# Approximating definite integrals; or how to derive Simpson's and other rules

Brailey Sims

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The definite integral

$$\int_a^b f(x)\,dx$$

corresponds to the total area of the regions between the graph of $f(x)$ on $[a, b]$ and the $x$-axis (with regions below the x-axis assigned negative areas), consequently it has a number of special properties. Two important properties are:

$$\int_a^b [f(x) + g(x)]\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx \qquad \text{– } \textit{additivity}$$

and

$$\int_a^b cf(x)\,dx \;=\; c\int_a^b f(x)\,dx \qquad \text{– } \textit{scalar homogeneity}$$

A simpler, more easily evaluated, expression that also enjoys these two properties is

$$\mathcal{A}_a^b f(x) \;:=\; Af(a) \;+Mf(m) \;+\; Bf(b),$$

where $m \;:=\; \frac{a+b}{2}$, and $A$, $M$ and $B$ are constants.

Additivity follows since,

$$\begin{aligned}\mathcal{A}_a^b[f(x)+g(x)] \;&=\; A[f(a)+g(a)] \;+\; M[f(m)+g(m)] \;+\; B[f(b)+g(b)]\\ &=\; Af(a) \;+Mf(m) \;+\; Bf(b) \;+\; Ag(a) \;+Mg(m) \;+\; Bg(b)\\ &=\; \mathcal{A}_a^b f(x) \;+\mathcal{A}_a^b g(x).\end{aligned}$$

Scalar homogeneity is verified by a similar, but even simpler, calculation.

Our strategy is to choose values of $A$, $M$ and $B$ so that $\mathcal{A}_a^b f(x)$ approximates $\int_a^b f(x)\,dx$.

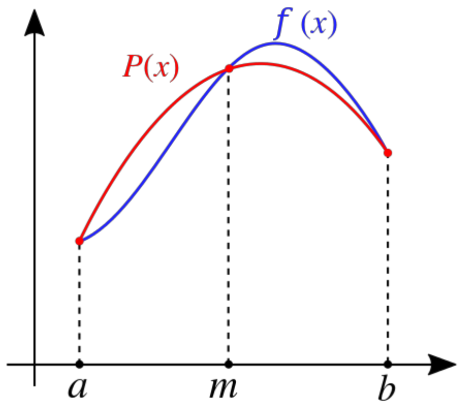


Figure 1: Simpson's Rule

Remark (1): Technically, we are regarding the definite integral as a linear functional from the space of (continuous) functions $f$ on $[a, b]$ onto the real numbers, and are seeking to approximate it by a simpler linear functional comprised of a linear combination of point evaluations; that is functionals of the form $f \mapsto f(c)$ with $c \in [a, b]$.

To obtain **Simpson's rule**, we do this by making the 'approximation' exact when $f(x)$ equals 1, $x$, and $x^2$. That is, we require,

$$A \;+\; M \;+\; B \;=\; \int_a^b 1\,dx \;=\; b - a \tag{1}$$

$$Aa \;+\; M(a+b)/2 \;+\; Bb \;=\; \int_a^b x\,dx \;=\; (b^2 - a^2)/2 \tag{2}$$

$$Aa^2 \;+\; M(a+b)^2/4 \;+\; Bb^2 \;=\; \int_a^b x^2\,dx \;=\; (b^3 - a^3)/3, \tag{3}$$

three simultaneous equations in the three unknowns $A$, $M$ and $B$.

Remark (2): Appealing to additivity and scalar homogeneity we see that these requirements ensure the approximation is exact for all quadratic polynomials (linear combinations of 1, $x$ and $x^2$).

Rearranging (1) as $M = b - a \;-\; (A + B)$, using this to eliminate $M$ from (2), followed by some simple algebra leads to $B = A$. Substituting into (3) using both of these yields $(b-a)^2 A/2 \;=\; (b^3 - a^3)/3 \;-\; (b-a)(a+b)^2/4$, which after a little interesting algebra gives $A = B = (b-a)/6$ and $M = 4(b-a)/6$. Introducing the *step size* $h = (b-a)/2$ we have the approximation,

$$\int_a^b f(x)\,dx \;\approx \frac{h}{3}[f(a) \;+\; 4f(m) \;+\; f(b)].$$

Remark (3): From Remark (2) we also see that the approximation equals $\int_a^b P(x)\,dx$, where $P(x)$ is the quadratic taking the same values as $f(x)$ at $a$, $m$ and $b$.

Remark (4): If the above algebraic machinations prove too tedious, one could always solve for $A$, $M$ and $B$ in the special case when $a = 0$ and $b = 1$ to obtain an approximation to $\int_0^1 g(t)\,dt$, and apply it when $g(t) := (b-a)f(a + (b-a)t)$, which using the substitution $t = \frac{x-a}{b-a}$ equals $\int_a^b f(x)\,dx$.

To derive the simpler **trapezoidal rule**,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a) + f(b)],$$

we disregard the $f(m)$ term (set $M = 0$) and solve for $A$ and $B$ by requiring the approximation be exact for $f(x)$ equal to 1 and $x$, thereby rendering it exact for all linear functions. This leads to the pair of simultaneous equations,

$$\begin{aligned} A + B &= (b-a) \\ aA + bB &= (b^2 - a^2)/2. \end{aligned}$$

Remark (5): It should be clear how these derivations could be extended to obtain approximations to the definite integral that are exact for polynomials of a higher degree (or indeed for any linear combination of a finite number of specified functions; for instance, linear combinations of 1, $\sin(x)$ and $\cos(x)$ on $[0, 2\pi]$), and how to derive modified approximation rules that involve values of the function at points other than the limits of the integration or their midpoint.

Remark (6): We might ask: what characteristic feature of the definite integral distinguishes it from the simpler approximating linear functionals we have constructed? The answer resides in the property,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

School of Mathematical and Physical Sciences
The University of Newcastle, 2308, Australia
brailey.sims@newcastle.edu.au