

# CIRCULAR MOTION

JACQUI RAMAGGE AND BRAILEY SIMS

## 1. INTRODUCTION

The aim of this lecture is to

- present an overview of circular motion (perhaps with a new slant),
- remind you of what you should know,
- help diagnose what you need to work on,
- illustrate the interplay between seemingly unrelated HSC topics.

We will look at the mechanics of motion with an emphasis on circular motion. Mechanics is the study of motion and consists of

- kinematics — the study of abstract motion without reference to matter or force, and
- dynamics — the study of the relationship between motion and force.

Circular motion has many applications and is historically of great importance. It was the first motion analyzed by Newton in his *Principia Mathematica*; almost all of the first book is devoted to circular motion.

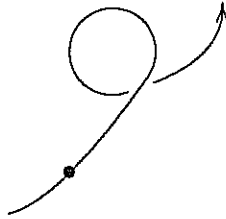
Most of you will be familiar with motion along a line. When looking at motion in a plane you may have tackled the problem by breaking the motion down into vertical and horizontal components. This approach allows you to use the results you know about motion in a line to deduce results about motion in the plane. We show how to use complex numbers to model circular motion and discuss the benefits (and drawbacks!) this has over the more traditional approach.

---

These notes were prepared by Jacqui Ramagge to be delivered as part of an HSC study day held at the University of Newcastle. They were based on a set of notes written for a similar purpose by Brailey Sims.

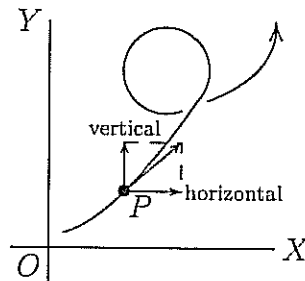
## 2. MOTION IN THE PLANE

Suppose we want to describe the path of an object in the plane such as that depicted below.



We could

- use Cartesian coordinates,
- locate a point  $P$  by giving its horizontal and vertical distance from the origin  $O$ ,
- resolve the motion (the velocity, acceleration, or whatever) into **horizontal** and **vertical** components.



Alternatively we could

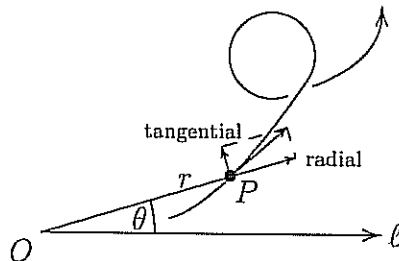
- choose an origin  $O$  and a fixed direction  $\ell$  (corresponding to a half-line from  $O$ ),
- specify the position  $P$  by giving

$$r = \text{"length"} OP = \text{radial distance from the origin to } P$$

and

$$\theta = \text{angle } OP \text{ makes with } \ell \text{ measured anticlockwise } \odot,$$

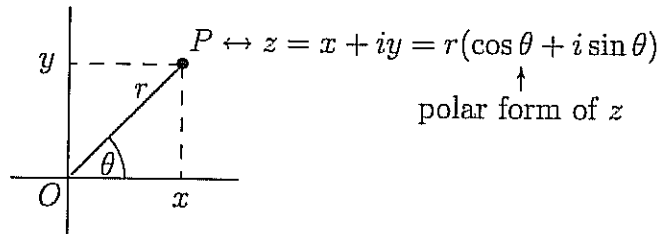
- resolve motion into **radial** and **tangential** components.



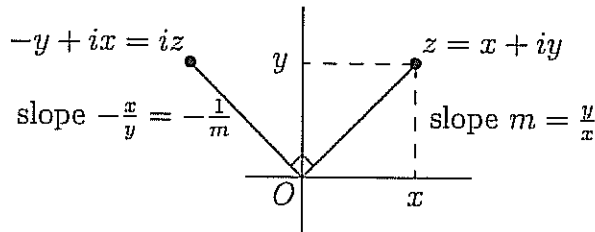
Note that with this model the directions of the components change with  $P$ . The radial direction is in line with  $OP$  and directed outwards (unless the component is negative

in which case it is directed inwards). The tangential direction is perpendicular to the radial direction and is positive in the anticlockwise direction

To help analyze the motion we use **complex numbers** to represent points on the plane (positions) as illustrated.



It will also be useful to recall that multiplying a complex number  $z$  by the imaginary number  $i$  (where  $i^2 = -1$ ) rotates  $z$  through a right angle about  $O$  in an anticlockwise direction.



We can now analyze motion where  $P$  is a function of time  $t$  (and hence so are  $r$  and  $\theta$ ).

$$P \leftrightarrow z = x + iy = r(\cos \theta + i \sin \theta)$$

**velocity** = rate of change of  $P$  with respect to  $t$

$$\begin{aligned} \leftrightarrow \dot{z} &= \frac{dz}{dt} = \text{derivative of } z \text{ with respect to time } t \\ &= \frac{d}{dt} (r(\cos \theta + i \sin \theta)) \end{aligned}$$

$$= \dot{r}(\cos \theta + i \sin \theta) + r(-\sin \theta + i \cos \theta)\dot{\theta}$$

(using product and chain rules)

$$= \dot{r}(\cos \theta + i \sin \theta) + ir(\cos \theta + i \sin \theta)\dot{\theta}$$

$$= \begin{array}{l} \text{complex number} \\ \text{parallel to } z \end{array} + \begin{array}{l} \text{complex number} \\ \text{perpendicular to } z \end{array}$$

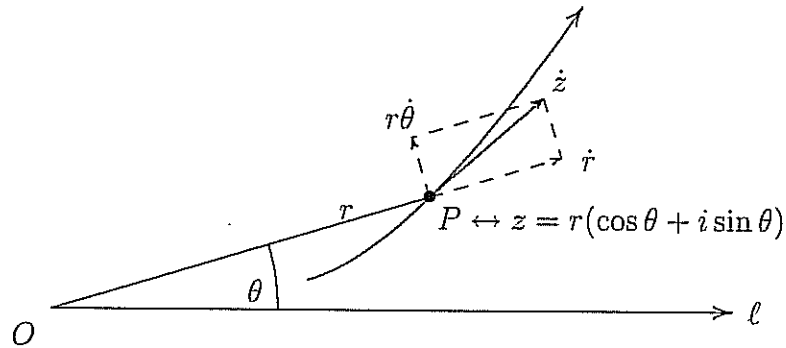
$$= \begin{array}{l} \text{radial component} \\ \text{of velocity} \end{array} + \begin{array}{l} \text{tangential component} \\ \text{of velocity} \end{array}$$

Thus the components of velocity are

$$\dot{r} \text{ directed radially away from } O$$

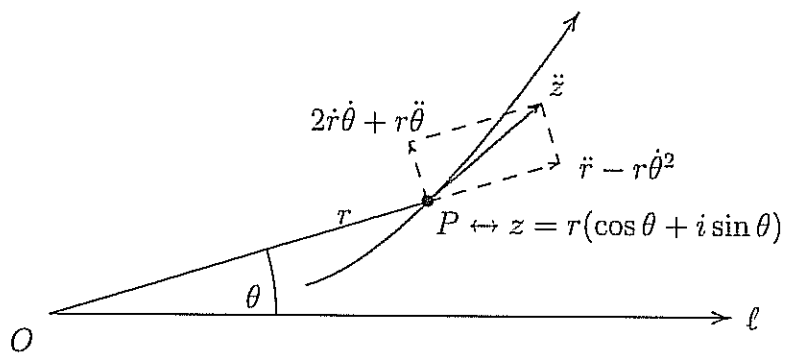
and

$$r\dot{\theta} \text{ at an angle } \frac{\pi}{2} \text{ (anticlockwise) to the radial direction } OP.$$



Acceleration may be analyzed similarly to obtain

$$\begin{aligned}
 \text{acceleration } \leftrightarrow \ddot{z} &= \frac{d}{dt}(\text{velocity}) \\
 &= \underbrace{\left( \ddot{r} - r\dot{\theta}^2 \right)}_{\text{radial component}} + i \underbrace{\left( 2\dot{r}\dot{\theta} + r\ddot{\theta} \right)}_{\text{tangential component}} (\cos \theta + i \sin \theta)
 \end{aligned}$$



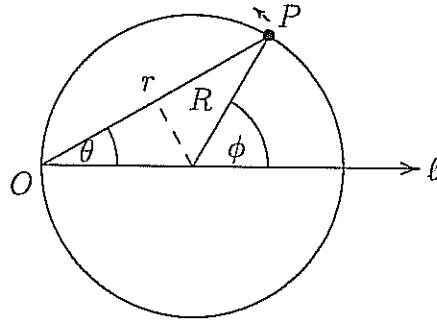
Any motion in the plane can be analyzed this way, but in general the resulting equations will be difficult to “solve”.

We now specialize to **motion around a circle**.

## 3. KINEMATICS OF CIRCULAR MOTION

Even in this case the equations are messy without a judicious choice of origin  $O$ . Two good choices are

- $O$  on the perimeter of the circle,
- $O$  at the centre of the circular motion.

3.1.  $O$  on the perimeter.

Since  $P$  is constrained to move on the circle,  $R$  doesn't change with time so  $\dot{R} = 0$ . The variables  $\phi$ ,  $\theta$  and  $r$  all change with time. Now

$$\theta = \frac{\phi}{2} \quad \text{and} \quad r = 2R \cos \theta.$$

So we have

$$\dot{\theta} = \frac{\dot{\phi}}{2} \quad \text{and} \quad \dot{r} = -2R \sin \theta \dot{\theta}.$$

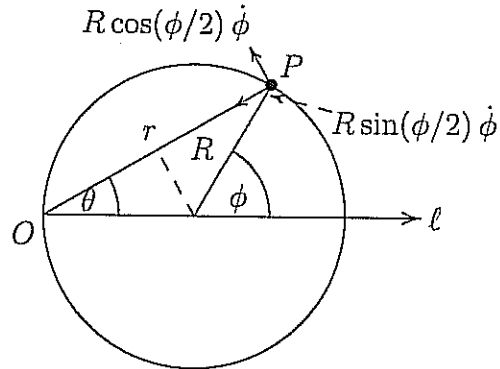
This makes the radial velocity

$$\dot{r} = -2R \sin \theta \dot{\theta} = -2R \sin(\phi/2) \frac{\dot{\phi}}{2} = -R \sin(\phi/2) \dot{\phi}$$

and the tangential velocity

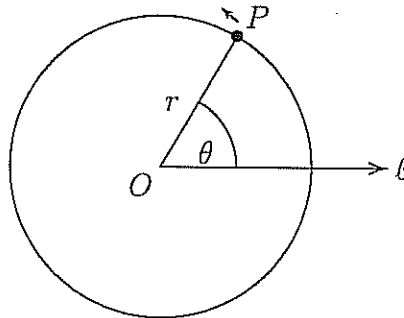
$$r\dot{\theta} = 2R \cos \theta \dot{\theta} = 2R \cos(\phi/2) \frac{\dot{\phi}}{2} = R \cos(\phi/2) \dot{\phi}.$$

We summarize these facts in the following diagram.



*Exercise 3.1.* Find the acceleration of  $P$  in terms of  $\phi$  and  $R$ .

3.2.  $O$  at the center of circular motion.



In this case  $r$  doesn't change because the motion is constrained to the circle so the radial velocity is  $\dot{r} = 0$ .

The

tangential velocity is  $r\dot{\theta}$ .

The

radial acceleration is  $-r\dot{\theta}^2$ ,

the negative sign indicating that radial acceleration is towards the centre. The

tangential acceleration is  $r\ddot{\theta}$ .

In the special case of **uniform circular motion** where the

angular velocity  $\dot{\theta} = \omega$ , a constant

we have

$$\text{tangential velocity } v = r\omega,$$

and the only acceleration is inward radial acceleration

$$\alpha = r\omega^2 = \frac{v^2}{r}.$$

Note: the constant angular velocity  $\omega$  corresponds to

$$\begin{aligned} \omega &= \frac{2\pi}{T} \quad \text{where } T \text{ is the period of one revolution} \\ &= 2\pi f \quad \text{where } f \text{ is the frequency; the number of revolutions per unit of time.} \end{aligned}$$

#### 4. DYNAMICS OF UNIFORM CIRCULAR MOTION

Newton's second law of motion says that

$$\text{Force} = \text{mass} \times \text{acceleration.}$$

Using polar coordinates and putting  $O$  at the centre of the circular motion we see that a particle of mass  $m$  will execute uniform circular motion if and only if it is subjected to a **centripetal** (centrally directed) force of magnitude

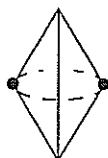
$$F = mr\dot{\theta}^2 = \frac{mv^2}{r}.$$

For non-uniform circular motion a tangential force must also operate.

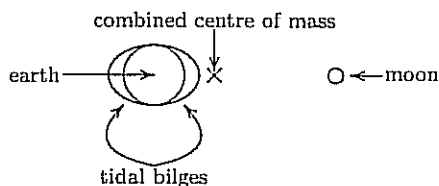
The identification of these forces is the key to the analysis of circular motion.

Applications of the dynamics of circular motion include the following examples.

*Examples 4.1.* 1. Conical pendulums — the action of governors in steam engines.

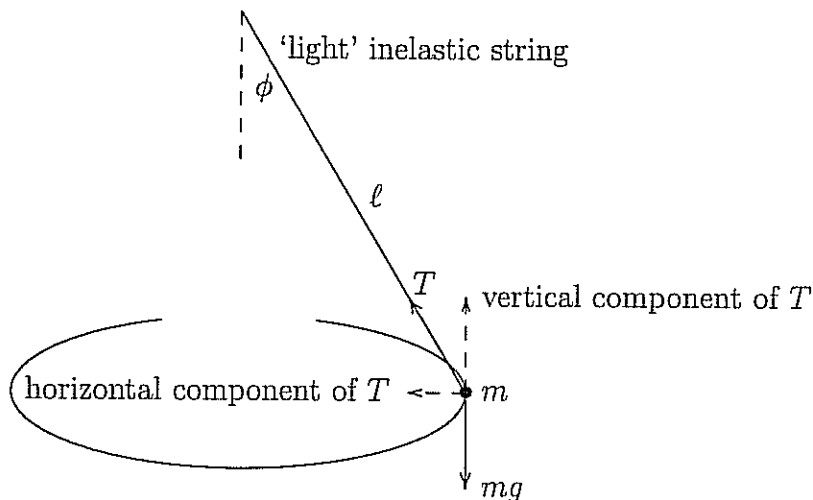


2. Forces on a vehicle rounding a curve — analysis of skidding.
3. Circular motion under inverse square laws for a central force — various applications including
  - gravitation; satellites, Kepler's third law,
  - electrostatic attraction; Bohr model of the hydrogen atom.
4. Change in gravity due to latitude.
5. Centrifuge — fun parks.
6. Tides — Roche's limit.



We analyze some of these in detail.

#### 4.1. Conical Pendulum.



In the above diagram,  $T$  is the tension in the string. Its vertical component balances  $mg$  to maintain the motion in a plane, its horizontal component supplies the centripetal force. Thus

$$\begin{aligned} T \cos \phi &= mg && \text{and} \\ T \sin \phi &= m\ell \sin \phi \omega^2 \end{aligned}$$

since  $r = \ell \sin \phi$ . So

$$\frac{mg}{\cos \phi} = m\ell \omega^2$$

and hence

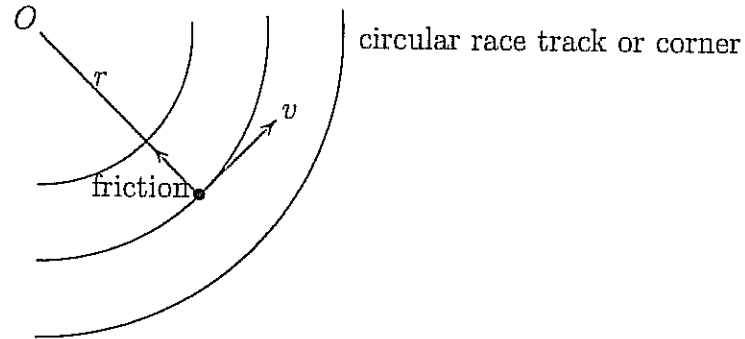
$$\omega^2 = \frac{g}{\ell \cos \phi}.$$

So, as  $\omega$  increases  $\cos \phi$  decreases, and hence  $\phi$  tends towards  $\pi/2$ . This means that the faster you spin the object, the more the mass rises. The period of oscillation is

$$T = 2\pi \sqrt{\frac{\ell \cos \phi}{g}} \approx 2\pi \sqrt{\frac{\ell}{g}} \quad \text{for small } \phi.$$



4.2. Skidding.

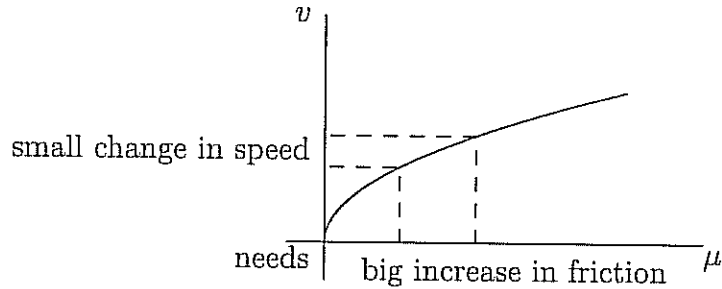


We assume there is no friction in the direction of motion, only towards the centre. This is an oversimplification, but it is sufficient for our purposes.

Here the centripetal force is due to friction and we have

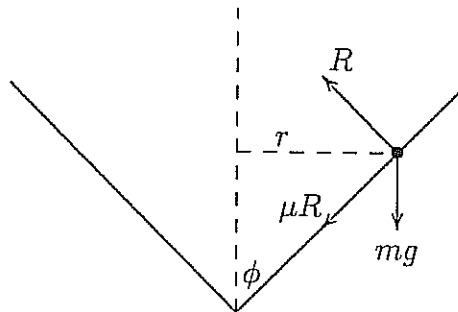
$$\frac{mv^2}{r} = \mu mg$$

where  $\mu$  is the friction coefficient and  $\mu \leq \mu_{max}$ . So  $v = \sqrt{\mu gr}$ ,



and skidding occurs if  $\mu = \frac{v^2}{rg} > \mu_{max}$ .

4.3. Banked Tracks and Fun Parks.



The vertical forces are

$$mg = R \sin \phi - \mu R \cos \phi.$$

The horizontal forces are

$$R \cos \phi + \mu R \sin \phi = mr\omega^2 = \frac{mv^2}{r}.$$

So

$$r\omega^2 = \frac{v^2}{r} = g \frac{\cos \phi + \mu \sin \phi}{\sin \phi - \mu \cos \phi}.$$

The extreme cases correspond to

- $\phi = \pi/2$ ; corresponds to a flat track and we get  $v^2 = \mu gr$  as before.
- $\phi = 0$ ; corresponds to spinning in a cylinder and we get  $v = \sqrt{\frac{rg}{\mu}}$  and

increase in  $v \leftrightarrow$  decrease in  $\mu$ .

- frictionless case,  $\mu = 0$ ; corresponds to  $\frac{v^2}{r} = g \cot \phi$ . So

increase in  $v \leftrightarrow$  increase in  $\cot \phi \leftrightarrow$  decrease in  $\phi$ .

Note that these results are independent of how far up the incline the mass is!

This explains why velodrome tracks get steeper towards the outside. You position yourself on the track where the incline is appropriate to your speed. The faster you go, the further up the track you position yourself.

#### REFERENCES

- [1] I. Newton, *Principia Mathematica*, 1687.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE, NSW 2308, AUSTRALIA.  
*E-mail address:* jacqui@maths.newcastle.edu.au

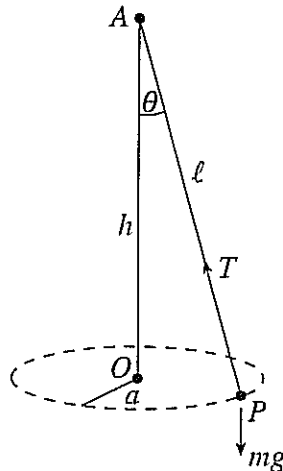
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE, NSW 2308, AUSTRALIA.  
*E-mail address:* bsims@maths.newcastle.edu.au

**QUESTION 5** Use a SEPARATE Writing Booklet.

**Marks**

- (a) The roots of  $x^3 + 5x^2 + 11 = 0$ , are  $\alpha$ ,  $\beta$  and  $\gamma$ . 3
- (i) Find the polynomial equation whose roots are  $\alpha^2$ ,  $\beta^2$  and  $\gamma^2$ .
- (ii) Find the value of  $\alpha^2 + \beta^2 + \gamma^2$ .

- (b) 4



A conical pendulum consists of a bob  $P$  of mass  $m$  kg and a string of length  $\ell$  metres. The bob rotates in a horizontal circle of radius  $a$  and centre  $O$  at a constant angular velocity of  $\omega$  radians per second. The angle  $OAP$  is  $\theta$  and  $OA = h$  metres. The bob is subject to a gravitational force of  $mg$  newtons and a tension in the string of  $T$  newtons.

- (i) Write down the magnitude, in terms of  $\omega$ , of the force acting on  $P$  towards centre  $O$ .
- (ii) By resolving forces, show that  $\omega^2 = \frac{g}{h}$ .

Question 5 continues on page 7.

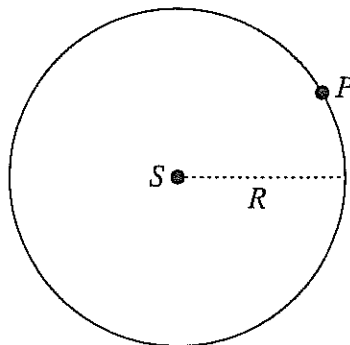
**QUESTION 4.** Use a SEPARATE Writing Booklet.

**Marks**

- (a) (i) Suppose that  $k$  is a double root of the polynomial equation  $f(x) = 0$ . Show that  $f'(k) = 0$ . 7
- (ii) What feature does the graph of a polynomial have at a root of multiplicity 2?
- (iii) The polynomial  $P(x) = ax^7 + bx^6 + 1$  is divisible by  $(x-1)^2$ . Find the coefficients  $a$  and  $b$ .
- (iv) Let  $E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ . Prove  $E(x) = 0$  has no double roots.

(b)

5



A planet  $P$  of mass  $m$  kilograms moves in a circular orbit of radius  $R$  metres around a star  $S$ . Coordinate axes are taken in the plane of the motion, centred at  $S$ . The position of the planet at time  $t$  seconds is given by the equations

$$x = R \cos \frac{2\pi t}{T} \quad \text{and} \quad y = R \sin \frac{2\pi t}{T},$$

where  $T$  is a constant.

- (i) Show that the planet is subject to a force of constant magnitude,  $F$  newtons.
- (ii) It is known that the magnitude of the gravitational force pulling the planet towards the star is given by

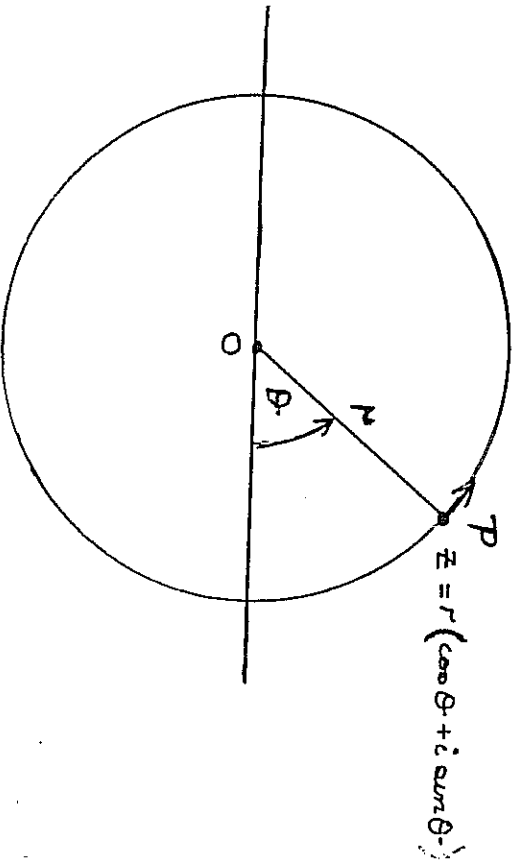
$$F = \frac{GMm}{R^2},$$

where  $G$  is a constant and  $M$  is the mass of the star  $S$  in kilograms. Find an expression for  $T$  in terms of  $R$ ,  $M$  and  $G$ .

# CIRCULAR MOTION

## KINEMATICS

We analyse the motion of a point  $P$  constrained to lie on a circle of given radius  $r > 0$ , and for convenience, centre at the origin  $O$ .



The position of  $P$  at time  $t$  is completely determined by the angle  $\theta = \theta(t)$ , and may be represented by the complex number  $z(t) = r(\cos \theta(t) + i \sin \theta(t))$ .

We assume that  $\theta$  (and hence  $z$ ) is a differentiable, therefore continuous, function of  $t$ .

In particular then  $\theta$  increases by  $2\pi$  when  $P$  makes a complete revolution in the anti-clockwise direction.

The velocity  $V$  of  $P$

= time rate of change of position

$$= \frac{dz}{dt}$$

$$= \frac{d}{dt} r(\cos \theta + i \sin \theta) \quad (r \text{ a constant})$$

$$= r \left( -\sin \theta \frac{d\theta}{dt} + i \cos \theta \frac{d\theta}{dt} \right)$$

$$= r \frac{d\theta}{dt} (i^2 \sin \theta + i \cos \theta)$$

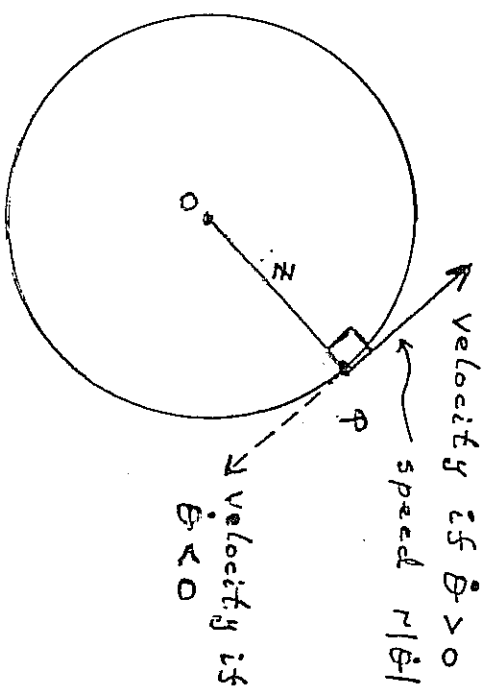
$$= r \frac{d\theta}{dt} (\cos \theta + i \sin \theta) i$$

parallel to  $z$

rotated through  
a right angle

Thus, the velocity is perpendicular to  $z$ ; that is, tangential to the circle, and has magnitude (speed)

$$|V| = \left| \frac{dz}{dt} \right| = r \left| \frac{d\theta}{dt} \right| = r |\dot{\theta}|.$$



$\omega = \dot{\theta} = \frac{d\theta}{dt}$  is the angular velocity of P about O,

If  $\omega = \dot{\theta}$ , and consequently  $|\omega|$  is constant we have uniform circular motion. This happens if and only if the 'angular acceleration'  $\ddot{\theta} = 0$ .

We restrict ourselves to this case.

Then,  $\omega = 2\pi / T$  ← time for 1 revolution  
 =  $2\pi f$  ← frequency; Number of revolutions per unit of time.

The acceleration  $a$  of P

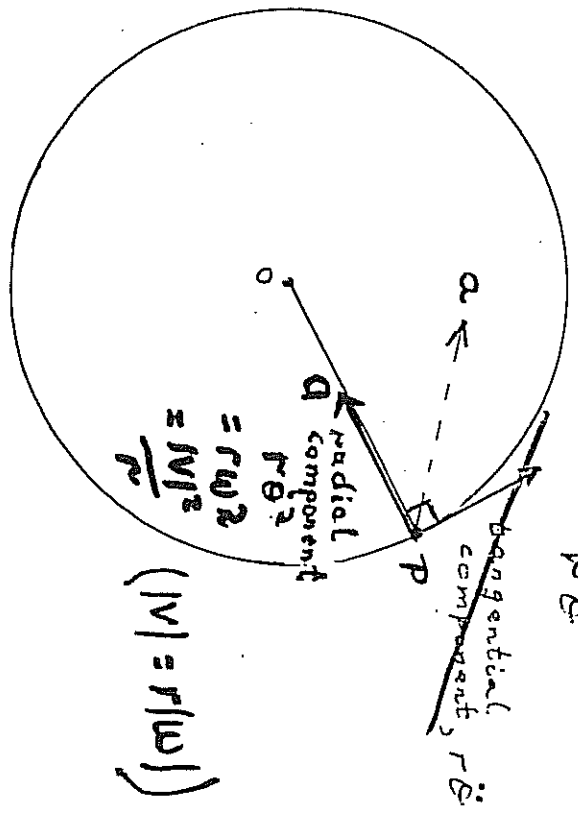
= time rate of change of velocity

=  $\frac{dv}{dt} \left( = \frac{d^2z}{dt^2} \right)$

=  $\frac{d}{dt} r \dot{\theta} (\cos\theta + i \sin\theta) i$

=  ~~$r \ddot{\theta} (\cos\theta + i \sin\theta) i$~~  + 0 for uniform circular motion.  
 $r \dot{\theta} (-\sin\theta \dot{\theta} + i \cos\theta \dot{\theta}) i$

=  ~~$r \ddot{\theta} (\cos\theta + i \sin\theta) i - r \dot{\theta}^2 (\cos\theta + i \sin\theta) i$~~   
 tangential component of magnitude  $r|\dot{\theta}|$   
 inwardly directed radial component of magnitude  $r\dot{\theta}^2$



# DYNAMICS of uniform circular motion.

From Newton's second law:

the change of motion [ie velocity] is proportional to the motive force impressed and is made in the direction of the right [straight] line in which that force is impressed  
ie Force = mass  $\times$  acceleration,

we see that a particle of mass  $m$  will execute uniform circular motion if and only if it is subjected to a centrally directed force of magnitude

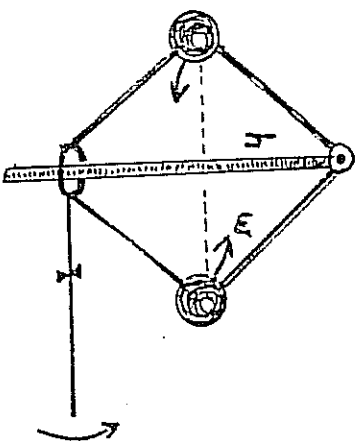
$$F = m r \dot{\theta}^2 = \frac{m v^2}{r}$$

— centripetal [centrifugal] force.

## APPLICATIONS

1) Conical Pendulum

— governors



2) forces on a vehicle rounding a curve — skidding  
— Road cambers

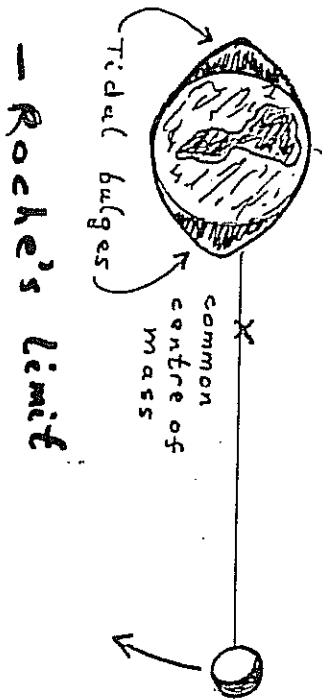
3) Circular motion under inverse square laws for a central force  
— gravitation; satellites,  
Kepler's 3rd. Law

— electrostatic attraction;  
Bohr model of hydrogen atom.

4) Change in gravity due to latitude

5) Centrifuge — fun parks.

6) Tides



- Roche's limit

7) motion of an electric charge in a magnetic field.

- Biot-Savart law
- Cathode ray tubes,
- cyclotrons

8) ...