CIRCULAR MOTION (Part 1)

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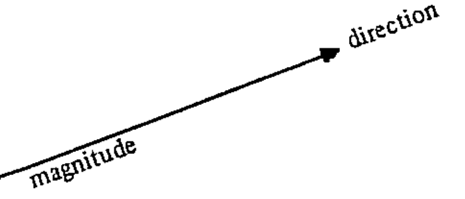
This is the first part of a two part article on circular motion. The second part will be included in Vol 7 No 3.

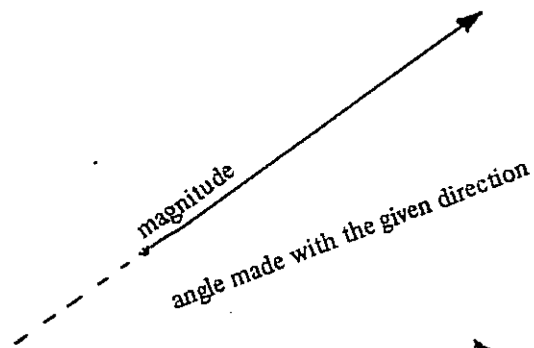
Prelude

Throughout we shall be concerned with vector quantities; quantities which have both a "magnitude" and a "direction". [For example: *force, velocity* (the magnitude of which we call *"speed"*), *acceleration* and, as we shall see, *position*]. A vector quantity changes if either its magnitude or its direction is changed. This observation lies at the heart of our analysis of circular motion, where it is directions rather than the more familiar case of magnitudes which undergo change. A vector quantity may be conveniently represented by an arrow pointing in its direction and with a length equal to the magnitude.

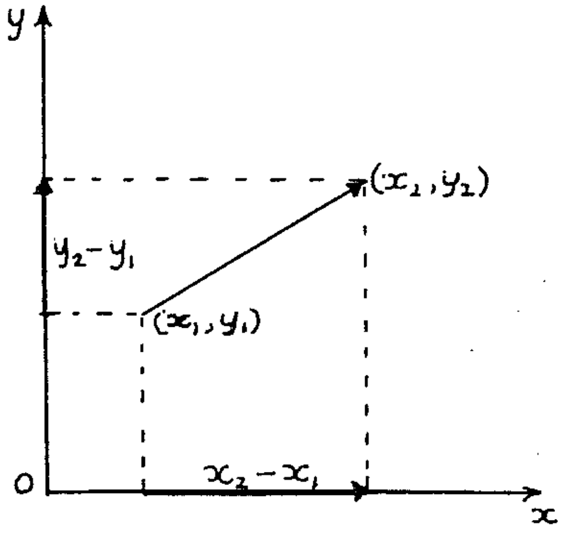


For circular motion, all the vector quantities of interest to us are constrained to have directions lying in a common plane and so may be described in either of the following ways :

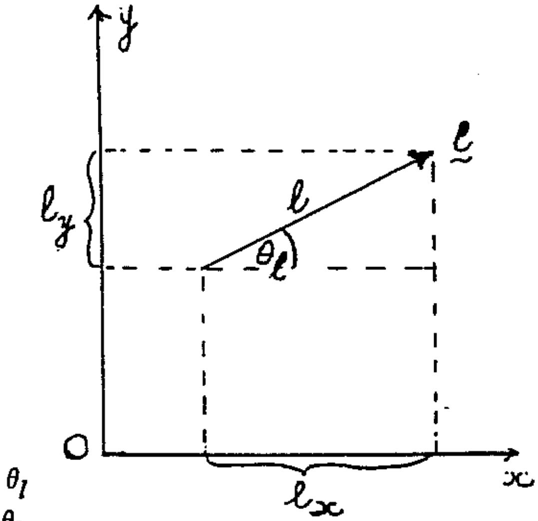
(1) By specifying the magnitude (length) and the angle the arrow makes with a given direction. [This requires us to specify a "sense of rotation" for measuring the angle – conventionally we take anticlockwise as the positive sense and measure in radians.]



(2)By specifying the components of the vector with respect to a given set of cartesian axes (see diagram below).



These two representations are related. If we agree to denote vector quantities by underlining with a tilde (_) [for example: a for acceleration, v for velocity, f for force] and for the vector quantity l write l for its magnitude [thus v will denote speed] and l_x, l_y, θ_l respectively for the x and y components relative to the given set of axes and the angle made with the given direction, which for convenience we take to be the positive x-direction, then we have-



$l_x = l \cos \theta_l$
$l_y = l \sin \theta_l$
$l = \sqrt{l_x^2 + l_y^2}$
and

$\sin \theta_l = l_y/l$
$\cos \theta_l = l_x/l$ from which θ_l is uniquely determined.

Note also that the slope of the arrow is $\tan \theta_l = l_y/l_x$, so two vectors l and n will be:

perpendicular if $l_y/l_x = -n_x/n_y$;

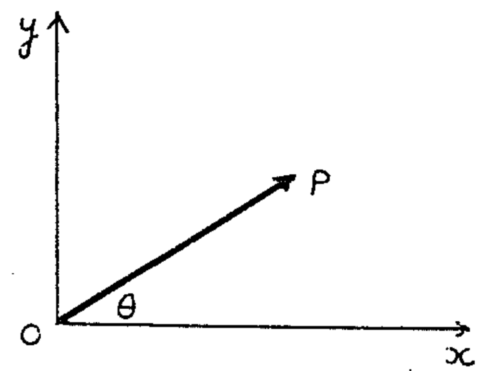
parallel if $l_x/l_y = n_x/n_y$;

in which case they will have the same direction when l_x and n_x (equivalently l_y and n_y) have the same sign and oppositely directed when l_x and n_x have opposite signs.

KINEMATICS OF CIRCULAR MOTION

Position as a vector quantity

If we take a given direction and a given point O, then the position of any other point P in the plane is uniquely specified by the arrow (position vector) with tail at O and head at P.



If the distance between O and P is r, then the position vector has magnitude r and will make an angle θ with the given direction.

Choosing O as the origin of our set of Cartesian coordinate axes and the given direction for the positive x-direction we see that the components of the position vector are

$$\text{x-component} = r\cos\theta,$$

$$\text{y-component} = r\sin\theta$$

and that these components are respectively the x and y coordinates of P relative to our axes.

If the point P moves with time, then in general both r and θ will vary as functions of the time t; that is

$$r \equiv r(t) \quad \text{and} \quad \theta \equiv \theta(t).$$

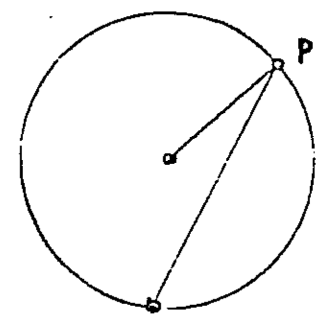
The time rate of change of θ, $\dot{\theta}$, is termed the <u>angular velocity</u> of P about O, sometimes denoted by ω. [Here, as elsewhere "dot" denotes differentiation with respect to t,

$$\text{thus} \quad \dot{\theta} \equiv \frac{d\theta}{dt} \quad]$$

Similarly $\frac{d\omega}{dt} = \ddot{\theta}$ is the angular acceleration of P about O.

Note: Both the angular velocity and angular acceleration depend on the choice of O but unlike the position vector do not depend on our choice for the given direction provided it remains fixed throughout the motion.

[EXERCISE: For a point P constrained to move on a circle, show that when O is chosen to be the centre of the circle the angular velocity is twice that when O is a point on the circumference.]



Henceforth we restrict our attention to motion in a circle and for simplicity take for its centre the origin O. Under these assumptions the position vector of the moving particle has r equal to a constant (the radius of the circle of motion) and so

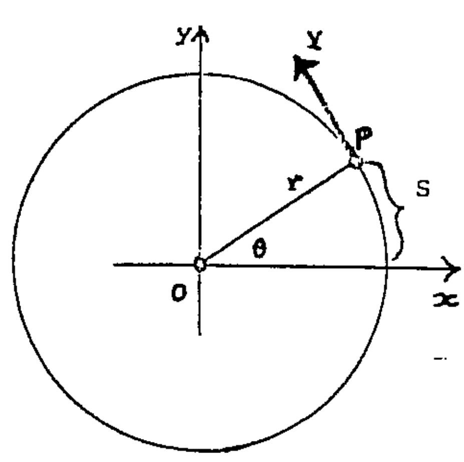
$$\dot{r} = \frac{dr}{dt} \equiv 0.$$

The velocity vector

The velocity vector $\underset{\sim}{v}$ must be directed tangentially to the circle of motion (otherwise our particle would have a <u>radial component</u> of velocity and so move off its circular path) – see diagram – and has a magnitude (speed):

$$v = \left|\frac{ds}{dt}\right| = \left|\frac{d}{dt}(r\theta)\right| = r\,|\dot{\theta}| = r\,|\omega|$$

(here s is arc length on the circle measured, in the sense of rotation, from a fixed point of the circle to the moving particle)



[Note: the absolute value signs are necessary in case the particle is rotating in the opposite sense to our chosen sense of rotation in which case $\omega = \dot{\theta}$ would be negative.]

These results may also be obtained by differentiating the components of the position vector to obtain those of the velocity.

x-component of velocity is

$$v_x = \dot{x} = \frac{d}{dt}(r\cos\theta) = -r\sin\theta\,\dot{\theta}$$

y-component of velocity is

$$v_y = \dot{y} = \frac{d}{dt}(r\sin\theta) = r\cos\theta\,\dot{\theta}.$$

From which we see that

$$\frac{v_y}{v_x} = -\frac{\cos\theta}{\sin\theta} = -\frac{\text{x-component of position}}{\text{y-component of position}},$$

so v is perpendicular to the position vector (that is, tangentially directed) and has magnitude

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)} = r\,|\dot{\theta}|.$$

If ω, and consequently v is constant, then the particle moves about the circle with uniform speed and we have the important special case of **uniform circular motion.**

Acceleration

Since the direction of the velocity changes with position (and consequently with time if our particle is to move at all) an acceleration must be present. Indeed, if our particle possessed mass and were not accelerating, then it could not be subject to any external force and so, by Newtons first law: *Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it;* the particle could not continue to move in a circular path. The components of the acceleration vector $\underset{\sim}{a}$ are obtained by differentiating those of $\underset{\sim}{v}$.

x-component of acceleration is

$$a_x = \dot{v}_x = \frac{d}{dt}(-r\sin\theta\,\dot{\theta}) = -r(\cos\theta)\dot{\theta}^2 - r(\sin\theta)\ddot{\theta}$$

Similarly,

y-component of acceleration is

$$a_y = -r(\sin\theta)\dot{\theta}^2 + r(\cos\theta)\ddot{\theta}.$$

In the case of uniform circular motion ($\dot{\theta}$ is constant and so $\ddot{\theta} \equiv 0$) these reduce to

$$a_x = -r(\cos\theta)\dot{\theta}^2$$

and

$$a_y = -r(\sin\theta)\dot{\theta}^2$$

so

$$\frac{a_y}{a_x} = \frac{\sin\theta}{\cos\theta} = \frac{\text{y-component of position}}{\text{x-component of position.}}$$

Thus, in this case, the acceleration vector is parallel, but oppositely directed (a_x has opposite sign to r cos θ, the x-component of position), to the position vector. That is, **for uniform circular motion the acceleration is directed radially inward and has magnitude**

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{r^2\cos^2\theta\,\dot{\theta}^4 + r^2\sin^2\theta\,\dot{\theta}^4} = r\dot{\theta}^2 = r\,\omega^2.$$

For general circular motion, besides this inwardly directed radial component of magnitude $r\omega^2$, there is a second component of the acceleration with

an x-component equal to $-r(\sin\theta)\ddot{\theta}$

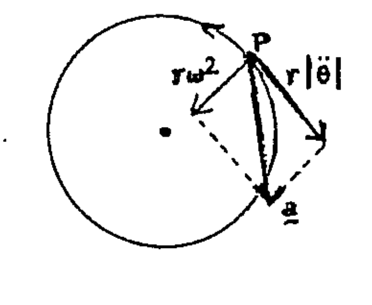
and

a y-component equal to $r(\cos\theta)\ddot{\theta}$.

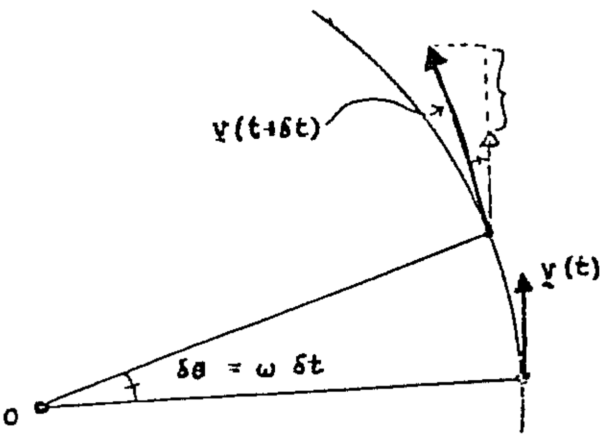
From which we deduce that this is a tangentially directed component of the acceleration of magnitude r $|\ddot{\theta}|$, r times the angular acceleration.

Summarizing we have:

Acceleration: Inwardly directed radial component of magnitude $r\omega^2$. Tangential component of magnitude r $|\ddot{\theta}|$



These expressions can also be deduced by considering a vector diagram showing the velocity at time t, $\underline{v}(t)$ and at a subsequent time $t + \delta t$ and considering limits as $\delta t \to 0$



From the diagram we see that:

Change in tangential component of $\underline{v}$ is

$$v(t+\delta t)\cos\delta\theta - v(t) \cong v(t+\delta t) - v(t)$$

Change in radial component is

$$v(t+\delta t)\sin\delta\theta \cong v(t+\delta t)\,\delta\theta$$

(as, to first order terms, $\cos\delta\theta \cong 1$ and $\sin\delta\theta \cong \delta\theta$)

and so,

Tangential component of acceleration is

$$\lim_{\delta t \to 0} \frac{v(t+\delta t)-v(t)}{\delta t} = \frac{d}{dt}(v(t)) = \frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta}$$

Radial component is

$$\lim_{\delta t \to 0} v(t+\delta t)\frac{\delta\theta}{\delta t} = v(t)\frac{d\theta}{dt} = (r\dot{\theta}) \times \dot{\theta} = r\dot{\theta}^2 .$$

In what follows we will restrict our attention to the case of uniform circular motion, however in the absence of this simplification we remark that a sometimes useful expression is

$$r\ddot{\theta} = \frac{d}{d\theta}(\tfrac{1}{2}r\dot{\theta}^2)$$

which leads to a "first (energy) integral."

DYNAMICS OF UNIFORM CIRCULAR MOTION

As we have already noted, in order that a massive particle execute circular motion it must be acted on by an external force. By Newton's second law: *The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed*, and the expression for acceleration derived in the last section we have that, **a particle of mass m constrained to move uniformly in a circle is subject to a centrally directed force of magnitude**

$$F = mr\dot{\theta}^2 = mr\omega^2 = \frac{mv^2}{r}$$

This applied force is known as the **centripetal force** ('centre seeking'). The reaction to this force, felt at the centre ("Pivot") is sometimes termed "centrifugal force" ('centre fleeing').

In the next issue, Dr Sims will take up the notion of the dynamics of uniform circular motion to conclude the article.

CIRCULAR MOTION (Part 2)

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This is the concluding part of the article which began in Volume 7 No. 2.

DYNAMICS OF UNIFORM CIRCULAR MOTION

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APPLICATIONS

The importance of an analysis of circular motion is in the great number and diversity of its applications, some of which are outlined below. Historically circular motion was one of the first motions to be studied. In Newton's epoch making *Philosphiae Naturalis Principia Mathematica* circular motion is given prime place. It is the first motion to be examined after the statement of his general laws of motion. Thus, as Proposition IV Theorem IV (The 4th out of 70 propositions) we find:

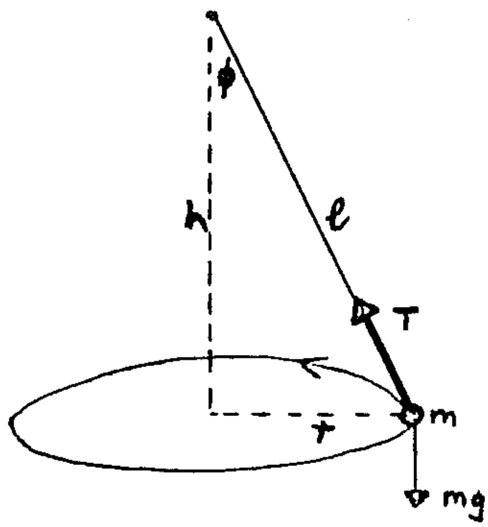
> *[illegible]e centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are to each other as the squares of the arcs described in equal times divided respectively by the radii of the circles.*

After proving this, Newton goes on to obtain 9 corollaries the first of which reads:

> *Therefore, since those arcs are as the velocities of the bodies, the centripetal forces are as the squares of the velocities divided by the radii.* *

1. The Conical Pendulum.

We consider a massive particle suspended by a light string (or rod) and moving in a horizontal plane - see diagram.



Resolving the tension T in the string into verticle and horizontal components and balancing forces we have that

$T \cos\phi = mg$ and

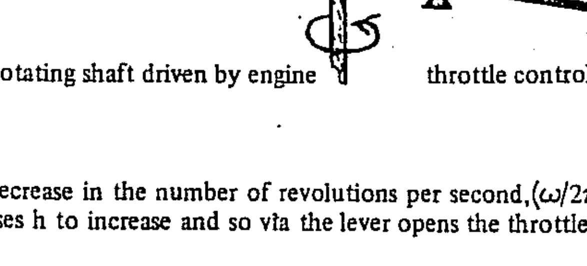
$T \sin\phi = \text{centripetal force} = mr\,\omega^2$

Thus $\tan\phi = r/h = r\,\omega^2/g$ or $\omega^2 = g/h$

The **period** of this motion, the time required to complete one complete revolution $2\pi/\omega$, is therefore $2\pi\sqrt{(h/g)}$

[Note: The period is independent of the mass m, and for small amplitude motion ($r < l$) we may use the approximation $h \doteq l$ to obtain; Period $\doteq 2\pi\sqrt{(l/g)}$

The conical pendulum provides the principle underlying the operation of **governers** (speed limiters) such as seen on old fashioned steam-engines and hidden inside many modern pieces of machinery.



A decrease in the number of revolutions per second, $(\omega/2\pi)$, causes h to increase and so via the lever opens the throttle.

* Here, as elsewhere, Newton uses proportions, as at that time "standard units" had not been introduced.

and so the orbital period (planetary year) is

$$T = 2\pi / \omega = 2\pi \sqrt{r^3/GM}\ .$$

That is; T^2 is proportional to r^3, a relationship which we should recognise as Kepler's third Law of planetary motion in the special case of a circular orbit.

We should also note that the orbital velocity is $v = \sqrt{(GM/r)}$

A similar analysis applies to the Bohr model of a hydrogen atom except that the centripetal force is supplied by the Coulombic attraction ke^2/r^2 and a "quantum effect" is introduced by arbitrarily requiring that the angular momentum of the electron (mvr) only assumes values which are integral multipliers of some universal constant (Planck's constant h).

Special case: telecommunications satellites.

In order that a satellite remain stationary above a given point on the earth's surface it must have an orbital period T of one day. Thus, if such a satellite orbits at a height h above the surface of the earth we have

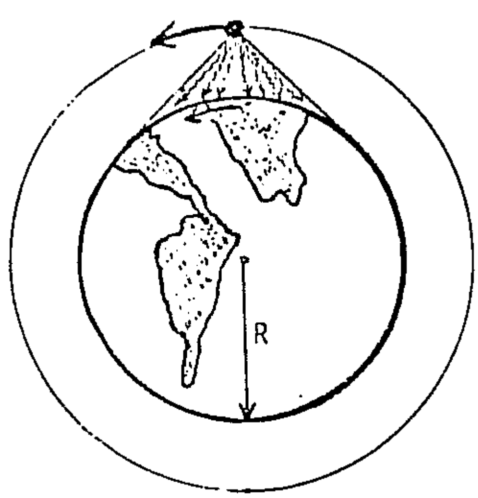
$$T = 2\pi\sqrt{[(R+h)^3/GM]}\ ,$$

where R is the radius of the earth and M its mass.

Since the gravitational attraction between the earth and a body of mass m on its surface is GmM/R^2 which also equals mg we see that $GM = gR^2$. Substituting this in the expression for T and rearranging we obtain

$$h = (\sqrt[3]{(gT^2/4\pi^2R)} - 1)R \quad \cong 0.5R$$

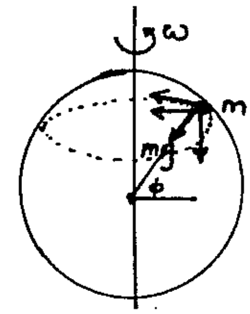
$$= 3180\text{ Km (using } R = 6360\text{ km).}$$



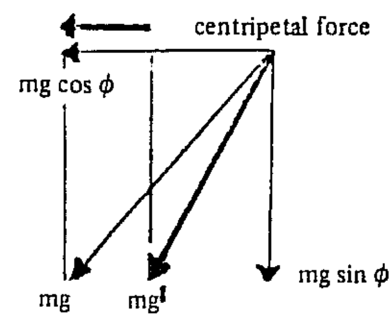
It is the fact that such satellites are restricted to a single altitude which effectively limits their number if "congestion" is to be avoided.

4. Change in Gravity due to Latitude

Although the gravitational attraction of the earth on a body on the surface is mg, the "effective gravitational force" (as measured by the acceleration of falling bodies) varies with latitude ϕ. In general it is not directed toward the earth's centre and has a magnitude mg^1, less than mg.



If we resolve mg into components $mg \sin\phi$ parallel to the earth's axis of rotation and $mg \cos\phi$ perpendicular to the axis, then we see that part of this last component is necessary to supply the centripetal force needed to maintain the particle in the general circular motion of the earth. This centripetal force has magnitude $m(R\cos\phi)\,\omega^2$, where ω is the angular velocity of the earth (2π radiansper day).



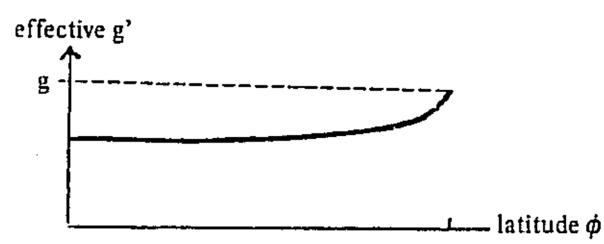
Thus the "effective gravitational force" has components

$mg \sin\phi$ and

$$mg\cos\phi - m(R\cos\phi)\,\omega^2 = m(g - R\,\omega^2)\cos\phi$$

From which the "effective gravitational force" may be computed. For example it has magnitude

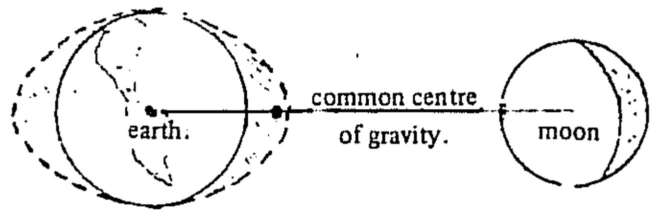
$$mg^1 = m\sqrt{g^2 - (2gR\,\omega^2 - R^2\,\omega^4)\cos^2\phi}$$



5. Further Miscellaneous Applications

A number of further situations, which involve circular motion, are listed below along with a few brief comments on some of them. You may like to try and analyse these situations further. You should also try to think of other examples to add to the list.

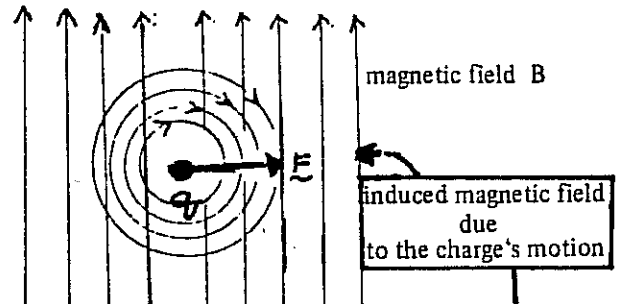
(i) The action of a centrifuge.
(ii) The formation of tides. In the earth-moon system both planets move in nearly circular orbits about their common centre of mass.



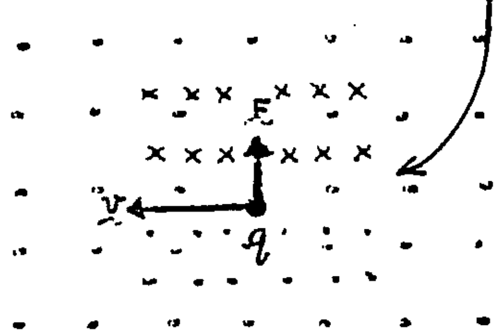
tidal buldges

At points on the earth's surface the same distance from the centre of mass as the earth's centre the lunar attraction very nearly equals the centripetal force needed to maintain them as part of the earths circular path around the common centre of gravity. However for points on the earth near the moon the lunar pull is stronger and the required centripetal force less. At points farther from the moon the reverse is the case, the lunar pull is weaker and the centripetal force greater. The nett effect at these points is a lowering of the local value of g and the consequent development of tidal bulges above them to maintain the pressure balance within) oceans (at least this would be the case were it not for me rotation of the earth about its axis which causes the bulges to lag due to frictional type effects).

(iii) When an electrically charged particle moves with velocity v through a perpendicular magnetic field of strength B it experiences a force F of magnitude qvB acting in a direction which is perpendicular both to the particle's velocity and the magnetic field (Biot-Savart Law).



,ositively charged particle moving into the page.



magnetic field B directed out of the page

Consequently, the particle can move in a circular path for which

$$mv^2/r = qvB \qquad \text{or} \qquad r = mv/(qB)$$

This effect is fundamental to the design and operation of Cathode ray tubes, television picture tubes and cyclotrons.

EXERCISES.

1. A massive particle suspended by a "light" rod from a pivot P, moves with constant speed in a horizontal circle. If the rod has a length of 1 m and makes an angle of 30^0 with the verticle from P to the centre of the circle, find
(a) the time taken to complete one complete revolution
(b) the speed of the particle.

   If the speed of the particle is doubled, what angle will the rod now make with the verticle? (Use $g = 9.8$ m/sec^2)

2. What is the maximum speed at which a car could travel around a circular curve of radius 150 m if the coefficient of friction between the tyres and road surface is 0.6 (a typical value). What will the maximum speed be if the coefficient of friction is reduced by 10%.

3. A circular curve of highway is designed for traffic moving at 80 km/hr. If the radius of the curve is 120 m at what angle should it be banked to the horizontal.

4. Given that Jupiter makes one revolution of the sun every 11.87 years and that the radius of the earth's orbit about the sun is 149×10^6 km determine the radius of Jupiter's orbit.

5. What is the orbital period and speed of a satellite moving in a circular orbit at an altitude of 250 km above the earth. (Take the earth's radius to be 6371 km).

6. In the Bohr model of a hydrogen atom the electron moves in a circular orbit about the nucleus. If the radius of the orbit is 5.3×10^{-11} m and the electron executes 6.6×10^{15} revolutions per second, find
(a) the speed and acceleration of the electron
(b) the centripetal force needed to maintain the electron in its circular path. (The mass of an electron is 9.1×10^{-31} kg.)

7. The acceleration due to gravity at the north pole is $g = 9.832$ m/sec^2. What is its "effective" magnitude g' at
(a) the equator
(b) a latitude of 35^0.
(The earth's radius is approximately 6371 km.)

8. Calculate the reduction in g due to the "tidal" influence of the moon at the points on the earth's surface which are nearest and farthest from the moon.

   [Use: Lunar mass equals 0.012 that of the earth.
   The earth's radius is 6371 km.
   The distance from the earth to the moon is 38×10^4 km.]