PROOF by INDUCTION

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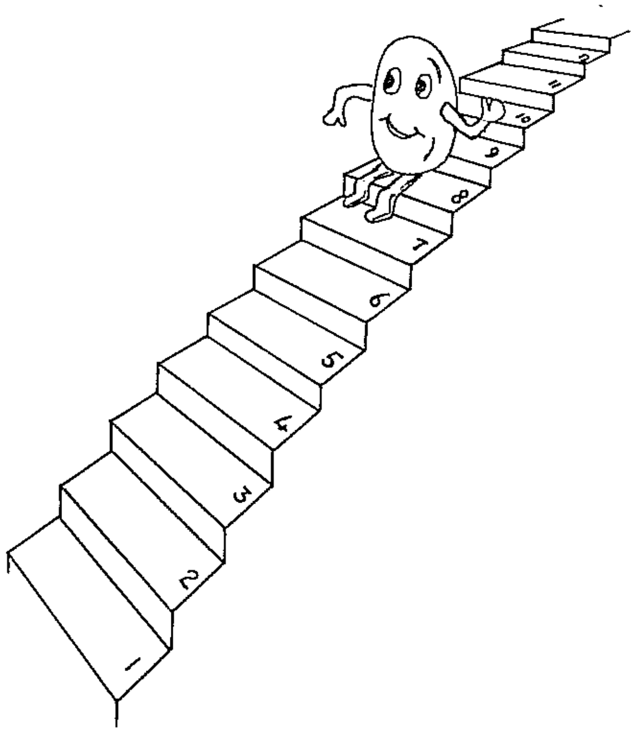

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1. Introduction to the problem

A *proposition* is a declarative sentence which is either **true** or **False** (but not both).

Examples of propositions are:

(1) 'tweety is a bird',

(2) '$2^{12345} - 1$ is a prime number',

(3) 'if sylvester is a cat then sylvester will try to eat tweety',

(4) 'the sum of the interior angles of any triangle is π', and

(5) 'if n is a natural number then $1 + 2 + 3 + \cdots + n = n(n+1)/2$'.

Here and elsewhere, by a *natural number* we mean a counting number, or strictly positive integer; that is, one of $1, 2, 3, 4, 5, \cdots, n, n+1, \cdots$.

Note: For a statement to be a proposition it is not necessary that we know whether it is true or false, it is enough that we know it must be one or the other. For example,

(6) 'some carnivorous dinosaurs were warm blooded', or

(7) 'the 100 billionth digit in the decimal expansion of π is 7'

are both propositions, but at the moment no one knows whether they are true statements or false statements. Palaeontologists are still arguing about the first one, and only the first 50 billion digits in the decimal expansion of π have been calculated (the 50 billionth digit is 2 and the digit before it is 4 — something for Douglas Adams fans to contemplate!)

By way of contrast, examples of statements which are not propositions include,

(8) '2+3'

(9) 'the high rate of inflation', and

(10) 'this statement is false'.

In algebra we often use a symbol to represent a more complicated expression, for example

$$s = \sum_{k=1}^{n} \frac{1}{k(k+1)},$$

so too, we will use capital letters to denote propositions. For example P might stand for the proposition: 'humans will land on mars by 2005'.

Much of mathematics is concerned with establishing the truth of certain propositions, often of the form 'if P then Q', where P and Q are themselves propositions, which we refer to as theorems.

By a *proof* of a proposition we understand a rigorous logical argument that demonstrates its truth. The level of rigor should be such as to unequivocally convince ourselves and others of the truth of the proposition.

Sometimes we are given a family of propositions, one for each natural number;

$$P(1), P(2), P(3), \cdots, P(n), P(n+1), \cdots,$$

and wish to establish the truth of all of them.

Example 1: Establish the truth of $P(n)$, for all natural numbers n, where $P(n)$ is the proposition,

$$P(n): \qquad \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

To see that this is so, first note that $1/k(k+1) = 1/k \ - \ 1/(k+1)$, then

$$P(1): \quad \sum_{k=1}^{1} \frac{1}{k(k+1)} = 1 - \frac{1}{2},$$

$$P(2): \quad \sum_{k=1}^{2} \frac{1}{k(k+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3},$$

$$P(3): \quad \sum_{k=1}^{3} \frac{1}{k(k+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4},$$

$$P(4): \quad \sum_{k=1}^{4} \frac{1}{k(k+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) = 1 - \frac{1}{5},$$

etc.

Sometimes the general argument, like the 'telescoping argument' indicated above, is so clear from the first few instances that we are convinced of its validity. But, this may not always be the case, and the first few instances may be misleading.

Example 2 [Due to Leonhard Euler, 1707–1783]:

$$E(n): \qquad n^2 + n + 41 \quad \text{is a } \textit{prime} \text{ number.}$$

Here we have,

$$
\begin{aligned}
&E(1): \quad 1^2+1+41=43, \text{ a prime number, so } E(1) \text{ is true.}\\
&E(2): \quad 2^2+2+41=47, \text{ a prime number, so } E(2) \text{ is true.}\\
&E(3): \quad 3^2+3+41=53, \text{ a prime number, so } E(3) \text{ is true.}\\
&E(4): \quad 4^2+4+41=61, \text{ a prime number, so } E(4) \text{ is true.}\\
&E(5): \quad 5^2+5+41=71, \text{ a prime number, so } E(5) \text{ is true.}\\
&\dots
\end{aligned}
$$

This would suggest that $E(n)$ is indeed true for all natural numbers n and remarkably it is true for $n = 1, 2, 3, \cdots, 39$, but a moments reflection shows that $E(41)$ cannot be true, nor is $E(40)$.

$$E(40): \quad 40^2+40+41=41^2, \text{ which is not a prime number, so } E(40) \text{ is false.}$$

Example 3: This example concerns the binomial expression $x^n - 1$. It had been observed that,

$$
\begin{aligned}
x-1 &= x-1,\\
x^2-1 &= (x-1)(x+1),\\
x^3-1 &= (x-1)(x^2+x+1),\\
x^4-1 &= (x-1)(x+1)(x^2+1),\\
x^5-1 &= (x-1)(x^4+x^3+x^2+x+1),\\
x^6-1 &= (x-1)(x+1)(x^2+x+1)(x^2-x+1),\\
&\dots
\end{aligned}
$$

and it had been conjectured that the following proposition was true for every natural number n.

$P(n)$: $x^n - 1$ can be factored into a product of lower degree polynomials all of whose non-zero coefficients are either $+1$ or -1.

However, in 1941 V. Ivanov showed that one of the factors of $x^{105} - 1$ is

$$
\begin{aligned}
&x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-\mathbf{2}x^{41}-x^{40}-x^{39}+x^{36}+x^{35}+x^{34}\\
&+x^{34}+x^{33}+x^{31}-x^{28}-x^{26}-x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}\\
&+x^{14}+x^{13}+x^{12}-x^{9}-x^{8}-\mathbf{2}x^{7}-x^{6}-x^{5}+x^{2}+x+1
\end{aligned}
$$

which cannot be further factored into polynomials with integer coefficients.

Example 4: Perhaps the most famous example is that of **Fermat's last theorem.**

It was known from antiquity that the Pythagorean identity $x^2 + y^2 = z^2$ has an infinite number of natural number solutions: $(3, 4, 5)$, $(5, 12, 13)$, $\cdots$, known as Pythagorean triads. In around 1637 Pierre de Fermat (1601–1665) made a note in the margin of his copy of Diophantus' *Arithmetica* (where the Pythagorean identity is dealt with) stating that the equation

$$x^n + y^n = z^n$$

has no natural number solutions for any natural number exponent $n \geq 3$. He further claimed to have "discovered a truly marvelous proof of this, which, however, the margin is not large enough to contain". A proof when $n = 4$ was later found among Fermat's papers, and perhaps he also knew a proof for the case $n = 3$. (Certainly this was known by the mid 1700's, proved by Euler. A proof for $n = 5$ was given by Dirichlet in 1828, and in 1840 Lamé and Lebesgue established the result for $n = 7$.) Fermat may have been convinced of the statement's truth by his proof for the case $n = 4$ (and possibly $n = 3$), but no general proof was ever found among his papers. Because Fermat was highly regarded for his ingenuity in proving numerous difficult theorems in number theory, many of the best mathematicians of the 18'th, 19'th and 20'th centuries sought to prove his 'theorem'. In the process much valuable mathematics was created. But, a proof continued to elude mathematicians until 1994 when the 42 year old Andrew Wiles, by a *tour de force* occupying the previous 7 years and using some of the most sophisticated modern mathematics, brought the quest to an end. His proof was published in 1995 and occupied the entire 130 pages of the May issue of the prestigious international journal *Annals of Mathematics.*

Although it eventually proved to be true, the history of Fermat's last theorem illustrates perfectly the pitfalls of 'jumping' to a general conclusion from the truth of a few initial cases. This is further underscored by the case of *Euler's conjecture.* In a vein similar to Fermat's last theorem, Euler claimed that the equation

$$x^4 + y^4 + z^4 = w^4$$

has no natural number solutions, but the method of proof for Fermat's theorem in the case $n = 4$ did not apply. Despite mounting evidence for its truth and numerous attempts to prove it, the problem remained undecided until 1988 when it was solved by Noam Elkies, who discovered that

$$2,682,440^4 + 15,365,639^4 + 18,796,760^4 = 20,615,673^4$$

(you might like to try and verify this). In fact Elkies proved that there are an infinite number of such solutions.

Thus, despite holding for all values of x, y, z and w up to a million, Euler's conjecture, unlike Fermat's, turned out to be false.

Perhaps the right question should be: For what values of n, if any, does $x^n + y^n + z^n = w^n$, or $x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n$, fail to have any natural number solutions? For a lively discussion of these questions see the last two pages of chapter 4 in Paul Hogffman's book *The Man Who Loved Only Numbers* [Fourth Estate, London, 1998].

2. The method of mathematical induction

The method of *mathematical induction*, with which we will be dealing, is a simple procedure for rigorously establishing the truth of all the propositions in a family:

$$P(1), P(2), P(3), \cdots, P(n), P(n+1), \cdots.$$

Ironically, it was Fermat who pioneered the use of induction.

A proof by mathematical induction requires us to establish two propositions, namely that

(1) $P(1)$ is true, and

(2) For each natural number n the truth of $P(n+1)$ follows from the truth of $P(n)$; that is, if $P(n)$ is true then so also is $P(n+1)$.

Note: Proposition (2) does not require us to establish the truth of $P(n)$, nor does it prove that $P(n+1)$ is true. We only establish that were $P(n)$ true, then so also would be $P(n+1)$.

Remark: Proposition (2) is often abbreviated to $P(n) \Longrightarrow P(n+1)$, here and elsewhere '$\Longrightarrow$' should be read as 'implies'. Also, (2) is sometimes referred to as *the inductive step* and the assumption that $P(n)$ is true, from which we deduce the truth of $P(n+1)$, is *the inductive hypothesis* but this is just useless verbiage.

Example. By way of illustration let us reprove example 1 of section 1 rigorously using the method of mathematical induction:

Here we wish to establish the truth of

$$P(n): \qquad \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1},$$

for all natural numbers n.

(1) For $P(1)$: $\displaystyle\sum_{k=1}^{1} \frac{1}{k(k+1)} = 1 - \frac{1}{2}$, we have

$$\text{LHS} = \frac{1}{1 \times 2} = \frac{1}{2} = RHS.$$

Therefore $P(1)$ is true.

(2) If $P(n)$ is true; that is, $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$,

then for

$$P(n+1): \qquad \sum_{k=1}^{n+1} \frac{1}{k(k+1)} = 1 - \frac{1}{n+2}$$

we have

$$\begin{aligned}
LHS &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \;+\; \frac{1}{(n+1)(n+2)} \\
&= 1 - \frac{1}{n+1} \;+\; \frac{1}{(n+1)(n+2)}, \quad \text{since } P(n) \text{ is assumed true,} \\
&= 1 - \left\{ \frac{(n+2)}{(n+1)(n+2)} \;-\; \frac{1}{(n+1)(n+2)} \right\} \\
&= 1 - \frac{1}{n+2} \\
&= RHS.
\end{aligned}$$

Thus, $P(n+1)$ is true if $P(n)$ is true. This, together with (1), completes the proof.

3. Why is the method of mathematical induction valid?

The validity of the method of mathematical induction relies on two simple facts about the natural numbers.

(i) Every natural number n has a unique *immediate successor* $s(n) = n + 1$, and

(ii) Every natural number is an *eventual successor* of 1. That is, given any natural number n, if we start at 1 and repeatedly apply the successor operation we will after a finite number of applications ($n - 1$ of them) arrive at n,

$$n = \underbrace{s(s(\cdots s}_{n-1}(1)\cdots)) = ((\cdots(1 + \underbrace{1) + 1 + \cdots) + 1}_{n-1}).$$

This allows us to *totally* (or *linearly*) *order* the natural numbers: we say $n < m$ (or, $m > n$) if m is an eventual successor of n. Given any two distinct natural numbers n and m, either $n < m$, or $m < n$.

It also allows us to show that relative to this total ordering the natural numbers are *well ordered*. That is, every non-empty subset A of the natural numbers has a unique smallest element. To see this consider the following 'pseudo-computer programme'.

```
  n = 1
1 If n is in A go to 2, Else
  n = n + 1
  go to 1
2 Return the answer:  'n is the smallest element of A'
  Stop
```

This programme is guaranteed to halt after a finite number of steps (less than or equal to a steps, where a is any element chosen from the non-empty set A), proving that the natural numbers are well ordered.

Note: The real numbers are totally ordered by the usual order relation '$<$', but they do not admit a successor operation, nor are they well ordered by '$<$'.

Proof of the method of mathematical induction: let $P(1), P(2), P(3), \cdots$ be a family of propositions for which we have established (1) and (2) above. We will argue by contradiction. Suppose that for some natural number n_0 the proposition $P(n_0)$ is false, and let A be the set of all natural numbers n for which $P(n)$ is false. Then A is non-empty (n_0 is in A), and so A has a smallest element, m say. Now, m cannot be 1, since by (1) we know that $P(1)$ is true, so 1 is not in A. Therefore m must be the immediate successor of a (smaller) natural number, $m - 1$. Since m is the smallest natural number for which $P(n)$ is false, it must be the case that $P(m - 1)$ is true, but then by (2) with $n = m - 1$

we have that $P(m) = P([m-1]+1)$ is true. This contradiction shows that no such n_0 could have existed, and therefore $P(n)$ is true for all natural numbers n.

Aside: The above proof in fact applies to show that the method of induction may be used to establish the truth of a family of propositions, $P(\lambda)$, whenever λ ranges over a well ordered set Λ, provided (1) is replaced by

(1′) $P(\lambda)$ is true, for all λ in Λ which are not the successor of some other element in Λ (in the natural numbers 1 is the only element with this property),

and provided (2) is replaced by,

(2′) For each λ in Λ, if $s(\lambda)$ exists, we have $P(\lambda) \Longrightarrow P(s(\lambda))$.

Here $s(\lambda)$, the successor of λ, is defined to be the unique smallest element of the set $\{\alpha : \alpha > \lambda\}$, if this set is empty λ has no successor.

Thus it is possible to do induction over any nonempty subset of the natural numbers. For example, over the set of even positive integers, or over the set of prime numbers. In more advanced work this yields a powerful method of proof known as *transfinite induction*.

Examples (2) and (3) of section 1 show that establishing proposition (2) is an essential part of a proof by mathematical induction. Proposition (1) is also essential. Failure to establish it can lead to nonsensical conclusions, here is an example.

A 'proof' that $P(n): \quad n = n+1$ is true for all natural numbers n.

If $P(n)$ is true; that is, $n = n+1$, then adding 1 to both sides of the equation we have $n+1 = (n+1)+1$, so $P(n+1)$ is also true. Thus, $P(n) \Longrightarrow P(n+1)$, and the result follows by induction.

Of course the fallacy here is that $P(1): \quad 1 = 2$ fails to be true.

4. Induction in action – Some examples

Example 1: Let s_n denote the sum of the first n odd numbers; that is,

$$s_n = 1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n}(2k-1).$$

Prove that for each natural number n we have $s_n = n^2$.

Solution:

(1) Observe that the result is true for $n = 1$. Indeed,

$$s_1 = 1 = 1^2.$$

(2) If $s_n = n^2$ we show $s_{n+1} = (n+1)^2$. Now,

$$\begin{aligned} s_{n+1} &= \sum_{k=1}^{n+1} 2k - 1 \\ &= \sum_{k=1}^{n} sk - 1 \ + \ 2(n+1) - 1 \\ &= s_n + 2(n+1) - 1 \\ &= n^2 + 2n - 1 \\ &= (n+1)^2, \end{aligned}$$

as required.

This proves the result, by mathematical induction.

Note: You should **never end a proof by induction with an attempted explanation** such as: '*since the proposition is true for 1 it is therefore true for 2 and then for 3 and then for 4 etc.*' This may give a heuristic feeling as to why the method of mathematical induction works, but it effectively reduces the argument back to the sort of non-rigorous, possibly fallacious, initial cases argument that induction is intended to replace. The whole point of induction is to replace such informal arguments by a rigorous demonstration. As we saw in the last section, the proof of the method of mathematical induction has nothing to do with climbing through cases, climbing ladders, falling down stairs, domino effects, or other such nonsense.

Exercise 1: Let x be a real number. Prove by mathematical induction that for all natural numbers n,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Example 2: Prove by the method of mathematical induction that,

$$s_n = 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Solution: Again, the first step is to prove the result for $n = 1$:

$$s_1 = 1^4 = 1 = \frac{1(1+1)(2+1)(3+3-1)}{30} = \frac{30}{30} = 1.$$

Next we show that if the assertion is true for n it will also be true for $n + 1$; that is, we have to show

$$s_{n+1} = \frac{(n+1)(n+2)(2n+3)(3n^2+9n+5)}{30},$$

assuming that

$$s_n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

To see this, we may proceed as follows.

$$\begin{aligned} LHS = s_{n+1} = s_n + (n+1)^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 \\ &= \frac{n+1}{30}\left[6n^4 + 39n^3 + 91n^2 + 89n + 30\right] \end{aligned}$$

In these kinds of questions, it is hard to factorize the term inside the brackets to obtain the RHS, so the best thing to do is to expand the RHS, indeed a little algebra shows that,

$$\begin{aligned} RHS &= \frac{(n+1)(n+2)(2n+3)(3n^2+9n+5)}{30} \\ &= \frac{n+1}{30}\left[6n^4 + 39n^3 + 91n^2 + 89n + 30\right] = LHS. \end{aligned}$$

Hence, the assertion is true.

Exercise 2: Prove by the method of mathematical induction that,

$$s_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

for each natural numbers n.

At this stage you should be treating each of the examples as a problem (with solution). Read the problem, try to do it, and only then look at the solution.

Example 3: Prove that for all natural numbers n we have,

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + (n-1)n = \frac{(n-1)n(n+1)}{3}.$$

Solution: For $n = 1$, we have $0 = 0$, so the result is true for $n = 1$.

Now, suppose it is true for n; that is, $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = (n-1)n(n+1)/3$, then

$$\begin{aligned} 1 \times 2 + 2 \times 3 + \cdots + (n-1)n + n(n+1) &= \frac{(n-1)n(n+1)}{3} + n(n+1) \\ &= \frac{n(n+1)[(n-1)+3]}{3} \\ &= \frac{n(n+1)(n+2)}{3}, \end{aligned}$$

so the result will also be true for $n+1$.

Thus, by mathematical induction, the result is true for all natural numbers n.

Exercise 3: Prove that $\sum_{k=1}^{n} n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$, for all natural numbers n.

Example 4: Prove that the sum of the cubes of three consecutive natural numbers is divisible by 9. That is, $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9, for all natural numbers n.

Solution: The sum of the cubes of the first three natural numbers is

$$1^3 + 2^3 = 3^3 = 1 + 8 + 27 = 36$$

which is divisible by 9. Hence, the assertion is valid for $n = 1$.

Suppose the sum $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. We have to prove that under this supposition $(n+1)^3 + (n+2)^3 + (n+3)^3$ will also be divisible by 9. Here is the proof.

$$\begin{aligned}(n+1)^3 + (n+2)^3 + (n+3)^3 &= (n+1)^3 + (n+2)^3 + n^3 + 9n^2 + 27n + 27 \\ &= \left[n^3 + (n+1)^3 + (n+2)^3\right] + 9(n^2 + 3n + 3).\end{aligned}$$

Now, since $[n^3 + (n+1)^3 + (n+2)^3]$ is assumed divisible by 9 and $9(n^2+3n+3)$ is certainly divisible by 9, it follows that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is also divisible by 9.

Hence, by the method of mathematical induction, the result is true.

Example 5: Prove that for any natural number $n > 1$,

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}.$$

Solution: Here we are asked to establish the result for the subset $2, 3, 4, \cdots, n, \cdots$ of the natural numbers. Since 2 is the only element of this set which is not the successor of some other element, we must first establish,

(1′) $s_2 > 13/24$.

But, $s_2 = 1/3 + 1/4 = 7/12 = 14/24 > 13/24$, as required.

Next, suppose $s_n > 13/24$. We wish to prove that this implies $s_{n+1} > 13/24$. Now

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{n+2} + \frac{1}{n+3} \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \\ &- \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right] \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2(n+1)(2n+1)} \\ &> 0 \end{aligned}$$

Therefore, $s_{n+1} > s_n$, so if $s_n > 13/24$ then $s_{n+1} > s_n > 13/24$.

Hence, it follows from the principle of mathematical induction that $s_n > 13/24$.

5. Variants of the method of mathematical induction, and further examples

Sometimes it is not possible to prove that if $P(n)$ is true then so too is $P(n+1)$, but it may be possible to prove,

($2''$) If the k propositions $P(n-[k-1])$ and $\cdots$ and $P(n-1)$ and $P(n)$ are all true then so too is $P(n+1)$. Here k is some given natural number.

This, in conjunction with,

($1''$) $P(1)$ and $P(2)$ and $\cdots$ and $P(k)$ are all true,

serves to prove that $P(n)$ is true for all natural numbers n. The proof of this is similar to that given in section 3 (for the case $k = 1$), and is left for the reader to ponder. The next two examples illustrate this type of inductive proof (both for the case when $k = 2$).

Example 1: Prove that if $v_1 = 3$ and $v_2 = 5$, and for every natural number $n \geq 2$ the relation

$$v_{n+1} = 3v_n - 2v_{n-1}$$

holds, then

$$v_n = 2^n + 1,$$

for all natural numbers n.

Solution: $2^1 + 1 = 3 = v_1$ and $2^2 + 1 = 5 = v_2$, so the assertion is valid for $n = 1$ and $n = 2$.

Assume that

$$v_{n-1} = 2^{n-1} + 1 \quad \text{and} \quad v_n = 2^n + 1.$$

We have to prove that

$$v_{n+1} = 2^{n+1} + 1.$$

Now,

$$v_{n+1} = 3v_n - 2v_{n-1} = 3(2^n + 1) - 2(2^{n-1} + 1) = 3 \cdot 2^n + 3 - 2^n - 2 = 2^{n+1} + 1.$$

Hence, by the method of mathematical induction the assertion is true.

Example 2: Prove that $A_n = \cos n\theta$ given that $A_1 = \cos\theta$, $A_2 = \cos 2\theta$ and for every natural number $n > 2$ the relation:

$$A_n = 2(\cos\theta)A_{n-1} - A_{n-2}$$

holds.

Solution: We readily observe that the assertion is valid for $n = 1$ and $n = 2$.

If

$$A_{n-2} = \cos(n-2)\theta \quad \text{and} \quad A_{n-1} = \cos(n-1)\theta,$$

then we want to prove that

$$A_n = \cos n\theta.$$

Now we have

$$\begin{aligned} A_n &= 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta \\ &= 2\cos\theta(\cos n\theta\cos\theta + \sin n\theta\sin\theta) - (\cos n\theta\cos 2\theta + \sin n\theta\sin 2\theta) \\ &= 2\cos n\theta\cos^2\theta + \sin n\theta\sin\theta\cos\theta - \cos n\theta\cos 2\theta - \sin n\theta\sin 2\theta \\ &= \cos n\theta(2\cos^2\theta - \cos 2\theta) = \cos n\theta \end{aligned}$$

Hence, the assertion is true.

Given a family of propositions: $P(1), P(2), P(3), \cdots, P(n), P(n+1), \cdots$, it may only be possible to prove,

(2‴) If $P(n)$ and all of its predecessors are true, then $P(n+1)$ is also true. That is,

$$[P(1)\ \mathbf{and}\ P(2)\ \mathbf{and}\ P(3)\ \mathbf{and}\ \cdots\ \mathbf{and}\ P(n)] \Longrightarrow P(n+1).$$

This together with,

(1) $P(1)$ is true,

provides an inductive proof that $P(n)$ is true for all natural numbers n. This may be proved by an argument very similar to that used in section 3. Alternatively, we may deduce it as follows.

For each natural number n, define a new proposition,

$$Q(n): \quad P(1)\,\mathbf{and}\,P(2)\,\mathbf{and}\,\cdots\,\mathbf{and}\,P(n).$$

Now, for any two propositions R, S their conjunction $R\,\mathbf{and}\,S$ is true when and only when R is true and S is true. It follows that the $P(n)$ are true for all natural numbers n if (and only if) $Q(n)$ is true for all natural numbers n. And, a proof by mathematical induction that $Q(n)$ is true for all natural numbers n translates into establishing (1) and ($2'''$).

Example 3: Prove by mathematical induction that the sum of the interior angles of any (not necessarily convex) polygon with n sides is $(n-2)\pi$.

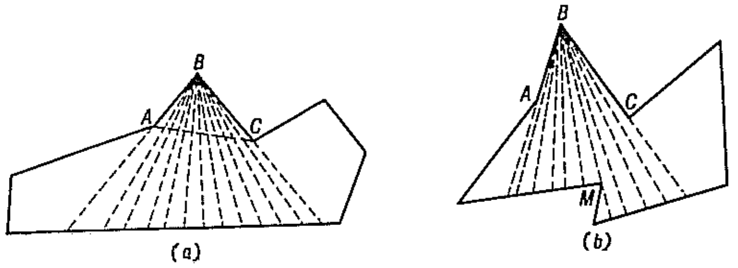
Solution: The allowable values for n are $3, 4, 5, \cdots$. When $n = 3$ the assertion becomes: the sum of the interior angles of a triangle is $(3-2)\pi = \pi$, the truth of which is well known.

Now, suppose that for $k = 3, 4, \cdots, n$ the sum of the interior angles of any polygon with k sides is $(k-2)\pi$. Under this supposition we want to show that the sum of the interior angles of any polygon $\mathcal{P}$ with $n+1$ sides is $([n+1]-2)\pi = (n-1)\pi$.

By a *diagonal* we mean a line joining any two non-adjacent vertices of the $\mathcal{P}$. Note that since $\mathcal{P}$ is not necessarily convex a diagonal can intersect some of its sides, or lie entirely outside it, none-the-less we can find a diagonal that cuts $\mathcal{P}$ into two polygons each having a smaller number of sides than $\mathcal{P}$. If $\mathcal{P}$ were convex it's clear that any diagonal would do. To show that it is still possible when $\mathcal{P}$ is non-convex let A, B and C be any three consecutive vertices. From the middle vertex B draw all possible interior rays. Two cases are possible:

(a) All the rays intersect the same side of the polygon. In this case the diagonal AC cuts $\mathcal{P}$ into an $(n-1)$-gon and the triangle ABC.

(b) The rays don't all intersect the same side of $\mathcal{P}$. In this case at least one of the rays must contain a vertex M of $\mathcal{P}$, and the diagonal BM will cut $\mathcal{P}$ into two polygons, each with fewer sides than $\mathcal{P}$.



In either case, we have been able to break $\mathcal{P}$ along a diagonal into two polygons $\mathcal{P}_1$ and $\mathcal{P}_2$. Let $\mathcal{P}_1$ have k_1 sides, then $3 \leq k_1 \leq n$ and $\mathcal{P}_2$ has $k_2 = n+3-k_1$ sides, which also satisfies $3 \leq k_2 \leq n$. By our supposition the sum of the interior angles of $\mathcal{P}_1$ and $\mathcal{P}_1$ are $(k_1-2)\pi$ and $(k_2-2)\pi$ respectively. Further, by our construction, the sum of the interior angles of $\mathcal{P}$ equals the sum of the interior angles of $\mathcal{P}_1$ plus the sum of the interior angles of $\mathcal{P}_2$. That is,

$$\begin{aligned}\text{sum of the interior angles of } \mathcal{P} &= (k_1 - 2)\pi \ + \ (k_2 - 2)\pi \\ &= ((k_1 - 2) + ([n + 3 - k_1] - 2))\,\pi \\ &= (n - 1)\pi\end{aligned}$$

as required.

This proves the validity of our assertion, by mathematical induction.

6. Additional Exercises

[I am indebted to James Taylor for the opportunity to steal from his collection of exercises.]

(1) *Sums of series*

Prove the following are true for all natural numbers n.

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k(k+2) = \frac{1}{6}\, n(n+1)(2n+7)$

(d) $\displaystyle\sum_{k=1}^{n} k^3 + 3k^5 = \frac{1}{2}\, n^3(n+1)^3$

(e) $\displaystyle\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$

(f) $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$

(h) $\sum_{k=1}^{n}(k^2+1)k! = n(n+1)!$

(j) $\sum_{k=1}^{n} k^2.2^k = 2^{n+1}(n^2-2n+3)-6$

(2) *Divisibility*

For all natural numbers n prove that

(a) $3^n - 1$ is divisible by 2

$4^n - 1$ is divisible by 3

$5^n - 1$ is divisible by 4, and in general: $p^n - 1$ is divisible by $p-1$

(b) $3^{2n} + 7$ is divisible by 8

(c) $2^{3n} - 3^n$ is divisible by 5

(d) $7^{2n} + 7^n + 4$ is divisible by 6

(e) $5^{2n} + 5^n + 2$ is divisible by 4

(f) $5^{3n} + 5^{2n} + 5^n + 1$ is divisible by 4

(g) $7^{2n} - 5^n$ is divisible by 4

(h) $4^n + 14$ is divisible by 6

(i) $9^{n+2} - 4^n$ is divisible by 5

(j) $3^n + 7^n$ is divisible by 10 if n is odd.

(3) *Inequalities*

Prove that for all natural numbers n,

(a) $5^n \geq 1 + 4n$

(b) $3^n \geq 1 + 2n$ and in general $(1+p)^n \geq 1 + np$.

(4) *Geometrical Applications*

(a) Prove by mathematical induction that the number of regions into which n straight lines divides a region never exceeds 2^n.

(b) Prove by mathematical induction that the exterior angles of a polygon always add to 2π.

(5) *Some Harder Problems*

(a) *The Postmaster's Problem*
A postmaster has an unlimited supply of 3 cent and 5 cent stamps. Prove by mathematical induction that he can make up the value of any postage greater than or equal to 8 cents.

(b) Let there be n points in space distributed in such a way that no more than three points lie on any single plane. Then the number of distinct planes each defined by three points is equal to $\dfrac{n(n-1)(n-2)}{6}$.

(c) Let there be n distinct objects. Prove by mathematical induction that the number of unordered pairs that can be selected is $\dfrac{n(n-1)}{2}$, for $n \geq 2$.

(d) Prove by mathematical induction that:

$$\tan^{-1} \frac{1}{2.1^2} + \tan^{-1} \frac{1}{2.2^2} + \ldots \tan^{-1} \frac{1}{2.n^2} = \frac{\pi}{4} - \tan^{-1} \frac{1}{2n+1}$$