MODELING WITH EXPONENTIAL FUNCTIONS

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1 Introduction

Exponential functions, functions of the form $y = ka^{\lambda x}$, where k, λ and $a > 0$ are constants (parameters) and x is a real variable, are among the simplest, yet most applicable of the transcendental functions, forming the basis for many mathematical descriptions (models) of real world situations.

When operating in an environment involving calculus it is, for technical reasons, usual to take $a = e := \lim_{n\to\infty}(1+1/n)^n \approx 2.71828$, the base of the natural logarithm, leading to the exponential function e^x (or exp(x)). However, in other contexts a different choice of a may be more expeditious. Indeed, as we will show below, the choice $a = 2$, and so $y = k2^{\lambda x}$, is especially prudent.

The Standard 2 Mathematics syllabus states:

"A4.2 [page 67 of the pdf version]: Non-linear relationships

Students:

*use an exponential model to solve problems

- graph and recognise an exponential function in the form $y = a^x$ and $y = a^{-x}$ $(a > 0)$ using technology

- interpret the meaning of the intercepts of an exponential graph in a variety of contexts

- construct and analyse an exponential model to solve a practical growth or decay problem."

Also, NESA has indicated that, building on work from 5.2 of the Standard 2 syllabus [see page 14 of the pdf] this encompasses students encountering functions of the type $y = c + ka^x$ [and $y = c + ka^{-x}$], in order to use "exponential models".

NOTE: Both $c + ka^x$ and $c + ka^{-x}$ are subsumed by the single expression $c + k2^{\lambda x}$ since λ can be either positive or negative, $a > 0$ can be written as $2^{\log_2(a)}$ and the $\log_2(a)$ absorbed as a factor of λ .

Such models, and the underlying rate assumptions (differential equations) from which they derive also provide excellent motivational and extension material for the Advanced and Extension courses, which could well be the basis for independent investigation and assessment tasks.

2 Practical Applications (Standard 2 and Advanced Courses)

2.1 Simple exponential models

$$y = ka^x \text{, or } y = ka^{-x},$$

where k and $a > 0$ are constants or the more convenient equivalent form

$$y = k2^{\lambda x}\text{, where } k \text{ and } \lambda \text{ are constants} \quad (1)$$

Below are some real-world situations to which simple exponential models can be applied.

(a) Malthusian population growth, $\lambda > 0$. Exponential growth describes the development of a quantity when at any given instant its rate of increase is directly proportional to the amount present at that instant. Thomas Malthus (1766-1834), an early pioneer of political economics, employed exponential growth to model the increase of human populations in times of plenty. Malthusian models have the form $P(t) = P_0 2^{\lambda t}$, where:

$t =$ time,

$P(t)$ is the population size at time t,

$P_0 = P(0)$ is the initial population size, and

λ is the specific growth rate (number of births per head per unit of time), sometimes called the Malthusian parameter.

(b) A nuclear chain reaction. When struck by a neutron the atomic nucleus of certain heavy elements splits (fissions) into the nuclei of two lighter elements with the release of a certain amount of energy, ε, and $\nu \geq 2$ new neutrons. Under these circumstances, initially bombarding a sufficiently large sample of the heavy element with neutrons triggers a chain reaction in which the total energy released after t seconds is $E = ka^t$, where k depends on ε and the intensity of the bombarding neutron beam, $a = 1 + (\nu - 1)p$, and p is the likelihood of a free neutron striking a heavy nucleus within a second . (Of course, the whole process will likely terminate catastrophically when only a small fraction of the heavy element has undergone fission.)

(c) Radioactive decay, $\lambda < 0$. In 1896 the French scientist Henri Becquerel found that uranium emits rays capable of fogging photographic plates in much the same way as light and the then recently discovered X-rays do. Marie and Pierre

Curie named this phenomenon *radioactivity* and furthered Becquerel's work by identifying other heavy elements that exhibited radioactivity. For their discoveries all three shared the 1903 Nobel Prize for Physics. In 1913 the New Zealand born British physicist Ernest Rutherford (known as the 'father of nuclear physics'; awarded the Nobel Prize for Chemistry in 1921) conclusively demonstrated that radioactivity involves the transmutation of an element's nuclei into those of a different element (the alchemist's dream); for example when a uranium nucleus emits an α-ray (essentially, a helium nuclei ejected at high speed) it becomes a nucleus of thorium. Earlier (1899) Rutherford determined that at any moment the amount of a radioactive element decays at a rate proportional to the quantity present at that moment, and so in accordance with the exponential formula,

$$m(t) = m_0 \times 2^{\lambda t},$$

where $m(t)$ is the mass present at the time t, $m_0 = m(0)$ is the initial quantity (quantity at $t = 0$) and $\lambda < 0$ is the negative rate constant.

The speed at which this decay happens is different for each element. We use the concept of half-life to indicate the rate at which decay occurs, that is the time, $-1/\lambda$, taken for half the mass to decay, see section 3.3. For example, ^{238}U, the most common isotope of uranium, has a half-life of about 4.47×10^9 years, while the inert gas radon-220 (a decay product of thorium) has a half-life of 55.6 seconds.

Discussion exercise: Nuclei of a radioactive element decay at random with the probability that any given nucleus will decay during a particular unit of time being constant. How is this probability related to the decay rate λ?

(d) Chemical kinetics. Certain chemical reactions proceed according to an exponential law similar to that for radioactive decay. For example, when dinitrogen pentoxide decomposes into nitrogen dioxide and oxygen at 45°C; $N_2O_5 \to 2NO_2 + O$,the concentration (in moles) of N_2O_5, denoted $[N_2O_5]$, satisfies,

$$[N_2O_5] = [N_2O_5]_0 \times 2^{-6.6\times 10^{-6} t},$$

where t is measured in seconds.

(e) Decomposition of litter on a forest floor. Fallen leaves and other organic matter form a layer of litter on the floor of a forest. Aerobic bacteria act to break the litter down into carbon dioxide and humus (the dark coloured, nitrogen rich, organic component of soil), a process known as composting. While the process is complicated the amount of bacteria adjusts quickly so that it remains at a constant level (the carrying capacity) in the undecomposed litter. This means that at any moment the rate at which material is composted is proportional to the amount of litter available, which therefore decays exponentially.

2.2 Modified exponential model

$$y = c + ka^x \text{ , or } y = c + ka^{-x}$$

where $a > 0$, c and k are constants.

Or, as suggested above, it is convenient to always employ powers of 2, and work with the unified, equally general, model

$y = y_a + k2^{\lambda x}$, where λ, y_a and k are constants. (2)

We use the subscript a on y_a as this is the height of the model's horizontal asymptote.

(a) Spread of mould on a piece of bread, or rust on a wheat crop. Suppose that an initially fresh slice of bread is left exposed so that airborne mould spoors settle at a fixed rate onto random points on the top surface of the slice. If A is the total area of the top surface, then a simple model describing the area of the top surface covered by mould at time t is:

$$m = m(t) = A(1 - e^{-\lambda t}),$$

where λ is related to (in fact, directly proportional to) both the density of spoors in the air and the rate at which mould cells replicate in situ on the bread. For a vindication of this model see section 4.2 (a).

As with the other two situations introduced below, this could form the basis of a student investigation. Daily photographs of the developing mould could be taken using a mobile phone, see Figure 1. The area of mould could be measured by superimposing a grid over each photograph and counting squares. The measurements could be plotted and, as detailed below, a model fitted.

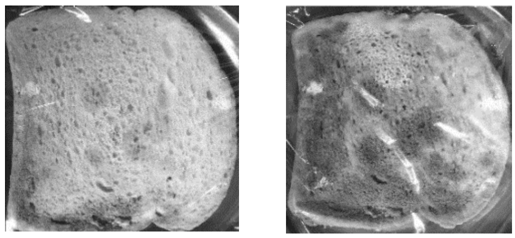

Figure 1: Growth of mould on a slice of bread. Left after 2 days, right after 4 days.

(b) How your soft drink warms or your coffee cools. Consider a cold [hot] object warming [cooling] to the ambient (room) temperature T_a .

Let $T(t)$ be the temperature of the object at time t and let its initial temperature at time $t = 0$ be $T_0 < T_a$ $[T_0 > T_a]$.

Newton's law of cooling tells us that

$$T(t) = T_a - (T_a - T_0)2^{-\lambda t},$$

Where λ is a positive constant related to the thermal conductivity of the surrounding media (air).

(c) Why your party balloon deflates, or Diffusion across a semi-permeable membrane. Diffusion is the mechanism whereby a substance (typically a gas or solute) disperses from regions of high concentration to those of lower concentration. Often the regions of high and low concentration are separated by a semi-permeable barrier/membrane (in which case the process is sometimes referred to as osmosis). The rate at which diffusion occurs is proportional to the difference in concentrations and the area of the surface through which the substance is permeating. Consequently, for a substance (helium say) enclosed by a rigid semi-permeable vessel (for example, a plastic bottle) from which it diffuses into a large exterior reservoir in which the concentration c_a remains effectively constant (for example, the atmosphere), the concentration inside the vessel varies with the time according to

$$c(t) = c_a + (c_0 - c_a)2^{-\lambda t} .$$

Modelling how a party balloon slowly deflates is more complex. As helium escapes, the helium concentration and the volume of the balloon, and hence the surface area through which the helium is diffusing, all decrease in a complicated way linked to the amount of helium remaining in the balloon and the elasticity of the rubber.

Diffusion is one of the most ubiquitous processes in nature; regulating everything from respiration, the uptake or excretion of a drug, the intake of water by a plant, to the leaking of energy from the core of a star to its photosphere and hence into space.

3 Fitting Models to Data

Sophisticated algorithms, akin to a mixture of Newton's method and linear regression (least square best fit), have been developed to determine values for the parameters in our models so that they best describe known data (a process known as *parameter identification*). However, we will assume that we have enough data points to let us sketch a curve that reasonably fits the data. We then seek parameter values for which our model will mimic the curve.

3.1 Estimating y_a from asymptotic behaviour

This is often the hardest parameter in (2) to estimate, requiring data that captures long term behaviour. We need out graph to extend far enough that we can read off the value to which it asymptotes, either as $x \to \infty \ (a < 0)$, or less commonly as $x \to -\infty \ (a > 0)$. Fortunately, however, in many situations the value of y_a is known a priori. For exponential models it is zero; for our mould growth it is the area of the top surface of the bread slice; while for our warming/cooling drink it is the ambient temperature of the room, which we can measure with a thermometer.

3.2 Estimating *k* from an initial condition

Once y_a is known (see section 3.1), setting $x = 0$ in (2) leads to $k = y(0) - y_a = y_0 - y_a$, allowing *k* to be determined from the initial condition. Since *x* is often time and what we take as the origin for time is somewhat arbitrary, we are at liberty to take it as the moment one of our observations was made. Usually, but not necessarily, when the first (earliest) observation was made.

3.3 Estimating λ using half-life

This is where working with powers of 2 really comes to the fore, allowing us to avoid the need for logarithms. We work with *half-lives*.

From (2), we have $y - y_a = (y_0 - y_a)2^{\lambda x}$, where y_a is known via section 3.1. We ask for what value, $x_{1/2}$, of *x*, is $y - y_a$ reduced to half its initial value of $y_0 - y_a$; that is when $y = (y_0 + y_a)/2$, a value that can be read directly from the graph of our data, see Figure 2. This occurs when

$$(y_0 - y_a)/2 = (y_0 - y_a)2^{\lambda x_{1/2}} ;$$

that is, when $1 = 2^{\lambda x_{1/2} + 1}$, whence $\lambda x_{1/2} + 1 = 0$, or

$$\lambda = -1/x_{1/2}.$$

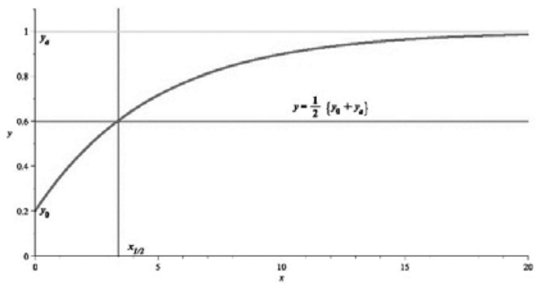

Figure 2: Modified exponential model with $\lambda = -0.3, y_a = 1$ and $k = -1$

When the situation being modelled required that α be positive, $x_{1/2}$ will lie to the left of the origin where there may be few or no data points. In this less common case it is better to work with the *double-life*, x_2, where $y - y_a = 2(y_0 - y_a)$, in which case $\lambda = 1/x_2$.

Using different choices for where the origin is located on the *x*-axis (see the remark in section 3.2), leads to different values for α. These may be averaged to obtain a more robust estimate. However, that all these values lie relatively near to one another is a good test that an exponential model is appropriate.

4 Underlying Differential Equations (Extension 1 Course)

4.1 Exponential model

The assumption that at any instant the rate at which a quantity $y = y(t)$ varies is proportional to the amount present at that instant is captured by the differential equation

$$\frac{dy}{dt} = \lambda y \text{, with initial condition } y(0) = y_0,$$

where λ is positive for exponential growth and negative for exponential decay. Verify by substitution, or otherwise, that

$$y(t) = y_0 e^{\lambda t}.$$

is a solution, and show that it is the only solution [Hint: suppose $x(t)$ is another solution and consider $\frac{d}{dt}\left(\frac{x}{y}\right)$].

4.2 Modified exponential model

$$dy/dt = \lambda(y_a - y),\ y(0) = y_0\ (\lambda > 0) \quad (3)$$

This may be solved either:

(i) using the change of variable $x = y - y_a$, so $dx/dt = -\lambda x$ and hence $x = Ce^{-\lambda x}$, or

(ii) using separation of variables (Extension 1 – C3.2) to obtain

$$\int \frac{dy}{(y_a - y)} = \int \lambda dt$$

Both ways lead to the solution,

$$y = y_a - (y_a - y_0)e^{-\lambda t}$$

In the following subsections we justify why some of the models introduced in section 2 are appropriate.

(a) Spread of mould on a slice of bread. Consider mould growing on a slice of bread under the circumstances described in section 2.2(a). A spoor that happens to land on a part of the bread that is already covered by mould does not lead to an increase in the area infested. On the other hand, one that lands where no mould is currently present becomes a nucleus for new mould growth. Thus, *at any moment, the rate at which mould develops, dm/dt, is proportional to the area of bread free of mould, A-m; that is,*

$$\frac{dm}{dt} = \lambda(A - m),\ \text{with } m(0) = 0 \quad (4)$$

and so, using the techniques of solution described in 4.2

$$m = A(1 - e^{-\lambda t}).$$

(b) Newton's law of cooling. Another instance of (3) in action is provided by Newton's law of cooling, which states that at any moment the rate of change of an object's temperature, dT/dt, is proportional to the difference between the ambient temperature, T_a, and that of the object, $T = T(t)$, at that moment, leading to the differential equation

$$dT/dt = \lambda(T_a - T),\ T(0) = T_0\ (\text{with } \lambda > 0).$$

4.3 Relation to models involving powers of 2

To convert from a model involving powers of e to one involving powers of 2 we need to introduce a factor of $\log_2(e) = 1/\log_e(2) \approx 1.443$ into λ. Thus, the model $y = 7e^{-0.3x}$ becomes

$$y = 7 \times 2^{-0.3\log_2(e)x} \approx 7 \times 2^{-0.43x}.$$

5 The Logistic Model (Extension 1 – C3.2)

Exponential growth or decay often provides a good description for small values of time, while the modified exponential model frequently works well for larger values of times when growth limiting factors have a greater effect. A superior description, resting upon the underlying assumptions of both, is offered by the logistic equation:

$$dy/dt = \lambda y(y_a - y),\ y(0) = y_0\ \ (\lambda > 0).$$

Separation of variables leads to,

$$\int \frac{dy}{(y_a - y)} = \int \lambda dt.$$

Which, after observing that

$$\frac{1}{y(y_a - y)} = \frac{1}{y_a}\left(\frac{1}{y} + \frac{1}{y_a - y}\right),$$

integrating and solving for y yields the logistic model:

$$y = \frac{y_a}{1 + Ce^{-\lambda y_a x}},\ \text{where } C = (y_a - y_0)/y_0. \quad (5)$$

The sigmoid curve of a logistic model provides an even better description for the growth of mould on our slice of bread. Rather than there being a constant level of airborne spores, spores are more likely to be released into the air from the mould already present on the bread. Thus, the density of airborne spores will be proportional to m, and so the constant factor λ in equation (4) is replaced by a factor of the form λm, leading to a logistic equation for m, $dm/dt = \lambda m(A-m)$.

While the logistic model is usually a better model, the drawback, besides having a more complicated solution, is in estimating a value for λ that will fit the data. (Can you see how this might be done? A hint lies in one step in the derivation of (5).)