NEW ENGLAND
MATHEMATICAL
ASSOCIATION

A SHORT ESSAY ON

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Brailey Sims
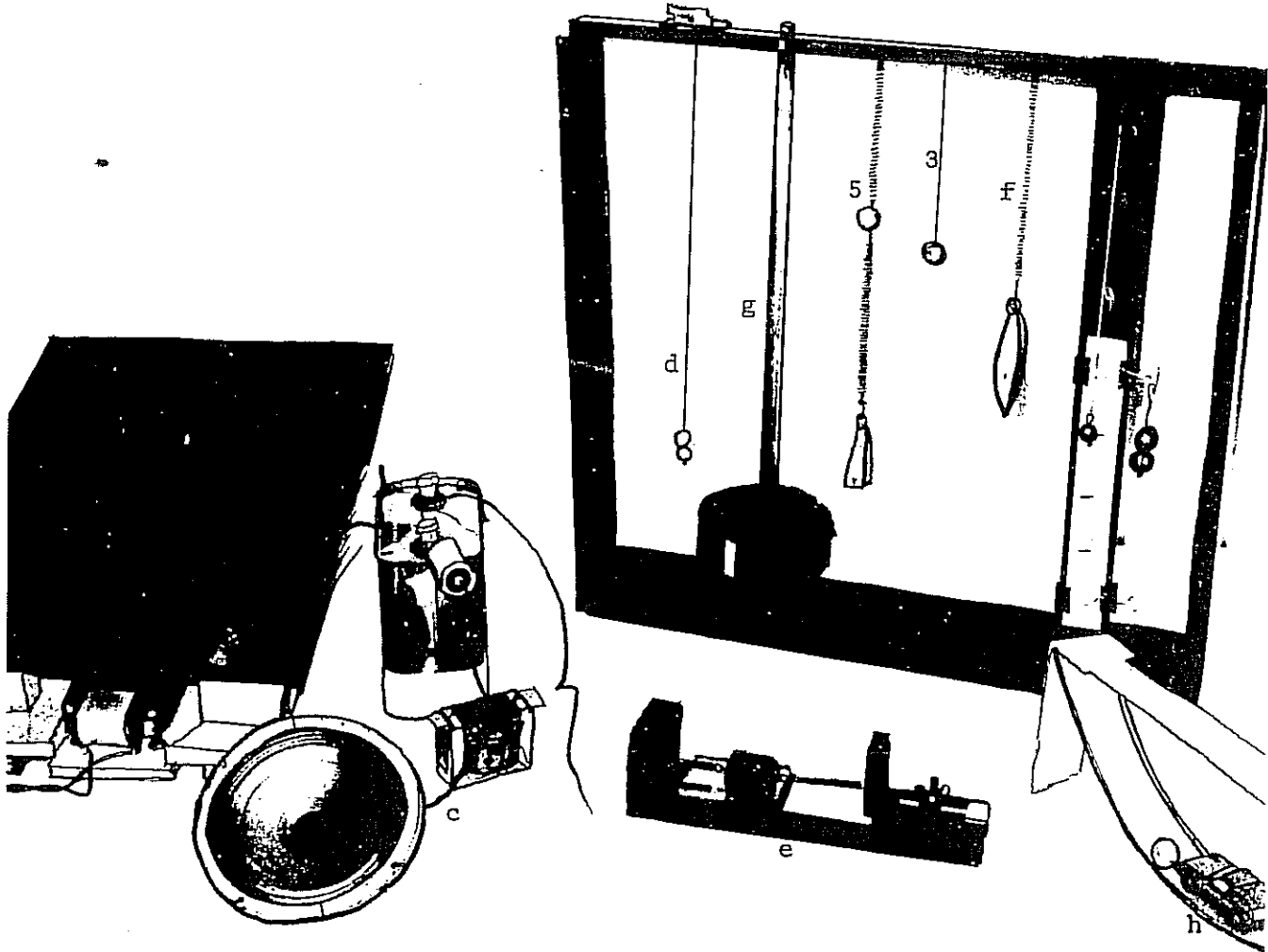
This essay is the first
in a series of articles
to be published by the
New England Mathematical
Association on specific
topics which it is hoped
will be of use to
teachers and of interest
to students.

The mathematics and discussion in this essay should be well within the ambit of a good senior school mathematics student undertaking a higher level course. The more difficult mathematical arguments are indicated by heavy rulings down the adjacent margin.

While many of the topics are not directly on the syllabus some have been chosen in the hope that they will provide useful and interesting extension material.   Others combine several different syllabus areas and are designed to illustrate the intimate connections and fruitful interplay existing between seemingly unrelated topics, which serves to give mathematics its coherence and power.

Considerable space has been devoted to physical examples of simple harmonic motion. It seems a sterile and useless exercise to undertake the study of a topic in applied mathematics without first acquiring a "physical feel" for the situation being modelled and some appreciation of its importance. For this reason I would urge that students and teachers encounter in the 'flesh' as many manifestations of the situation that they can.

Open questions have been left throughout the essay and I hope these will be pursued vigorously as will any others that may present themselves. Mathematics is as much the creation of questions as it is the answering of them.

February, 1975   B. S.

The cover depicts   simply constructed apparatus suitable for illustrating some of the examples discussed in the essay.

A Short Essay on Simple Harmonic Motion

B. SIMS

(Embodying a talk given to sixth form students at the Northern
Rivers Mathematical Association's Winter School Mathematics on
the 3rd of August 1974)

1. FORMAL DEFINITION AND THE MATHEMATICAL PROBLEM

Any quantity x, *depending on* t, *whose variation with* t *is* retarded
*by a second rate of change directly proportional to the quantity
itself is said to vary* <u>*Simple Harmonically*</u>.

Expressed symbolically x varies simple harmonically if it satisfies
the *Differential Equation*

$$\ddot{x} = -kx \quad \text{(where k > 0 is a constant of proportionality)}$$

$$\dots\dots 1)$$

Here, and throughout, the dot denotes differentiation with respect
to t, so $\ddot{x}$ is the second rate of change $\left(\dfrac{d^2x}{dt^2}\right)$ while the minus sign
represents retardation.

[1) is known as a 'differential equation' as it expresses a
relationship between x and some of its derivatives, in analogy with
a 'quadratic' equation wich relates x and its quadrature (square).
Other differential equations would be $\dot{x} = ax$, $\ddot{x} = g$]

Usually we require x, $\dot{x}$ to satisfy an initial condition. Mathematically
then, the problem is to determine as much as we can about those
functions x ≡ x(t) which satisfy

$$\ddot{x} = -kx \quad (k > 0)$$

and $\qquad x(0) = a$

$\qquad\qquad \dot{x}(0) = b$ (where a, b are specified constants) ...2)

In more general problems of this type it often proves impossible to
determine explicitly what the functions x are (as we shall see this
is not the case here), however even without knowing the solution we

can often deduce many properties which such functions must
have. Our first results typify this procedure. (That such
arguments are expected of students is exemplified by Question
8 of the 1974 paper B.)

Lemma: *If x varies simple harmonically then* $kx^2 + v^2$ *is a constant, where* $v = \dot{x}$.

Proof: Let $f(t) = kx^2 + v^2$ then $\dot{f} = 2kx\,\dot{x} + 2v\,\dot{v}$ but $v = \dot{x}$
so $\dot{v} = \ddot{x} = -kx$ and $\dot{f} = 2kxv - 2kxv = 0$. Hence, as a consequence
of the Mean Value theorem, f is a constant as required. ■

Note: This result is often 'deduced' from the erroreous formula

$\ddot{x} = \frac{1}{2}\frac{d}{dx}(\dot{x}^2)$, which to be valid requires $\dot{x}$ to be a differentiable
function of x — an unwanted (and unwarranted) assumption.

Corollary: *a function x varying simple harmonically is bounded,*
i.e. *there exists a constant* A $\geq$ 0 *such that* $-A \leq x(t) \leq A$ *for all*
t. (The smallest such A is termed the *amplitude* of the Motion)

Proof. From above $kx^2 + v^2 = c$, a constant, so $v^2 = c - kx^2$.
Now the L.H.S., v^2, is positive whence $c - kx^2 \geq 0$ or $x^2 \leq \frac{c}{k}$

and so $-\sqrt{\frac{c}{k}} \leq x \leq \sqrt{\frac{c}{k}}$ and taking $A = \sqrt{\frac{c}{k}}$ proves the result. ■

Corollary: *For a non-trivial function x varying simple harmonically*
.(i.e. $x \not\equiv 0$) x *and v are never simultaneously zero.*

Proof. This follows immediately from the above corollary, for
if x and v are simultaneously zero then $c = kx^2 + v^2 = 0$ and
so A = 0, whence $x \equiv 0$. ■

Even if we find a function x satisfying 2),we would still be in
doubt as to whether there wasn't yet another possible x lurking
undiscovered, hence the following theorem is of practical
importance.

Theorem: *There is at most one function x satisfying* 2) *ie a function satisfying given initial conditions and varying simple harmonically is 'unique'.*

Proof. Assume x and y are both solutions of 2. Then $z = x - y$ is such that $\ddot{z} = \ddot{x} - \ddot{y} = -kx + ky$, as both x and y satisfy 1

$$= -k(x-y) = -kz$$

So z is also varying simple harmonically. Further

$$z(0) = x(0) - y(0) = a - a = 0 \text{ and}$$

$$\dot{z}(0) = \dot{x}(0) - \dot{y}(0) = b - b = 0 \text{ as both x and y}$$

satisfy the same initial conditions. Thus both z and $\dot{z}$ are simultaneously zero at t=0, hence by the last corollary $z = 0$, showing $x = y$ and so only one solution is possible. Though we have uncovered many properties a function varying simple harmonically must have, one vital question remains unanswered. We have yet to prove that 2) has a solution, that a function varying simple harmonically exists. For more complex situations, proving a solution exists is an important, though often extremely difficult, problem. In our case however we can demonstrate the existence of a solution simply by finding it. This can be done in a great variety of ways, each illustrating an important technique. We conclude this analysis by looking at a few of these.

Firstly, note that if x_1 and x_2 are solutions of 1) satisfying

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0;$$

$$x_2(0) = 0, \quad \dot{x}_2(0) = 1 \quad \text{then}$$

$$x = ax_1 + bx_2 \quad \text{is the required unique solution of 2)}$$

It thus suffices to determine the two functions x_1, x_2.

It should be a familiar fact that many functions can be represented as an infinite power series, for example

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots + \frac{z^n}{n!} + \ldots \text{ for all real } z.$$

one might therefore ask is x_1 of the form

$$x_1(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n + \ldots \text{ for an appropriate}$$

choice of the coefficients a_0, a_1, a_2, ... and similarly for x_2.

Clearly $x_1(0) = 1$ requires that $a_0 = 1$, while if we assume the derivative of x may be obtained from term by term differentiation of the series (strictly, this requires proof and so at this stage must be treated as a purely formal procedure) $\dot{x}_1(0) = 0$ requires that $a_1 = 0$.

We are thus led to seek values of a_2, a_3, ... such that

$$x_1(t) = 1 + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + \ldots \text{ satisfies}$$

$\ddot{x} = -kx$, or formally

$$(2a_2 + 3.2a_3 t + 4.3a_4 t^2 + 5.4a_5 t^3 + \ldots + n(n-1)a_n t^{n-2} + \ldots)$$

$$= (-ka_2 t^2 - ka_3 t^3 - \ldots - ka_{n-2} t^{n-2} - \ldots)$$

Equating coefficients of equal powers of t gives

$$2a_3 = -k \qquad \text{or} \qquad a_2 = -\frac{k}{2}$$

$$3.2a_3 = 0 \qquad \text{or} \qquad a_3 = 0$$

$$4.3a_4 = -ka_2 \qquad \text{or} \qquad a_4 = -\frac{ka_2}{4.3} = \frac{k^2}{4.3.2}$$

$$5.4a_5 = -ka_3 \qquad \text{or} \qquad a_5 = \frac{ka_3}{5.4} = 0$$

$$\text{etc.}$$

It is now an easy exercise in mathematical induction to show from what we have and the *recurrence relationship* $a_n = \frac{-k}{n(n-1)} a_{n-2}$, obtained from equating the coefficients of t^{n-2} , that

$$a_{2n} = (-1)^n k^n \Big/ 2n!$$

$$a_{2n+1} = 0$$

and so we have, provided the series converges, and the above formal operations are justified

$$x_1(t) = 1 - \frac{k}{2} t^2 + \frac{k^2}{4!} t^4 - \frac{k^3}{6!} t^6 + \ldots$$

$$\text{or } x_1(t) = 1 - \frac{(\sqrt{k}t)^2}{2!} + \frac{(\sqrt{k}t)^4}{4!} - \frac{(\sqrt{k}t)^6}{6!} + \ldots$$

a series which should be recognised as that of $\cos \sqrt{k}t$.
We are led to suspect that $x_1(t) = \cos \sqrt{k}t$ the truth of
which is easily verified by direct substitution.

A similar argument (which it is left to the reader to
supply) leads to

$$x_2(t) = \frac{1}{\sqrt{k}} [(\sqrt{k}t) - (\sqrt{k}t)^3 \frac{3}{3!} + (\sqrt{k}t)^5 \frac{5}{5!} - (\sqrt{k}t)^7 \frac{7}{7!} + \ldots)$$

$$= \frac{1}{\sqrt{k}} \sin \sqrt{k}t \ .$$

Since x_2 is necessarily differentiable, and so continuous,
it follows that for t <u>near 0</u> we may proceed as follows,
from the lemma

$$v = \frac{dx_2}{dt} = \pm(1 - kx_2{}^2)^{\frac{1}{2}}$$

and since $\dot{x}_2(0) = 1 > 0$, we must choose the positive sign
giving

$$(1 - kx_2{}^2)^{-\frac{1}{2}} \frac{dx_2}{dt} = 1$$

$$\text{or } \int_0^t (1 - kx_2{}^2)^{-\frac{1}{2}} \frac{dx_2}{dt} \, dt = \int_0^t dt = t$$

By the change of variable theorem the L.H.S. is $\frac{1}{\sqrt{k}} \sin^{-1}\sqrt{k}x_2$

$$\text{or } x_2 = \frac{1}{\sqrt{k}} \sin \sqrt{k}t$$

Similarly one derives $x_1(t) = \cos \sqrt{k}t$.

So far we only know that these forms of x_1 and x_2 are
valid for values of t near 0, however direct substitution
shows they are in fact the required solutions.

Alternatively, interpreting 1) as asking for a function
which when differentiated twice becomes - k times itself,
we may after a little trial and error find that $x = C \sin \sqrt{k}t$
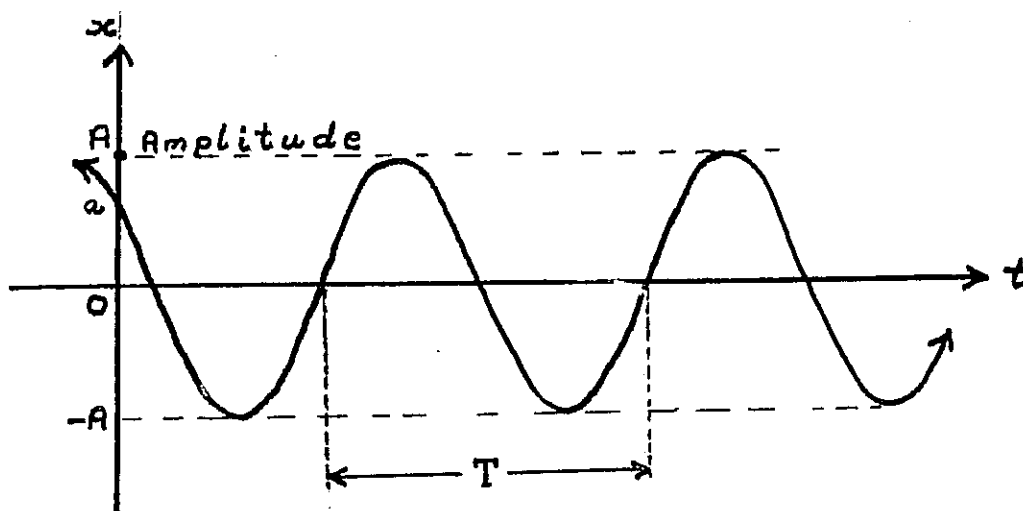or $C \cos \sqrt{k}t$ are suitable solutions and so

$$x_1(t) = \cos \sqrt{k}\,t$$

$$x_2(t) = \frac{1}{\sqrt{k}} \sin \sqrt{k}\,t \ .$$

The final conclusion from any of these approaches is that the unique solution of 2) is

$$x(t) = a \cos \sqrt{k}\,t + \frac{b}{\sqrt{k}} \sin \sqrt{k}\,t \qquad\qquad \dots\dots 3)$$

$$\text{or} \quad = \sqrt{a^2 + \frac{b^2}{k}} \sin (\sqrt{k}\,t + \xi) \qquad (\text{why?})$$

Thus a typical "space-time" curve for a quantity varying simple harmonically is

Since the curve reproduces itself after a time $T = {}^{2\pi}/\sqrt{k}$ has elapsed we speak of T as the *period* of the variation, and $\nu = \frac{1}{T}$ as the *frequency* (since $\frac{1}{T}$ complete executions of the motion are made in one unit of t).

Conversely, of course, any quantity x given by 3. is varying simple harmonically.

Geometrical Representation of Simple Harmonic Motion
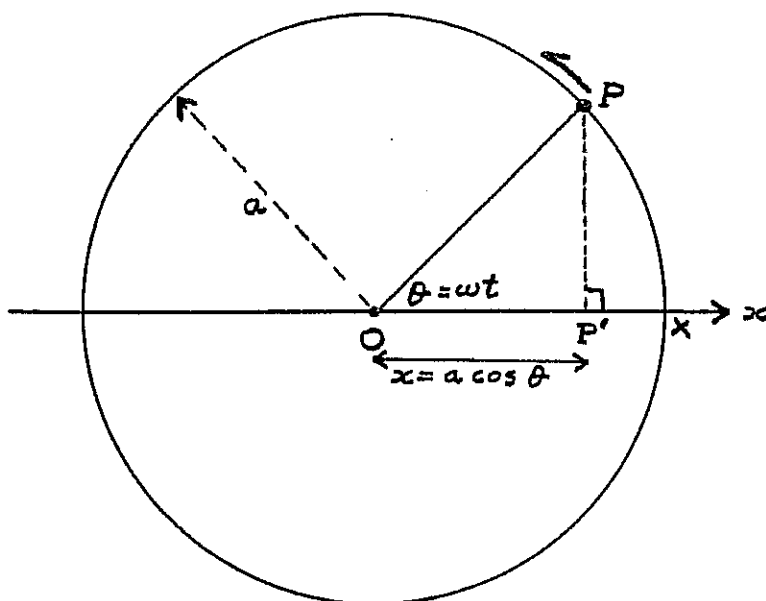
If a point P moves with uniform speed around a circle of radius a, as illustrated (i.e. OP sweeps through a constant angle of ω radians per unit of time), then after t units have elapsed

$\hat{XOP} = \theta = \omega t$ and so the shadow (projection) P' of P onto the line OX is at $x = a \cos \theta = a \cos \omega t$. We thus conclude that the position of P' varies simple harmonically along OX with period $T = \frac{2\pi}{\omega}$ (which, as expected, is the time needed for P to complete one revolution of the circle). The circle is often termed *the circle of reference*, and we have shown:

The projection of a point, moving with uniform speed around a circle onto a straight line varies simple harmonically.

When the variable t in 1) is time as in the above discussion, and as will be the case throughout the remainder of this essay, we say the quantity x executes *Simple Harmonic Motion*.

It was from the view point of a circle of reference that simple harmonic motion was first (and extensively) studied in the late 17th and throughout the 18th centuries, and is often still so examined in modern physics courses.

So far we have examined (and answered) the purely mathematical

questions posed by a quantity x varying according to the law

$$\ddot{x} = -kx \ .$$

For this to be of interest to the applied mathematician

(or theoretical physicist) we must establish the

occurrence of *natural* quantities which vary simple harmonically.

The next section presents eight diverse examples of such

instances.

2. PHYSICAL REALIZATIONS OF SIMPLE HARMONIC MOTION

(a) Our first examples of observed simple harmonic motion

are direct consequences of the previous geometric

interpretation .

During a revolution of one of Jupiter's satellites (the order of

10 days) the relative motion of the Earth and Jupiter is negligible,

further   all the solar systems motions occur, at least

approximately, in the same plane (the plane of the ecliptic),

thus a terrestrial astronomer sees the moons of Jupiter

executing simple harmonic motion across the face of their

parent planet. We find a similar situation when we observe

a binary star system whose orbital plane is edge on to

us (as is approximately the case for the α   Centauri

system – centred around the solar system's nearest stellar

neighbour, where both components move in simple harmonic motion

about their common centre of mass).
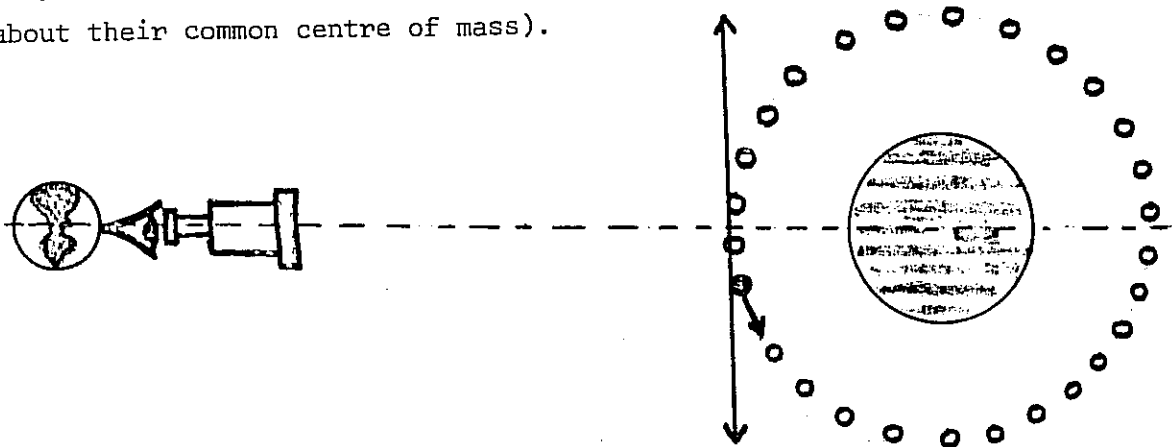
(b) The following hypothetical example formed a sub-part of the 1973 Level I mechanics question.

Consider an object dropped into an evacuated shaft "drilled" through the earth, and passing through its centre.
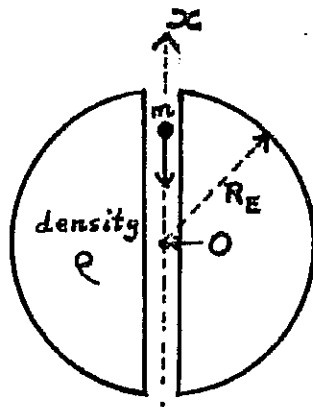
From Newton's law of universal gravitation it follows that the force on such a particle is
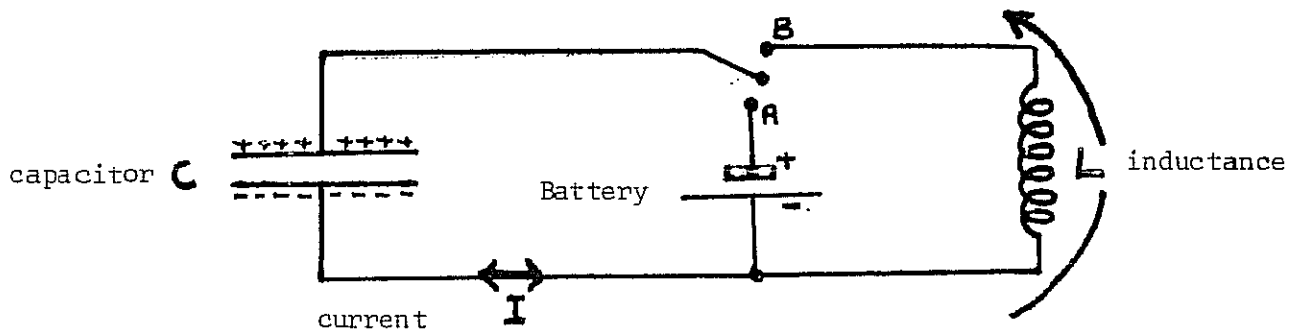
$$\frac{Gm \left(\frac{4}{3} \pi \rho x^3\right)}{x^2}$$

$$= \frac{4\pi G\rho m}{3} x$$

$$= -m\ddot{x}, \text{ by Newton's Force Law.}$$

i.e. $\ddot{x} = \frac{-4\pi G\rho}{3} x$.

So the particle will execute simple harmonic motion with $x = R_E \cos\sqrt{\frac{4\pi G\rho}{3}} t$ (since at t = 0 the object is at $x = R_E$ and $\dot{x} = 0$) and period $\sqrt{\frac{3\pi}{G\rho}}$.

(c) Resonant L-C circuits (the basis of tuned radio receivers etc.)

capacitor C   Battery   inductance L

current I

When the switch is in position A the capacitor charges, electrons
flowing from the + plate and onto the - plate. When the switch
is changed to B all the electrons on the - plate are attracted to
the + plate and flow through the inductance L (which has the
effect of slowing down the speed of their passage) to it,
ultimately creating an excess of electrons on the once + plate,
at which stage the situation is reversed and electrons flow back
in the opposite direction. The electrons continue to oscillate
back and forth in this way creating a current I in the wires which
may be shown to vary according to the differential equation

$$\ddot{I} = -\frac{C}{L} I$$

Thus I varies simple harmonically with respect to time at a frequency
of $\frac{1}{2\pi}\sqrt{\frac{C}{L}}$

(d) Oscillations of the Simple Pendulum
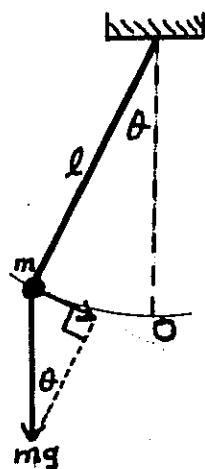
The distance of the bob from O is $\ell\theta$
(see diagram) so the acceleration of
the bob is $(\ell\theta)\ddot{} = \ell\ddot{\theta}$

$= -g \sin \theta$, by Newton's
Force Law. So if θ is sufficiently
small (less than .1 radians, ~ 5°)
that we have $\sin \theta \simeq \theta$ this becomes
$\ddot{\theta} = -\frac{g}{\ell} \theta$. The variation in θ is
approximately simple harmonic of
period $2\pi\sqrt{\frac{\ell}{g}}$.

This motion was the first studied by Galileo (1564-1642) in
the famous episode of the chandelier in Pisa Cathedral. So in a
real sense simple harmonic motion was the first type of motion
studied in the sense of "modern" science. Just as it was the
first hitherto unknown heavenly motion examined by Galileo through
the newly discovered telescope (example(a)). An observation which

contributed greatly to the decreasing prestige of aristotelian cosmology,   clearing the way for the growth of science as we know it today.

Apart from the rather hypothetical hole through the earth, all the examples considered so far have failed to yield us an example of a physical (massive) particle which is itself executing simple harmonic motion. To achieve this we would require a *force* varying proportionately with distance (since the R.H.S. of 1) must by Newton's Force Law, F=ma=m$\ddot{x}$, equal the force applied to the particle at position x.)

A large class of forces behaving in precisely this way was first discovered by Hooke (1635-1703) and announced to the world in the form of a latin cryptogram, *ut tensio sic vis.*[*]

These are the forces arising from the deformation of 'elastic' bodies such as strings, springs, wires etc. and formed the object of a more intensive study by Young (1773-1829). Hooke found experimentally that the force needed to extend an elastic body by a modest ammount x is directly proportional to x, a rule known as Hooke's Law, ie F = kx (where the proportionality constant k depends on the nature of the elastic body under consideration and is closely related to its Young's Modulus.)

A molecular justification of Hooke's Law is possible, however it is far too complicated to consider here (see Kronig *Textbook of Physics* pp 116-118 for a brief account.) We will content ourselves with the empirical result.

The existence of such forces allows us to give a number of examples of simple harmonic motion in the form of vibrating systems.

[*] extension is proportional to force

(e) A Horizontal Spring System (see diagram)
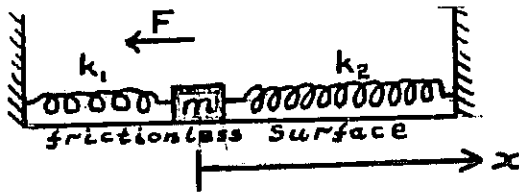
If when in equilibrium m is at 0, upon displacement from 0 by an amount x we have, for the force acting on m, $F = -k_1 x - k_2 x$ where the second term is due to the 'compression' of spring 2, and in this case the proportionality constants k_1 and k_2 are known as the *spring constants* of springs 1 and 2 respectively.

So $m\ddot{x} = -(k_1 + k_2)x$, whence $x = a \cos\sqrt{\dfrac{k_1+k_2}{m}}\, t$ (where m is initially displaced by an amount a and then released) and m executes simple harmonic motion of period $2\pi\sqrt{\dfrac{m}{k_1+k_2}}$

(For a brief discussion of the Simple Harmonic properties of the motion of such a system when the mass is vibrating on the surface along a line perpendicular to that of the springs see the author's note 'a need for tension' in the New England Mathematical Associations News Letter Volume 1, Number 1).

Note: In the above example, and in several of the subsequent ones (try to identify them yourself) it has been assumed that the masses of the springs are so slight compared to m that their contributions to the dynamics of the system can safely be neglected. The case of "heavy" springs is quite difficult requiring a careful analysis of the energies of the system and will not be considered.

(f) Small amplitude Vibrations of a loaded light vertical spring.
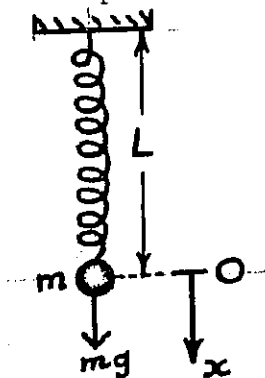
At the equilibrium position 0, $mg = k(L - \ell)$ where ℓ is the unstretched length of the spring.

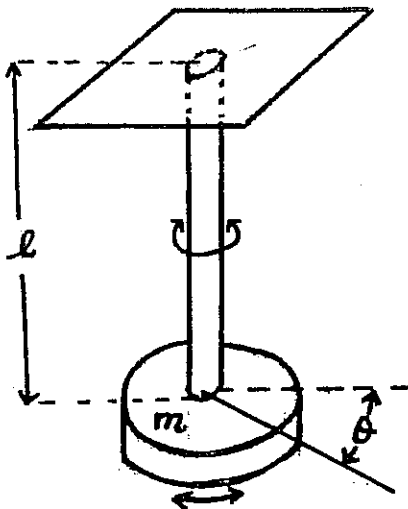m is pulled down a distance a (where $a \leqslant L - \ell$, see Section 3) and released.

Then, at any point x of the subsequent motion

$$m\ddot{x} = mg - k(L + x - \ell)$$
$$= [mg - k(L - \ell)] - kx$$
$$= -kx, \text{ as the term in square brackets}$$

is zero from above.

So m executes simple harmonic motion. What is its amplitude and period?

(g) The Torsional Pendulum (and many similar devices, such as a clock balance wheel).

The 'angular acceleration' of the bob equals $\ddot{\theta}$ and is proportional to the torque τ applied to it by the elastic arm. We thus try to relate τ to θ. While we cannot give a rigorous demonstration of the result we can at least make it plausible.

Regard the flexible arm as made up of a large number, N, of parallel coherent elastic fibres as illustrated
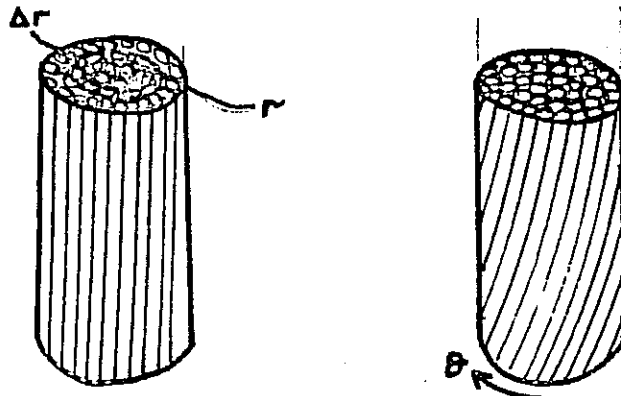
Then in a shell of radius r and thickness Δr there will be approximately $\dfrac{2 N r \Delta r}{R^2}$ fibres, where R is the radius of the arm. ∴ Peeling off such a shell of fibres we see, upon flattening it out,

that the effect of rotating the bob through θ is to displace the individual fibres as illustrated. From which we have

$$T \cos \alpha = \frac{mg}{N}$$

and so the torque due to the shell is

$$T \sin \alpha \times r \times \frac{2Nr\Delta r}{R^2}$$

$$= \frac{2mg}{R^2} \tan \alpha \; r^2 \; \Delta r$$

$$= \frac{2mg}{\ell R^2} \; r^3 \; \theta \; \Delta r \quad \text{as}$$

$$\tan \alpha = \frac{r\theta}{\ell} \; .$$

Now the total torque τ equals the sum of the torques due to each such shell,

$$\tau \simeq \sum_r \frac{2mg}{\ell R^2} \theta r^3 \; \Delta r \quad \text{which in}$$

the limit as the number of shells becomes infinite and $\Delta r \to 0$, gives a total torque of magnitude

$$\int_0^R \frac{2mg}{\ell R^2} \theta \; r^3 \; dr = \frac{mgR^2}{2\ell} \theta$$

oppositely directed to θ.

component of T producing the torque.

So $\ddot{\theta} \propto \tau \propto -\theta$ and the variation of θ is simple harmonic. In fact elementary dynamics of rotating bodies yields $I\ddot{\theta} = \tau = -\frac{mgR^2}{2\ell} \theta$ where I is the moment of inertia of the bob, equal to $mR_B^2/2$, R_B being the radius of the bob. And so motion has period

$$2\pi \frac{R_B}{R} \sqrt{\frac{\ell}{g}}$$

Compare this to (d) the case of a simple pendulum, noting that here no assumption concerning the size of θ has been made, only that the mass of the arm is negligible.
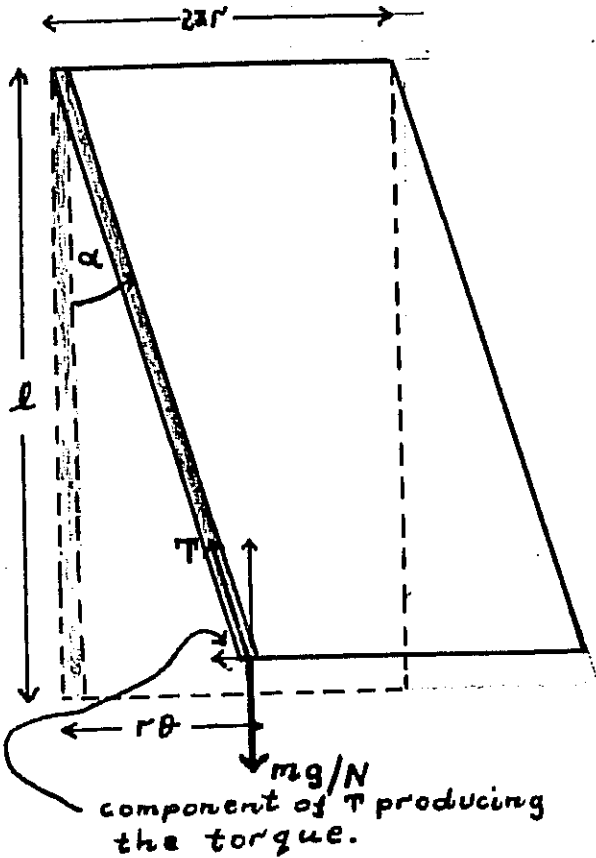
(h)   Consider an object sliding on the frictionless surface of a convex curve as shown.
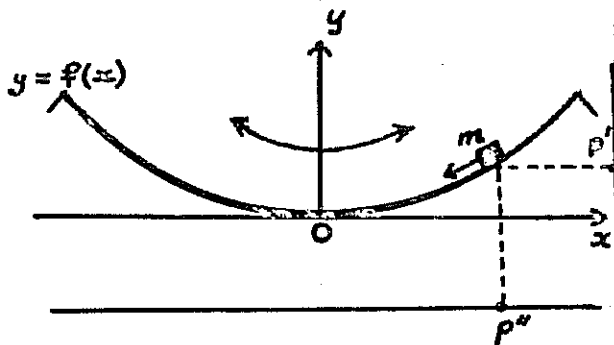
It can be shown that if the shape of the slide, y = f(x), is chosen correctly the distance along the slide from 0 to m varies simple harmonically.

The actual curve required turns out to be a *cycloid*, the curve traced out by a point on the circumference of a circle (radius $^A/_2$) rolling along under the line y = A, a result first established by Huygens (1629-1695) in order to improve the accuracy of clocks by using an isochronous pendulum (ie a pendulum whose period is truely independent of the amplitude of the swing).

The similar problem of asking for a curve y = f(x) so that the vertical projection P' of the mass's position moves simple harmonically turns out to have the same answer, viz. the curve is a cycloid. However the corresponding question for the horizontal projection P'' appears to have no simple answer. In fact, without modifying the problem to allow for collisions at a pair of perfectly reflecting barriers at the upper extremities of the curve, no curve exists.

The examples should help to vindicate the importance attached to simple harmonic motion both in physics and applied mathematics. In each of the remaining four sections a specific problem connected with simple harmonic motion is investigated.

3. OVER-EXTENDED LIGHT VERTICLE STRINGS
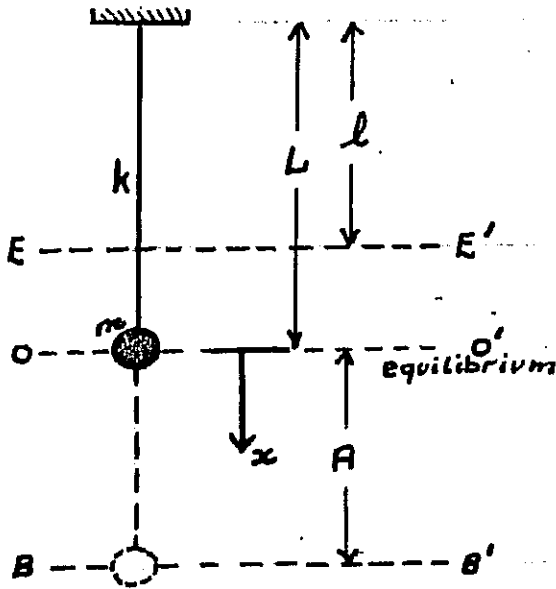
Here the symbolism is the same as in (f) so $mg=k(L-\ell)$ but $A>L-\ell$. For motion between BB' & EE' we have

$$m\ddot{x} = mg - k(L + x - \ell)$$

$$= -kx. \quad \underline{ie} \text{ motion is}$$

given by $x = A \cos \sqrt{\dfrac{k}{m}} \, t$

and

$$(\dot{x})^2 = \frac{k}{m}(A^2 - x^2), \text{ by lemma.}$$

So at $x = -(L - \ell)$ we have

$$\dot{x} = -\sqrt{\frac{k}{m}} \sqrt{A^2 - (L-\ell)^2}$$

$$= -\sqrt{\frac{k}{m}} \sqrt{A^2 - \frac{m^2 g^2}{k^2}} = v_0, \text{ say.}$$

Thus the bob continues its vertical motion, however the only force continuing to act is due to its weight as the string is now relaxed, exerting no force on m. $\underline{ie}$ beyond EE' the motion is that of a projectile fired vertically upwards with initial velocity v_0, in which case it continues to rise to a height of

$$-(L - \ell) - \frac{v_0^2}{2g} = -\frac{mg}{k} - \frac{k}{2mg}(A^2 - \frac{m^2 g^2}{k^2})$$

$$= -\frac{mg}{k} - \frac{k A^2}{2mg} - \frac{mg}{2k}$$

$$= -\left[\frac{3mg}{2k} + \frac{kA^2}{2mg}\right]$$

$$> A \text{ since } A > \frac{mg}{k} \text{ for } v_0 \text{ to exist .}$$

$\underline{ie}$ motion is not simple harmonic, the bob oscillating through a total distance of $A + \dfrac{3mg}{2k} + \dfrac{kA^2}{2mg}$. It is however periodic (see diagram on next page), of period greater than $\sqrt{\dfrac{k}{m}}$. An interesting exercise is to find this period in any specific situation of this type.
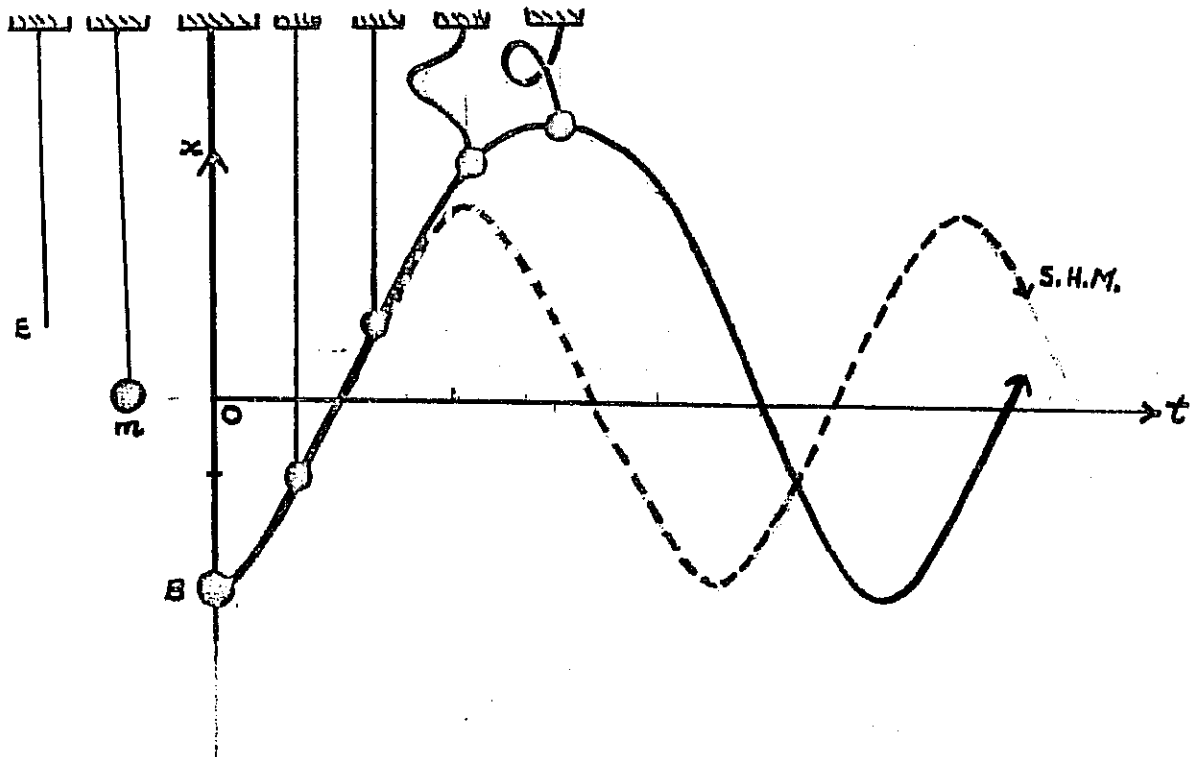
4.   MECHANICO-OPTICAL SIMPLE HARMONIC DEVICE

A uniformly rotating circular disk is viewed through a narrow
slit   aligned to form a diameter of the disk.
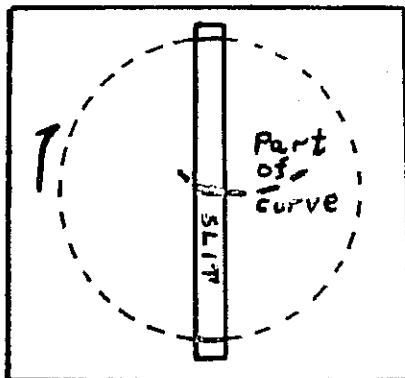
Drawn on the disk is a curve whose
intersection with the slit performs
simple harmonic motion along the
slit as the disk rotates.   Thus
when viewed we see a point executing
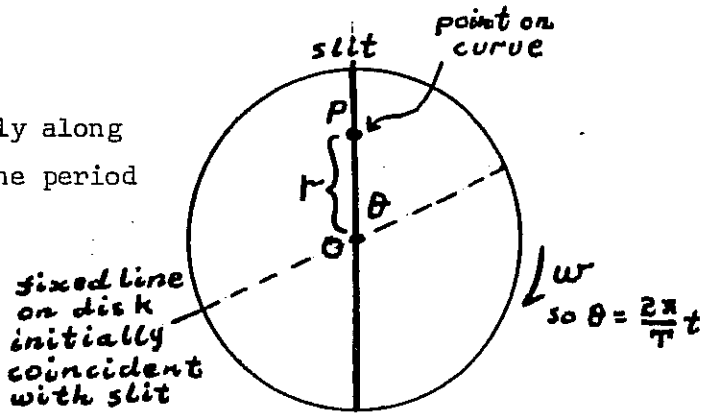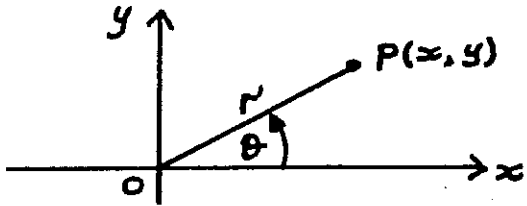simple harmonic motion up and down
the slit.

Problem: find the appropriate
curve to paint on the disk.

If r is to vary simple harmonically along
the slit with a period equal to the period
of revolution of the disk, T

we have

$r = R \cos \frac{2\pi}{T} t$

$= R \cos \theta.$

slit   *point on curve*

fixed line on disk initially coincident with slit

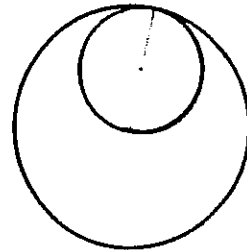*so* $\theta = \frac{2\pi}{T} t$

$P(x, y)$

So the curve is locus of the point
P where $r = R \cos \theta$

ie $\sqrt{x^2 + y^2} = r = R \cos \theta = \frac{Rx}{\sqrt{x^2 + y^2}}$
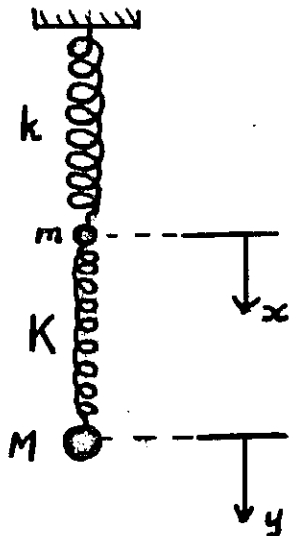
or $x^2 + y^2 = Rx$, $(x - \frac{R}{2})^2 + y^2 = \frac{R^2}{4}$

So the curve is a circle centre at $x = \frac{R}{2}$, $y = 0$, radius $\frac{R}{2}$.

5. COUPLED SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM

The example we will consider is illustrated below, a more
ambitious example might be the
body of a car riding on its
four springs (shock absorbers).

The question we will seek to
answer is can the illustrated
system vibrate in such a way
that both masses execute
simple harmonic motion of the
same period and what is this
period? Of course many other
questions could be asked, and
answered, for example what is the general motion of such a system?

Now the equations of motion are

$$M \ddot{y} = \text{force on } M = -K(y - x)$$

$$m \ddot{x} = \text{force on } m = K(y-x) - kx = -(K+k)x - Ky$$

and for simultaneous simple harmonic motion we want $x = a \cos \lambda t$

$$y = A \cos \lambda t$$

to be a solution for suitably chosen values of A, a and λ. Substituting these into the equations of motion gives

$$- \lambda^2 A \cos \lambda t = - \frac{K}{M} (A - a) \cos \lambda t$$

$$\lambda^2 a \cos \lambda = \frac{K}{m} (A - a) \cos \lambda t - \frac{k}{m} \cos \lambda t.$$

Cancelling $\cos \lambda t$ and rearranging yields $\left(\frac{K}{M} - \lambda^2 \right) A + \frac{K}{M} a = 0$

and

$$- \frac{K}{m} A + \left(\frac{K}{m} + \frac{k}{m} - \lambda^2 \right) a = 0$$

or as a matrix equation for A, a

$$\begin{bmatrix} \frac{K}{M} - \lambda^2 & \frac{K}{M} \\ \frac{K}{m} & \frac{K+k}{m} - \lambda^2 \end{bmatrix} \begin{bmatrix} A \\ a \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix}$$

But this equation will only have non-zero solutions for A,a if,

$$\det \begin{vmatrix} \frac{K}{M} - \lambda^2 & \frac{K}{M} \\ \frac{K}{m} & \frac{K + k}{m} - \lambda^2 \end{vmatrix} = 0 \quad (\underline{ie} \text{ the matrix is singular})$$

and this is precisely the condition that λ^2 be an eigenvalue of the matrix $\begin{bmatrix} \frac{K}{M} & \frac{K}{M} \\ \frac{K}{m} & \frac{K+k}{m} \end{bmatrix}$

In precisely this manner eigenvalues enter in an important way into advanced theories of mechanics.

Thus λ^2 is a root of

$$(\lambda^2)2 - \left(\frac{K}{M} + \frac{K}{m} + \frac{k}{m} \right) \lambda^2 + \frac{Kk}{mM} = 0$$

A straight forward argument shows this quadratic has two positive (and therefore real) roots.

We are therefore able to conclude that the sought after motion is always possible (at least in theory) and can in general take place at two distinct frequencies. (Why not four?) Perform these calculations in a few specific (ie numeric) examples also determine the ratio of A/a for these motions (what are you doing in terms of the theory of eigenvalues?)

6. RESISTED (DAMPED) HARMONIC MOTION

In almost all the physical examples of simple harmonic oscillations introduced in section 2 it can be observed that, due to the unavoidable presence of frictional and other resistive forces, the motion dies out.

If these forces are mainly due to air drag (as in the case of a freely pivoted simple pendulum, mass on the end of a spring etc.) then it is usual to take them proportional to the velocity but oppositely directed (see Lush and Smith, p.178), an assumption which is experimentally and hydrodynamically justifiable at least for relatively low speeds. (For higher speeds the assumption that the drag is proportional to the square of the velocity is often used.) In this case the differential equation of motion becomes

$$m\ddot{x} = -kx - r\dot{x} \quad \text{where } r \text{ is the drag}$$

coefficient.

We could apply arguments similar to those of section 1 and develop a theory along the same lines.

(Using an argument similar to that in the proof of the lemma show that in this case $kx^2 + v^2$ is a decreasing function of t.) Instead however we will simply present a typical example of the possible forms the solution x can have.

Hence, *show this equation can have a solution of the form*

$$x = Ae^{-\alpha t} \cos \lambda t$$

for appropriate values of α and λ Also sketch a typical graph of such a solution.

Now $\dot{x} = -\alpha Ae^{-\alpha t}\cos \lambda t - \lambda Ae^{-\alpha t}\sin \lambda t$

and $\ddot{x} = \alpha^2 Ae^{-\alpha t}\cos \lambda t + 2\alpha\lambda Ae^{-\alpha t}\sin \lambda t - \lambda^2 Ae^{-\alpha t}\cos \lambda t$

So if x is a solution we must have, for all values of t

$m\alpha^2 \cos \lambda t + 2\alpha\lambda m \sin \lambda t - \lambda^2 m \cos \lambda t = -k \cos \lambda t + r\alpha \cos \lambda t + r\lambda \sin \lambda t$

This will certainly be true if we can find λ and α so that

$$(m\alpha^2 - m\lambda^2 + k - r\alpha) \cos \lambda t$$

and $\qquad (2\alpha\lambda m - r\lambda) \sin \lambda t \qquad$ are both simultaneously zero

for all values of t.

From the last of which $\alpha = {}^{r}/2m > 0$, and then from the first

$$m\lambda^2 = k - {}^{r^2}/4m.$$

so $\qquad \lambda = \sqrt{\dfrac{k - r^2/4m}{m}} \qquad$ provided $r < 2\sqrt{mk}$

ie provided the damping is sufficiently small.

What becomes of the solution if $r \geqslant 2\sqrt{mk}$? (a solution does exist

and the corresponding motion is termed *critically damped*.)

We therefore have established that under some restrictions

appropriate values of α and λ can be found and so the equation has

a solution of the suggested form, a graph of which is given below.
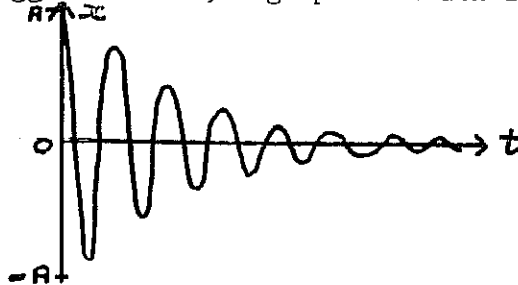
If on the other hand the resistive forces are largely

frictional in origin, it might well be appropriate to assume

they are constant in magnitude but oppositely directed to the

sense of the motion.

As an example let us consider the motion of a mass resting

on a rough platform and oscillating on the end of a horizontal

spring. Initially suppose the spring extended through a distance A and then released.
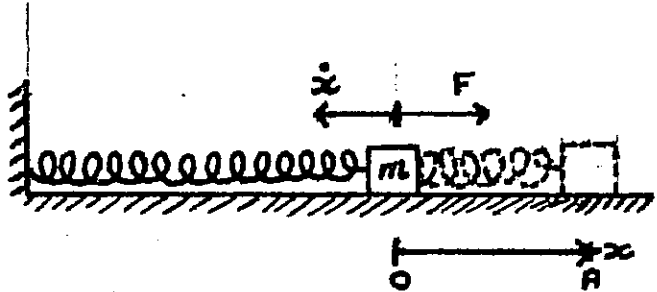
Under such assumptions the differential equation of motion would be

$$m\ddot{x} = -kx - \text{sgn}(\dot{x})F$$ where F is the magnitude of the frictional force and sgn is the function defined by

$$\text{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0 \end{cases}$$ (Graph this function)

so $m\ddot{x} = -kx - F$ whenever $\dot{x} > 0$

and $m\ddot{x} = -kx + F$ whenever $\dot{x} < 0$. Of course the motion will cease whenever $\dot{x}$ and the net force on the body are simultaneously zero, which from the laws of static friction, is the first instance, t_F, when $\dot{x}(t_F) = 0$ and $k|x(t_F)| \leqslant F$. Thus motion will only commence if kA>F, which we take to be the case.

Initially $\dot{x} < 0$, so $m\ddot{x} = -kx + F$. If we change the variable from x to $y = x - \frac{F}{k}$ this becomes

$$m\ddot{y} = -ky$$ and so from our previous results, and the initial conditions $y = (A - \frac{F}{k}) \cos \sqrt{\frac{k}{m}} \, t$

or $x = (A - \frac{F}{k}) \cos \sqrt{\frac{k}{m}} \, t + \frac{F}{k}$.

This will remain the equation of motion till $\dot{x}$ changes sign, ie till $t = \pi\sqrt{\frac{m}{k}}$ by which stage the mass is at position $x = -A + 2F/k$. The motion will only continue beyond this provided $|2F - kA| > F$. Which will certainly not be the case if $x > 0$. (Why?)

Should the motion continue, then the new differential equation
is $m\ddot{x} = -kx - F$ and by a suitable change of variable
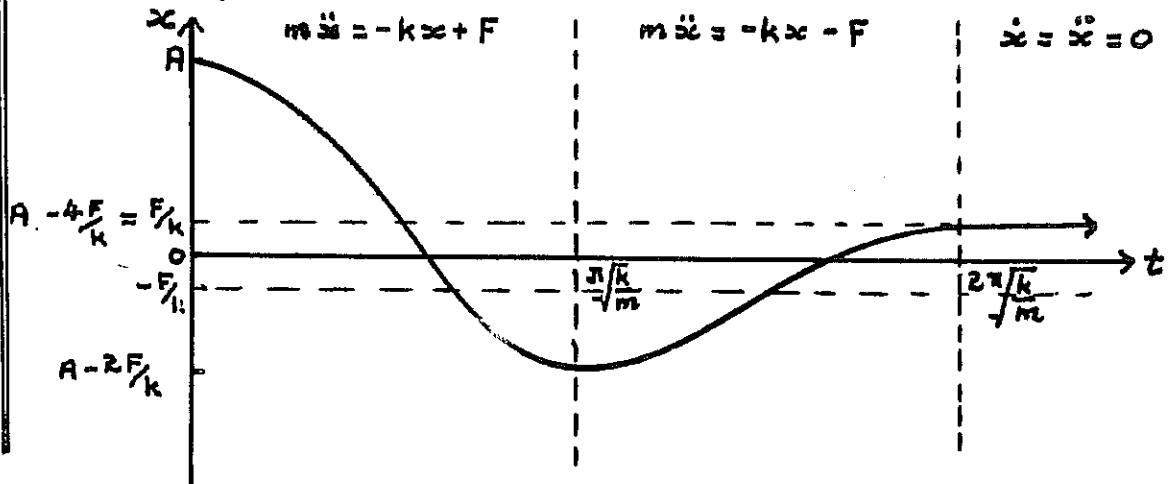$(y = x + {}^F/k)$ coupled with the new initial conditions

$$x(\pi\sqrt{\tfrac{m}{k}}) = -A + 2F/k \ , \quad \dot{x}(\pi\sqrt{\tfrac{m}{k}}) = 0 \text{ we obtain}$$

similarly

$$x = (A - {}^{3F}/k) \cos \sqrt{\tfrac{k}{m}}\, t \ - \ {}^F/k.$$

This again remains the equation of motion till $\dot{x}$ next changes
sign at $t = 2\pi\sqrt{\tfrac{m}{k}}$ by which $x = A - {}^{4F}/k$.

Continuing in this way we can build up a graph of the mass's
motion, such as the one illustrated below.

With the above example I will finish this short essay, not
because the topic is exhausted (far from it) but because I am,
leaving it to the readers to carry the discussion further and
to correct the many mistakes which I have undoubtedly committed
and which are the only features for which I can claim any
originality.

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