TIN CANS AND ALL THAT

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Question 10 of Paper B in the 1974 Higher School Certificate Mathematics examinations asked us to *minimize the area of sheet metal required to manufacture a tin can of given volume.*

Treating a can as a circular cylinder with closed ends, we have from figure 1:

(1) Area of sheet metal

= Surface Area of cylinder

= 2 × (Area of circular end) + (Area of "side")

$= S = 2\pi r^2 + 2\pi rh$

and

(2) Fixed Volume $= V = \pi r^2 h$

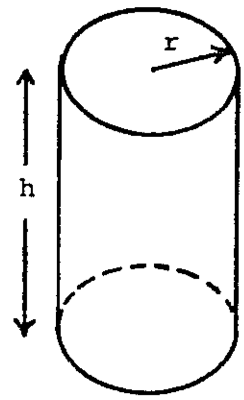


Figure 1

Our problem is to find r and h so as to minimize the expression (1) for S subject to the constraint (2). This is a problem of *optimization subject to constraints.* Such problems arise frequently from the application of mathematics to "practical" situations and are usually difficult to solve. In our case, however, the constraint may be absorbed by substituting (2) into (1) and the problem is readily solved.

Making the substitution we obtain

$$S = 2\pi r^2 + \frac{2V}{r}.$$

Since $S \to \infty$ as $r \to 0$ or $r \to \infty$ and S is a differentiable function of r for $r > 0$, the desired value of r will occur when $\frac{dS}{dr} = 0$. Now,

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}$$

and so equating this to zero we have

$$2\pi r^3 = V \quad \left(\text{or} \quad r = \sqrt[3]{\frac{V}{2\pi}}\right).$$

Using (2) we have

$$2\pi r^3 = \pi r^2 h \quad \text{or} \quad \frac{h}{r} = 2 \quad \left(\text{so} \quad h = 2\sqrt[3]{\frac{V}{2\pi}}\right).$$

Thus, the optimum shaped can (at least so far as minimizing the area of sheet metal used) is obtained by choosing r and h so as to give the required volume and also satisfy the relationship $h/r = 2$; that is, the can must have its diameter and height equal.

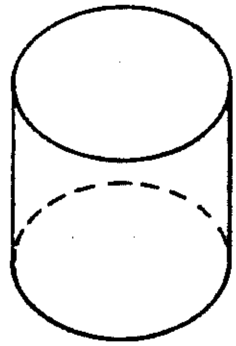


Figure 2

Optimal shape for minimizing area of sheet metal used

The above represents a crude model for minimizing the production cost of a tin can, however, the production cost involves other factors besides the cost of the sheet metal in the can. In order to more nearly minimize the cost, a more realistic model for the cost of production must be used. We will develop one such model which accounts for some of the other factors involved.

FACTORS CONTRIBUTING TO THE COST

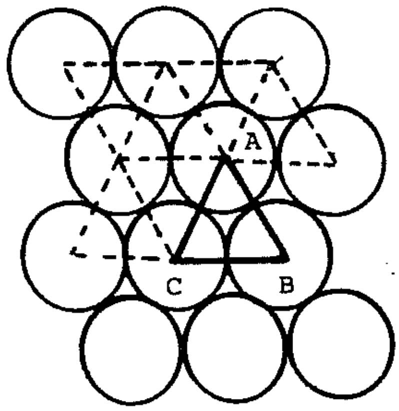
Cost of sheet metal used in production.

To simplify matters we will assume that the metal for the cans is cut from an infinitely expansive sheet (that is, wastage at the edges of the sheet are being neglected). Since it is possible to tessellate the plane with rectangles it is clearly possible to cut the sides of the cans without any wastage of metal, however the plane cannot be tessellated by circular disks and so when producing the circular tops and bottoms a certain amount of wastage is inevitable.

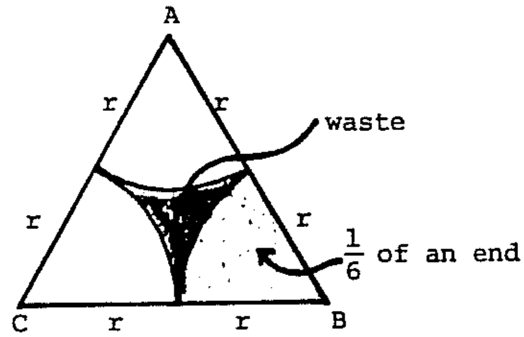
The most "efficient" way of packing circular disks in the plane is illustrated in Figure 3(a) and since the plane is covered by triangles of the form ABC we see from Figure 3(b) that the area of waste metal to that of metal used is

$$\frac{\text{Area of triangle ABC} - 3 \times (\text{Area of } \frac{1}{6} \text{ of a disk})}{3 \times (\text{Area of } \frac{1}{6} \text{ of a disk})}$$

$$= \frac{\sqrt{3}r^2 - \frac{1}{2}\pi r^2}{\frac{1}{2}\pi r^2} = \frac{2\sqrt{3}}{\pi} - 1 \doteq 0.1$$



(a)



(b)

Figure 3

Thus on the average the cost of sheet metal needed to produce each circular can end is approximately 1.1 times greater than the cost of the metal in the can end itself.

From this we see that the effective cost of the metal in the can is

$$c_m(2\pi rh + 2.2\pi r^2) \quad (3)$$

where c_m equals the cost of sheet metal per unit area.

Cost of cutting (stamping) the sheet metal to shape

Here it is reasonable to assume that the cost will be proportional to the length of the cut. Thus the cost to cut one circular can end will be $2\pi c_e r$, where c_e is the cost per unit length for a circular cut.

In producing the rectangular side for the can it is usual to use two different cutting operations. First the sheet is "slit" into long strips of appropriate width (h, say) and then each of these strips is guillotined into suitable lengths ($2\pi r$, say). There is no reason to expect the cost of these two operations to be the same per length of cut. Thus the total cost of cutting is

$$4\pi c_e r + c_h h + 2\pi c_r r \quad (4)$$

where c_h and c_r are appropriate costs per unit length of cut. (Note, the last two terms are not doubled since each of these cuts is effectively shared by the components for two cans.)

Cost of seaming

Having cut the correct shaped pieces of sheet metal it is now necessary to join them together. This involves a side seam, of length h, and two circular seams at the top and bottom of the can of total length $4\pi r$ (see figure 4).

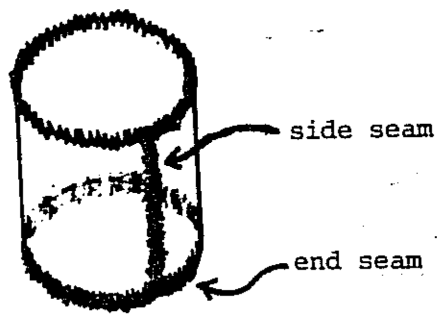


Figure 4

The cost of these two different types of seaming operations will, in general, be different, however it is reasonable to assume each is proportional to the length of the seam. The cost of seaming will therefore be given by

$$4\pi c_b r + c_s h, \tag{5}$$

where c_b and c_s are the respective costs per unit length of an end seam and a side seam.

TOTAL COST OF A CAN

The costs discussed above depend on the particular size and shape of can under production, besides these various fixed costs (labour costs, maintenance costs on the machinery, etc.) are involved. We will take these to be represented by an amount F per can. Adding this to the variable costs given in (3), (4) and (5) we obtain as a reasonable model for the *total production cost of a can*

$$C = c_m(2\pi rh + 2.2\pi r^2) + 2\pi ar + bh + F, \tag{6}$$

where

$$a = 2c_e + c_r + 2c_b$$

and

$$b = c_h + c_s.$$

Since the volume of the can V is known, our problem is to minimize C subject to the constraint (2). We may reduce C to a function of one variable r by substituting (2) into (6):

$$C = c_m\left(\frac{2V}{r} + 2.2\pi r^2\right) + 2\pi ar + \frac{bV}{\pi r^2} + F.$$

As before, the minimum of C occurs for a positive value of r at which $\frac{dC}{dr} = 0$. That is, we require r to satisfy

$$(7) \qquad c_m\left(-\frac{2V}{r^2} + 4.4\pi r\right) + 2\pi a - \frac{2bV}{\pi r^3} = 0$$

or, multiplying throughout by r^3

$$(8) \qquad 4.4\, c_m\pi r^4 + 2\pi ar^3 - 2c_m Vr - \frac{2bV}{\pi} = 0.$$

This fourth degree polynomial for r is more complicated than the corresponding relationship from our first model and cannot be solved so easily. Nonetheless, for given values of V, c_m, a and b an approximate value for r could be obtained numerically, using Newton's method, for example.

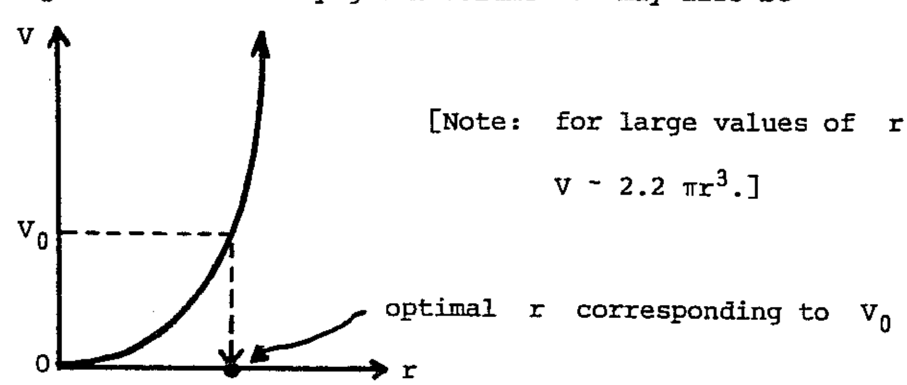
An alternative analysis is to rearrange (7) as

$$V = \frac{4.4\, c_m\pi r + 2\pi a}{\frac{2c_m}{r^2} + \frac{2b}{\pi r^3}}$$

or

$$(9) \qquad V = \frac{2\pi r^3(1.1\, c_m r + a)}{c_m r + \frac{b}{\pi}}.$$

Graphing V against r from (9) allows us to determine for any given value of r the volume V for which that value of r is optimal. By reading the graph "backwards" the optimal r for any given volume V may also be estimated.



[Note: for large values of r $V \sim 2.2\,\pi r^3$.]

Figure 5

A typical V versus r graph, where V and r are related by an expression of the form 9

From (9) and (2) we also have

$$\frac{h}{r} = 2\left(\frac{1.1\ c_m\ r + a}{c_m\ r + b/\pi}\right) .$$

As r (and hence, also V) becomes large we see that

$$\frac{h}{r} \to 2.2.$$

This is 1.1 times the constant value suggested by our first crude mode and so the limiting shape for large cans is similar to that obtained previously with a height 10% greater than the diameter. The departure from our first crude model is greater for small cans - see figure 6.

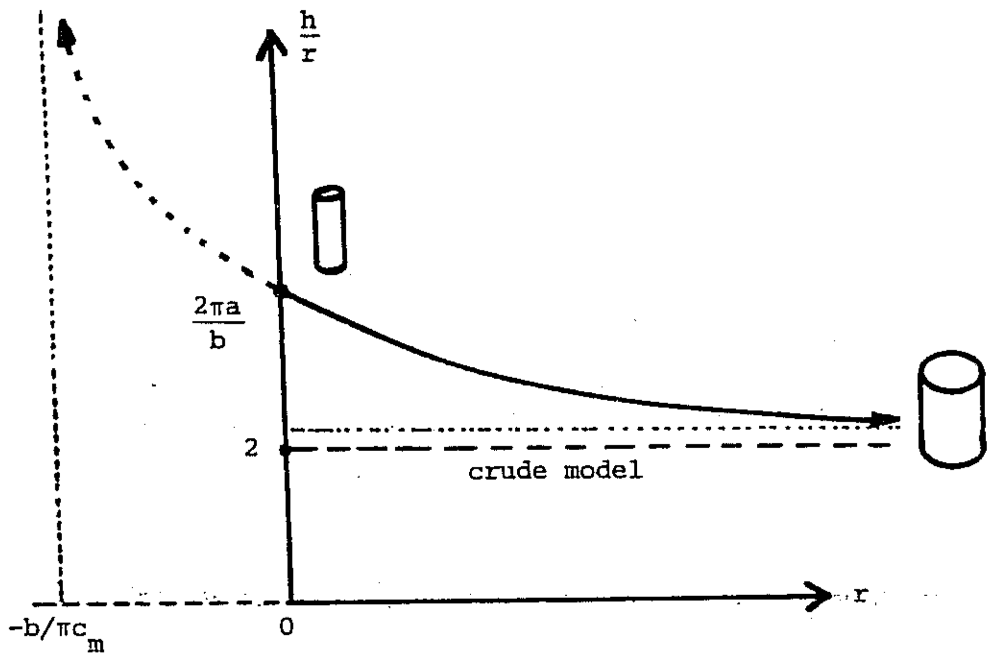


Figure 6

THE NEW BREED OF CANS

A number of soft-drinks and beers come in cans with the bottom and sides extruded from a single piece of metal. Such cans involve only a top seam (see Figure 7).

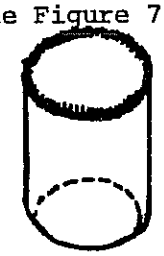


Figure 7

An analysis similar to that used above suggests the following cost of production for such a can

$$C = c_m'(2\pi rh + \pi r^2) + 1.1\ c_m\ \pi r^2 + 2\pi(c_e + c_b)r + F, \qquad (10)$$

here c'_m equals the cost per unit area for extruding the bottom and side. The other coefficients have been introduced previously.

Proceeding as before we have

$$C = \pi(1.1\ c_m + c'_m)r^2 + 2\pi dr + \frac{2c'_m V}{r} + F,$$

where

$$d = c_e + c_b.$$

$$\frac{dC}{dr} = 2\pi(1.1\ c_m + c'_m)r + 2\pi d - \frac{2c'_m V}{r^2}$$

and so the optimal radius r satisfies

$$\pi(1.1\ c_m + c'_m)r^3 + \pi dr^2 - c'_m V = 0$$

or

$$V = \frac{\pi r^2\left((1.1\ c_m + c'_m)r + d\right)}{c'_m}$$

and

$$\frac{h}{r} = \frac{(1.1\ c_m + c'_m)}{c'_m} + \frac{d}{c'_m r}\ .$$

The author researching for the current article

(Sketch by N. Talbot)

As $r \to \infty$, $\frac{h}{r} \to 1.1 \frac{c_m}{c'_m} + 1$ and so for large volumes such cans are again of approximately the same height as diameter. For small volumes the behaviour is markedly different from that of our other models. In this case, as $r \to 0$, $\frac{h}{r} \to \infty$ and so as the volume is decreased the optimal shaped can becomes progressively taller and thinner - the situation is illustrated in Figure 8.

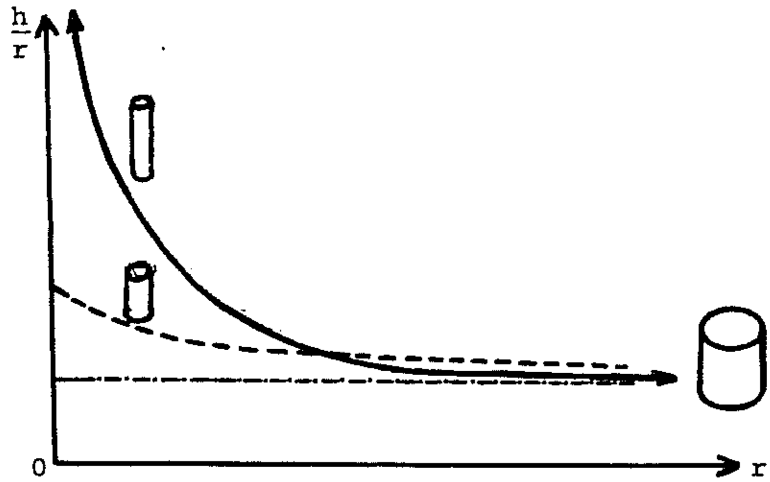


Figure 8

The dotted curves represent previous models, the heavy curve corresponds to cans with only a top seam

From these analyses we see that large cans should be of approximately the same height as diameter while smaller cans should be taller than they are wide, with beer cans (without a bottom seam) even taller and thinner than the corresponding ordinary can. You might like to inspect a number of different sized cans to see if these trends are present.

It may be salutory to note that, if an analysis of this kind led to a reduction in production cost of 1 cent per can, then on the cans discarded at the Sydney Cricket Grounds during a season of play a saving of some $50,000 would have been made.

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What relation do the numbers 17, 19, 37 and 46 have?

(See p. 19)

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