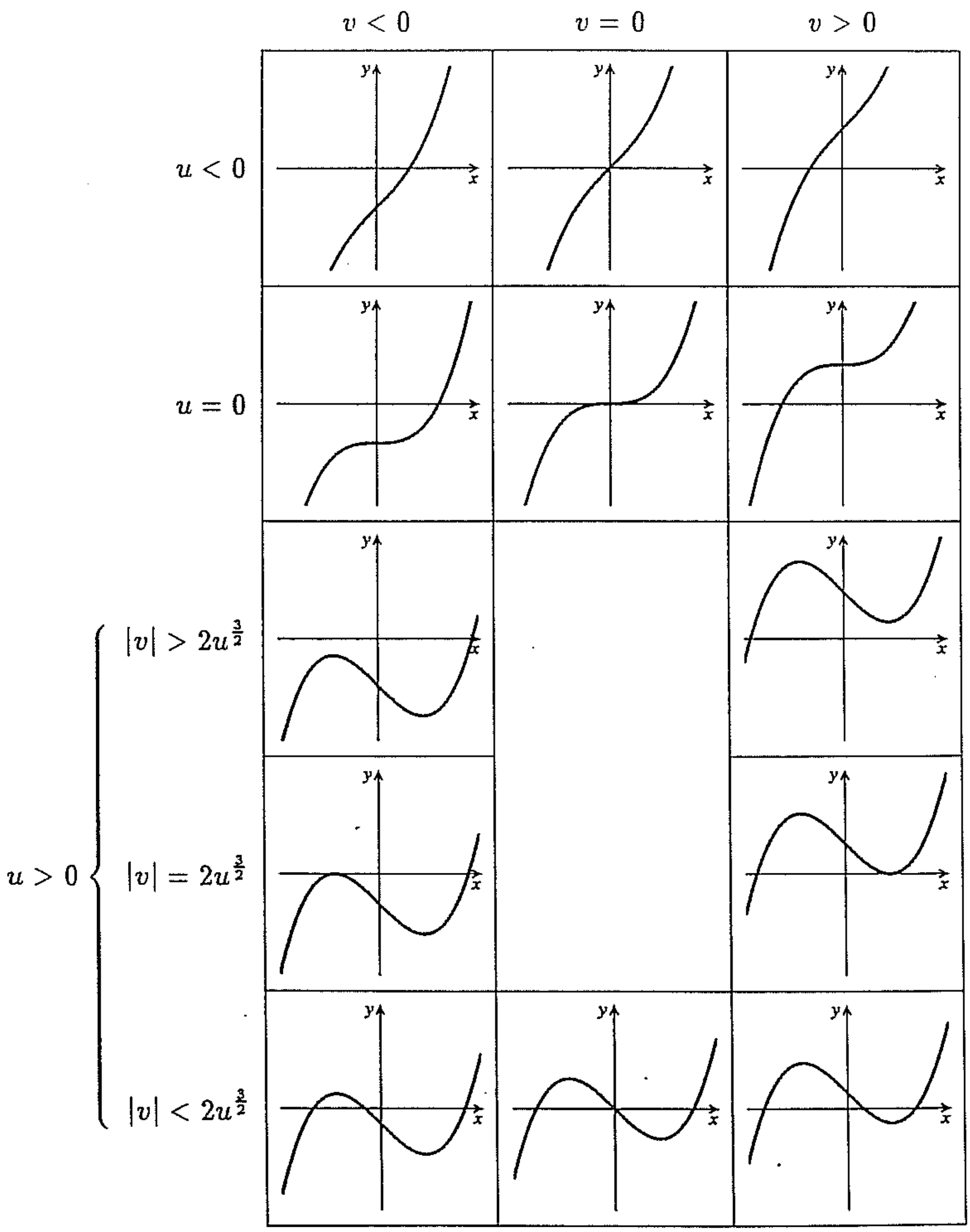


Figure 1. Graphs of $y = x^3 - 3ux + v$

Now, $p'(x) = 3x^2 - 3u$. Thus, if $u \leq 0$, $p'(x) \geq 0$ and p is (strictly) increasing, so $p(x)$ has one real root.

If $u > 0$, $p'(x) = 0$ at $x = \pm\sqrt{u}$, so $p(x)$ has local maximum and minimum at these points. The values of $p(x)$ at these points are

$$\mp u^{\frac{3}{2}} \pm 3u^{\frac{3}{2}} + v = \pm 2u^{\frac{3}{2}} + v\,.$$

We see that there will three real roots if and only if these values are of opposite sign; this is so if $|v| < 2u^{\frac{3}{2}}$. There will be only one real root if the values have the same sign, i.e. if $|v| > 2u^{\frac{3}{2}}$. In the remaining case, $|v| = 2u^{\frac{3}{2}}$, one of the stationary values will be zero; this gives a double root.

We conclude that $x^3 - 3ux + v$ has one real root whenever $u \leq 0$ or $|v| > 2u^{\frac{3}{2}}$, two real roots (with one being a repeated root) when $u > 0$ and $v = 2u^{\frac{3}{2}}$, and three real roots if $u > 0$ and $|v| < 2u^{\frac{3}{2}}$.

If we take into account the signs of the roots, we find there are thirteen cases of polynomials of the form $x^3 - 3ux + v$. Graphs of these are illustrated in Figure 1.

POLYNOMIALS

1. (a) Write down the relations which hold between the roots α, β and γ of the equation

$$ax^3 + bx^2 + cx + d = 0, \quad (a \neq 0),$$

and the coefficients a, b, c and d.

(b) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given that the roots α, β and γ of this equation satisfy $\alpha = \beta + \gamma$. Find α.

(c) Suppose the equation $x^3 + px^2 + qx + r = 0$ has roots λ, μ and ν which satisfy $\lambda = \mu + \nu$. Show that $p^3 - 4pq + 8r = 0$.

2. (a) Suppose that k is a double root of the polynomial equation $f(x) = 0$. Show that $f'(k) = 0$.

(b) What feature does the graph of a polynomial have at a root of multiplicity 2?

(c) The polynomial $P(x) = ax^7 + bx^6 + 1$ is divisible by $(x-1)^2$.
Find the coefficents a and b.

(d) Let $E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$.
Prove $E(x) = 0$ has no double roots.

(e) Consider the following statements about a polynomial $Q(x)$.

i. If $Q(x)$ is even, then $Q'(x)$ is odd.

ii. If $Q'(x)$ is even, then $Q(x)$ is odd.

Indicate whether each of these statements is true or false. Give reasons for your answers.

(f) Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number w is a root of $P(z) = 0$.

i. Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$.

ii. Find one of the roots of $P(z) = 0$ in exact form.

iii. Hence find all the roots of $P(z) = 0$.

3. Suppose that a and b are real numbers and $d \neq 0$. Consider the polynomial

$$P(z) = z^4 + bz^2 + d.$$

The polynomial has a double root at α.

(a) Prove that $P'(z)$ is an odd function.

(b) Prove that $-\alpha$ is also a double root of $P(z)$.

(c) Prove that $d = \frac{b^2}{4}$.

(d) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}i$?

(e) For what values of b does $P(z)$ have real roots?

4. Consider the polynomial equation

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

Where a, b, c and d are all integers. Suppose the equation has a root of the form ki, where k is real, and $k \neq 0$.

(a) State why the conjugate $-ki$ is also a root.

(b) Show that $c = k^2 a$.

(c) Show that $c^2 + a^2 d = abc$.

(d) If 2 is also a root of the equation, and $b = 0$, show that c is even.

5. Let $f(t) = t^3 + ct + d$, where c and d are constants.
Suppose that the equation $f(t) = 0$ has three distinct real roots, t_1, t_2 and t_3.

(a) Find $t_1 + t_2 + t_3$.

(b) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$.

(c) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$. Show that

$$27d^2 + 4c^3 < 0.$$

6. Let $x = \alpha$ be a root of the quartic polynomial

$$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1,$$

Where A and B are real. Note that α may be complex.

(a) Show that $\alpha \neq 0$.

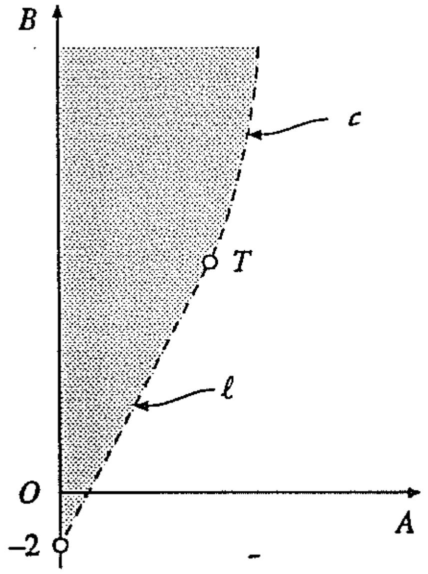
(b) Show that $x = \alpha$ is also a root of

$$Q(x) = x^2 + \frac{1}{x^2} + A\left(x + \frac{1}{x}\right) + B.$$

(c) With $u = x + \frac{1}{x}$, show that $Q(x)$ becomes

$$R(u) = u^2 + Au + (B - 2).$$

(d) For certain values of A and B, $P(x)$ has no real roots. Let $\mathcal{D}$ be the region of the AB plane where $P(x)$ has no real roots and $A \geq 0$.



The region $\mathcal{D}$ is shaded in the figure. Specify the bounding straight line segment l and curved segment c. Determine the coordinates of T.

7. Find all roots of the equation

$$18x^3 + 3x^2 - 28x + 12 = 0,$$

given that two of the roots are equal.

8. Let $x = \alpha$ be a root of the quartic polynmomial

$$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$$

where $(2 + B)^2 \neq 4A^2$.

(a) Show that α cannont be 0, 1, or -1.

(b) Show that $x = \dfrac{1}{\alpha}$ is a root.

(c) Deduce that if α is a multiple root, then its multiplicity is 2 and

$$4B = 8 + A^2.$$

9. Let $w = \cos\left(\dfrac{2\pi}{9}\right) + i\sin\left(\dfrac{2\pi}{9}\right)$.

   (a) Show that w^k is a solution of $z^9 - 1 = 0$, where k is an integer.

   (b) Prove that

$$w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1.$$

   (c) Hence show that

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}$$

10. Suppose that $z^7 = 1$ where $z \neq 1$.

   (a) Deduce that

$$z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0.$$

   (b) By letting $x = z + \dfrac{1}{z}$ reduce the equation in (a) to a cubic equation in x.

   (c) Hence deduce that

$$\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right) = \frac{1}{8}$$