Complex Numbers: Problems

1. (a) Evaluate i^{1998}.

   (b) Let $z = \dfrac{18+4i}{3-i}$

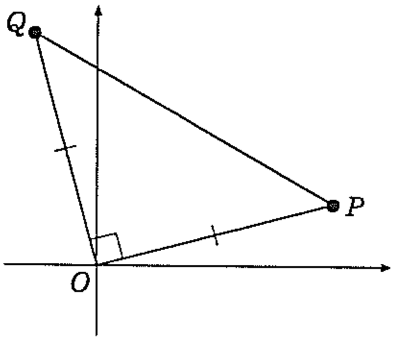
   i. Simplify $(18+4i)\overline{(3-i)}$.

   ii. Express z in the form $a+ib$, where a and b are real numbers.

   iii. Hence, or otherwise, find $|z|$ and $\arg(z)$.

   (c) Sketch the region in the complex plane where the inequalities $|z-2+i| \le 2$ and $\Im(z) \ge 0$ both hold.

   (d) The points P and Q in the complex plane correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and $\angle POQ$ is a right angle.



   Show that $z^2 + w^2 = 0$.

   (e) i. By solving the equation $z^3 + 1 = 0$, find the three cube roots of -1.

   ii. Let λ be a cube root of -1, where λ is not real. Show that $\lambda^2 = \lambda - 1$.

   iii. Hence simplify $(1-\lambda)^6$.

2. (a) i. Express $\sqrt{3} - 1$ in modulus argument form.

   ii. Hence Evaluate $(\sqrt{3}-1)^6$.

   (b) i. Simplify $(-2i)^3$.

   ii. Hence find all complex numbers z such that $z^3 = 8i$. Express your answers in the form $x + iy$.

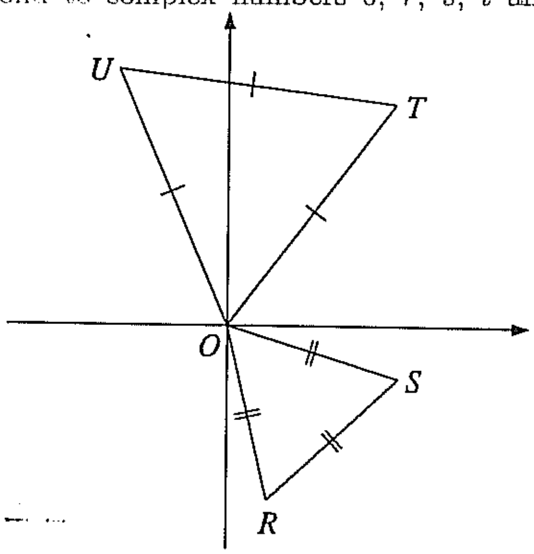
   (c) Sketch the region where the inequalities $|z-3+i| \le 5$ and $|z+1| \le |z-1|$ both hold.

   (d) Let $w = \dfrac{3+4i}{5}$ and $z = \dfrac{5+12i}{13}$, so that $|w| = |z| = 1$.

   i. Find wz and $w\overline{z}$ in the form $x = iy$.

ii. Hence find two distinct ways of writing 65^2 as the sum of $a^2 + b^2$ where a and b are integers and $0 < a < b$.

(e) The diagram shows points O, R, S, T and U in the complex plane. These points correspond to complex numbers 0, r, s, t and u, respectively. The



triangles ORS and OTU are equilateral.

Let $w = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$.

i. Explain why $u = wt$.

ii. Find the complex number r in terms of s.

iii. Using complex numbers show that the length of RT and SU are equal.

3. (a) Suppose that c is a real number, and that $z = c - i$.

Express the following in the form $x = iy$, where x and y are real numbers:

i. $\overline{iz}$;

ii. $\dfrac{1}{z}$

(b) On the Argand diagram shade the region specified by the conditions

$$\Re(z) \leq 4 \text{ and } |z - 4 + 5i| \leq 3.$$

(c) i. Prove by induction that

$$(\cos\theta = i\sin\theta)^n = \cos(n\theta) = i\sin(n\theta)$$

for all integers $n \geq 1$.

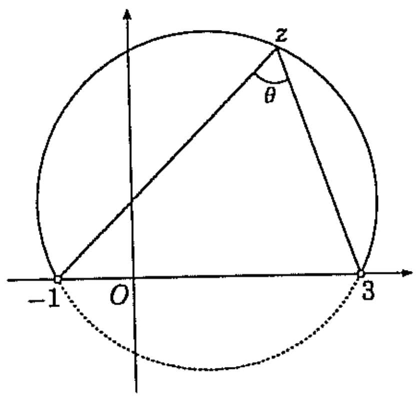
ii. Express $w = \sqrt{3} - i$ in modulus argument form.

iii. Hence express w^5 in the form $x = iy$, where x and y are real numbers.

(d) The diagram shows the locus of points z in the complex plane such that

$$\arg(z-3) - \arg(z+1) = \frac{\pi}{3}.$$

This locus is part of a circle. The angle between the lines from -1 to z and



from 3 to z is θ, as shown.

i. Explain why $\theta = \frac{\pi}{3}$.

ii. Find the center of the circle.

(e) Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

i. Show that w^k is a solution of $z^9 - 1 = 0$, where k is an integer.

ii. Prove that

$$w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$$

iii. Hence show that

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}.$$

4. (a) Let $w_1 = 8 - 2i$ and $w_2 = -5 + 3i$. Find $w_1 + \overline{w_2}$.

(b) i. Show that $(1-2i)^2 = -3 - 4i$.

ii. Hence solve the equation

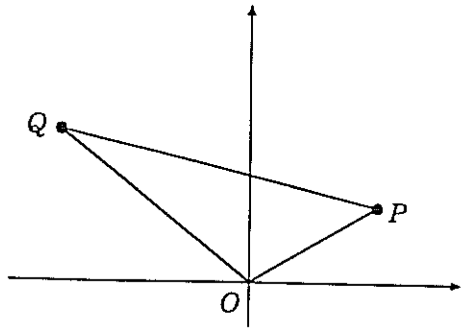
$$z^2 - 5z + (7+i) = 0$$

(c) Sketch the locus of z satisfying:

i. $\arg(z-4) = \frac{3\pi}{4}$

ii. $Im(z) = |z|$.

(d) The diagram shows a complex plane with origin O. The points P and Q represent arbitrary complex numbers z and w respectively. Thus the length of PQ is $|z-w|$.



i. Show that

$$|z-w| \leq |z| + |w|$$

ii. Construct the point R representing $z+w$. What can be said about the quadrilateral $OPRQ$?

iii. If $|z-w| = |z+w|$, what can be said about the complex number $\dfrac{w}{z}$?

5. (a) Let $z = a + ib$ where a and b are real. Find:

i. $\Im(4i - z)$.

ii. $\overline{3iz}$ in the form $x + iy$, where x and y are real.

iii. $\tan\theta$, where $\theta = \arg(z^2)$.

(b) Express in modulus-argument form:

i. $-1+i$,

ii. $(-1+i)^n$, where n is a positive integer.

(c) i. On the same diagram draw a neat sketch of the locus specified by each of the following:

A. $|z-(3+2i)| = 2$

B. $|z+3| = |z-5|$.

ii. Hence write down all values of z which satisfy simultaneously

$$|z-(3+2i)| = 2 \text{ and } |z+3| = |z-5|$$

iii. Use your diagrams in (i) to determine the values of k for which the simultaneous equations

$$|z-(3+2i)| = 2 \text{ and } z - 2i| = k$$

have exactly one solution for z.

6. (a) i. On an Argand diagram shade in the region determined by the inequalities

$$2 \leq Im(z) \leq 4 \text{ and } \frac{\pi}{6} \leq \arg(z) \leq \frac{\pi}{4}.$$

ii. Let z_0 be the complex number of maximum modulus satisfying the inequalities of (i). Express z_0 in the form $a + ib$.

(b) Let θ be a real number and consider

$$(\cos\theta + i\sin\theta)^3.$$

i. Prove that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

ii. Find a similar expression for $\sin 3\theta$.

(c) Find the equation in cartesian form of the locus of the point z if

$$\Re\left(\frac{z-4}{z}\right) = 0$$

(d) By substituting appropriate values of z_1 and z_2 into the equation $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$. Show that $\frac{\pi}{4} = \tan^{-1} 2 - \tan^{-1}\frac{1}{3}$.

(e) Let P, Q and R represent the complex numbers w_1, w_2 and w_3 respectively. What geometric properties characterize triangle PQR if $w_2 - w_1 = i(w_2 - w_1)$?