ACTA ARITHMETICA
Online First version

Heights and isogenies of Drinfeld modules

by

FLORIAN BREUER (Newcastle),
FABIEN PAZUKI (København and Bordeaux) and
MAHEFASON HERINIAINA RAZAFINJATOVO (Antananarivo)

1. Introduction. Let E and E' be two elliptic curves over a number field, linked by an isogeny $f : E \to E'$. Can we compare their heights? In the case of the Faltings height, a classical result [Fal83, Ray85] states that

$$(1.1) \qquad |h_{\mathrm{Falt}}(E) - h_{\mathrm{Falt}}(E')| \le \tfrac{1}{2} \ln \deg f.$$

A more elementary height is $h(j(E))$, the Weil height of the j-invariant of E. For this height, the second author [Paz19] proved

$$(1.2) \qquad |h(j(E)) - h(j(E'))| \le 9.204 + 12 \ln \deg f.$$

The proof of (1.2) involves modifying the Faltings height at the infinite places so that the result can be deduced from (1.1).

In the present paper, we consider function field analogues of these results. Consider two Drinfeld $\mathbb{F}_q[t]$-modules of rank $r \ge 2$ linked by an isogeny $f : \phi \to \phi'$. There are several notions of height of a Drinfeld module; the best analogue of the Faltings height was defined by Taguchi [Tag93], who also proved a variant of (1.1) for Drinfeld modules (see Lemma 4.4 below).

For the more elementary height h_G associated to the coefficients of a Drinfeld module, we prove an analogue of (1.2) of the form

$$(1.3) \qquad |h_G(\phi') - h_G(\phi)| \ \le\ \log_q \deg f + \left(\frac{q}{q-1} - \frac{q^r}{q^r-1}\right).$$

Our basic approach is somewhat similar to that in [Paz19], with some natural changes: we use analytic estimates based on the technology developed by Gekeler [Gek97, Gek17, Gek19], notably the fundamental domain for the

2020 *Mathematics Subject Classification*: Primary 11G09; Secondary 11G50, 14G17, 14G40.
Key words and phrases: Drinfeld modules, heights, isogenies, modular polynomials.
Received 29 October 2019; revised 15 January 2020.
Published online *.

moduli space of Drinfeld modules, Bruhat–Tits buildings, and we conclude by invoking Taguchi's Isogeny Lemma. Combined with a deeper result of David–Denis [DD99], this allows us to give a new proof of the finiteness of isomorphism classes of Drinfeld modules over a global function field within each isogeny class (see Corollary 3.4). A variant of (1.3) in the special case $r = 2$ allows us to deduce explicit estimates on the height of Drinfeld modular polynomials in the rank 2 case (see Proposition 6.5 and equation (6.9) below).

The layout of this paper is as follows. In Section 2 we define heights associated to Drinfeld modules. Our main results are stated in Section 3.

In Section 4 we introduce the notion of a reduced Drinfeld module and define Taguchi's height. Our main results are then proved in Section 5, where we compute various analytic estimates. Finally, we deduce the upper bound on the coefficients of Drinfeld modular polynomials (in the rank 2 case) in Section 6.

2. Heights of Drinfeld modules

2.1. Places. Let $A = \mathbb{F}_q[t]$ and $F = \mathbb{F}_q(t)$. To each place v of F we associate an absolute value $|\cdot|_v$ normalized as follows. A place of F corresponding to a monic irreducible polynomial $P \in A$ is called a *finite place*, and we have $|x|_v = q^{-(\deg P) v_P(x)}$ for $x \in F$. There is one more place, denoted $\infty \in M_F$, with $|x|_\infty = q^{\deg x}$.

For a finite extension K/F we denote by M_K the set of places of K. A place $v \in M_K$ is called *infinite* if it is an extension of ∞, otherwise it is called *finite*. The sets of finite and infinite places of K are denoted by M_K^f and M_K^∞, respectively.

To each place $v \in M_K$ we associate its absolute value normalized so that for every $x \in F$ we have $|x|_v = |x|_w$, where $w \in M_F$ lies beneath v.

To each place $v \in M_K$ we also associate the ramification index e_v (so $|K|_v \subset q^{(1/e_v)\mathbb{Z}}$), the residual degree f_v and the local degree $n_v = [K_v : F_v] = e_v f_v$.

We have the following two important properties:

- Product formula: For every $x \in K$, $\prod_{v \in M_K} |x|_v^{n_v} = 1$.
- Extension formula: For every $w \in M_F$, $[K : F] = \sum_{v|w} n_v$.

Note that in articles like [DD99] and [Tag93], the absolute values are normalized differently; the exponent e_v is included in $|x|_v$, so in their situation the product formula holds with n_v replaced by f_v.

Finally, for the remainder of this article, log always means the logarithm to base q.

We will associate to a Drinfeld module a number of different heights. Every height h will be decomposed into a sum of local heights,

$$h = \frac{1}{[K:F]} \sum_{v \in M_K} h^v.$$

We also write

$$h^f = \frac{1}{[K:F]} \sum_{v \in M_K^f} h^v \quad \text{and} \quad h^\infty = \frac{1}{[K:F]} \sum_{v \in M_K^\infty} h^v$$

for the finite and infinite components, respectively.

2.2. Naïve heights. Let ϕ be a Drinfeld $\mathbb{F}_q[t]$-module of rank r over K. We assume throughout this paper that $r \geq 2$ and that our Drinfeld modules are of generic characteristic. Then ϕ is characterized by

$$\phi_t(X) = tX + g_1 X^q + g_2 X^{q^2} + \cdots + g_r X^{q^r}, \qquad g_i \in K,\ g_r \neq 0.$$

We refer to the $g_1, \ldots, g_r$ as the *coefficients* of ϕ.

Let $d = \operatorname{lcm}\{q-1, q^2-1, \ldots, q^r-1\}$. For $k = 1, \ldots, r$, set

$$\text{(2.1)} \qquad j_k := \frac{g_k^{d/(q^k-1)}}{g_r^{d/(q^r-1)}} \in K, \quad \text{and} \quad J = J(\phi) = (j_1, \ldots, j_r).$$

Clearly $j_r = 1$. These are isomorphism invariants of ϕ.

Remark 2.1. These invariants differ from those defined by Potemine in [Pot98, (2.5)] in their exponents: we have chosen exponents such that each j_k has the same denominator, whereas Potemine used the least integer exponents for each j_k. Nevertheless, it follows from [Pot98, Theorem 2.2] that for each tuple $(a_1, \ldots, a_{r-1}) \in \bar{F}^{r-1}$, there are at most finitely many $\bar{F}$-isomorphism classes of Drinfeld modules ϕ with $J(\phi) = (a_1, \ldots, a_{r-1}, 1)$.

Now we define the *J-height* of ϕ:

$$\text{(2.2)} \qquad h_J(\phi) := h(J) = \frac{1}{[K:F]} \sum_{v \in M_K} n_v \log \max\{|j_1|_v, \ldots, |j_r|_v\},$$

which is just the logarithmic Weil height of the tuple J. This height, and its local components $h_J^v(\phi)$ for $v \in M_K$, are invariant under isomorphisms of ϕ.

In the special case $r = 2$, we see that $j_1 = j = g_1^{q+1}/g_2$ is the usual j-invariant and $h(J) = h(j)$ is the usual height of $j \in K$.

Next, consider the weighted projective space

$$W\mathbb{P} := \mathbb{P}(q-1, q^2-1, \ldots, q^r-1),$$

which is Proj of the graded polynomial ring $K[g_1, \ldots, g_r]$, where the indeterminates are assigned the weights $\deg g_k = q^k - 1$.

It is well-known that $W\mathbb{P}-V(g_r=0)$ is the coarse moduli space of rank r Drinfeld modules. Indeed, if ϕ' is another Drinfeld module with $\phi'_t(X) = tX+g'_1X^q+g'_2X^{q^2}+\cdots+g'_rX^{q^r}$, then $(g_1,\ldots,g_r)$ and $(g'_1,\ldots,g'_r)$ represent the same point in $W\mathbb{P}$ if and only if ϕ and ϕ' are isomorphic over some algebraically closed field.

One can define heights on weighted projective spaces in the obvious way, and the height associated to the point representing ϕ is called the *graded height* of ϕ:

(2.3)
$$h_G(\phi) := \frac{1}{[K:F]}\sum_{v\in M_K} n_v \log\max\{|g_1|_v^{1/(q-1)}, |g_2|_v^{1/(q^2-1)},\ldots,|g_r|_v^{1/(q^r-1)}\}.$$

For a finite place $v\in M_K^f$, the local component $h_G^v(\phi) = n_v\log\max_i|g_i|_v^{1/(q^i-1)}$ equals Taguchi's $v(\phi)$ (see [Tag93, §2]).

From the product formula, we see that

$$dh_G(\phi) = h_J(\phi), \tag{2.4}$$

and so again $h_G(\phi)$ is invariant under isomorphism. However, the local components $h_G^v(\phi)$ depend on the choice of ϕ in its isomorphism class.

PROPOSITION 2.2. *Let $F=\mathbb{F}_q(t)$ and let K/F be a finite extension. Let $C>0$. Then there are only finitely many $\bar{F}$-isomorphism classes of rank r Drinfeld modules ϕ defined over K such that $h_J(\phi)<C$ (respectively $h_G(\phi)<C$).*

Proof. The usual Northcott Theorem for the Weil height implies that there are only finitely many $(j_1,\ldots,j_{r-1})\in K^{r-1}$ for which $h(j_1,\ldots,j_{r-1},1) < C$. The result now follows from Remark 2.1 and the identity (2.4). ∎

3. Main results. Let $f:\phi\to\phi'$ be an isogeny of Drinfeld modules (still of generic characteristic) of degree $\deg f := \#\ker f$. We may associate to f a (not necessarily unique) dual isogeny $\hat{f}:\phi'\to\phi$ of degree $\deg\hat{f}\le(\deg f)^{r-1}$, such that $\hat{f}\circ f=\phi_N$, where $N\in A$ is an element of minimal degree for which $\ker f\subset\phi[N]$, and similarly $f\circ\hat{f}=\phi'_N$. In particular, $\deg N = \frac{1}{r}(\log\deg f+\log\deg\hat{f})\le\log\deg f$. See for example [DD99, Lemme 2.19].

Denote by $\bar{K}$ an algebraic closure of K. We now state our main result.

THEOREM 3.1. *Let $f:\phi\to\phi'$ be an isogeny of rank r Drinfeld modules over $\bar{K}$ and suppose* $\ker f\subset\phi[N]$.

(1) *We have*

$$|h_G(\phi')-h_G(\phi)|\le\deg N+\left(\frac{q}{q-1}-\frac{q^r}{q^r-1}\right). \tag{3.1}$$

(2) *Suppose $r = 2$. Then we have the following variant. Let $j = j_1(\phi)$ and $j' = j_1(\phi')$. Then*

$$h(j') - h(j) \le \frac{q^2-1}{2} \log \deg f + \frac{q^2-1}{2} \log\left[1 + \frac{1}{q} h(j')\right] + q. \tag{3.2}$$

This is the analogue of Theorem 1.1 of [Paz19].

Remark 3.2. If $r \ge 3$, we cannot hope to get a similar result replacing h_G with the height $h(j_k)$ of a single invariant, as the following example shows.

Fix a Drinfeld module ϕ and consider all isogenies $f : \phi \to \phi'$ of kernel $\ker f \cong A/tA$, where

$$\phi_t(X) = tX + g_1 X^q + X^{q^3}, \qquad \phi'_t(X) = tX + g'_1 X^q + g'_2 X^{q^2} + X^{q^3},$$
$$f(X) = f_0 X + X^q.$$

From $f \circ \phi_t = \phi'_t \circ f$, comparing coefficients of X^q and X^{q^3}, we obtain

$$g'_1 = f_0^{-q}(f_0 g_1 + t^q - t) \quad \text{and} \quad g'_2 = f_0 - f_0^{q^3}, \qquad \text{respectively.} \tag{3.3}$$

Since $\ker f \subset \ker \phi_t$, we write $\phi_t = P \circ f$ for some $P(X) = aX + bX^q + X^{q^2}$. Again, comparing coefficients gives

$$a = f_0^{-1} t, \qquad b = -f_0^{q^2} \quad \text{and} \quad f_0^{-1} t - f_0^{q^2+q} = g_1.$$

Thus f_0 is a root of

$$X^{q^2+q+1} + g_1 X - t = 0. \tag{3.4}$$

Conversely, every root f_0 of (3.4) produces an isogeny $f : \phi \to \phi'$ as above. For each such root, from (3.3) we obtain

$$g'_1 = -f_0^{q^2+1} + f_0^{-q} t^q, \qquad g'_2 = f_0 - f_0^{q^3}. \tag{3.5}$$

In particular, if $h(g_1)$ is very large, then at least one of the roots f_0 of (3.4) has large height, and thus so does the corresponding g'_2. Then $h(j_2(\phi'))$ is large, whereas $h(j_2(\phi)) = 0$ and $\deg f = q$.

The following result follows from Theorem 3.1 and [DD99, Thm. 1.3]:

Corollary 3.3. *There exists an effectively computable constant C, depending only on r and q, such that the following holds. Suppose ϕ and ϕ' are rank r Drinfeld $\mathbb{F}_q[t]$-modules, defined over a finite extension K/F, which are isogeneous over $\bar{K}$. Then*

$$|h_G(\phi') - h_G(\phi)| \le 10(r+1)^7 \log\bigl([K:F] h_G(\phi)\bigr) + C.$$

Proof. By [DD99, Thm. 1.3], there exists an effectively computable constant $c_2 = c_2(r,q)$ and an isogeny $f : \phi \to \phi'$ of degree

$$\deg f \le c_2\bigl([K:F] h(\phi)\bigr)^{10(r+1)^7}.$$

Here $h(\phi)$ denotes a height function defined in terms of the coefficients of ϕ by $h(\phi) = \max\{h(g_1), \dots, h(g_r)\}$. It is easy to see that $h(\phi) \le (q^r - 1)h_G(\phi)$, so

$$\begin{aligned}|h_G(\phi') - h_G(\phi)| &\le \deg N + \left(\frac{q}{q-1} - \frac{q^r}{q^r-1}\right)\\ &\le \log\deg f + \left(\frac{q}{q-1} - \frac{q^r}{q^r-1}\right)\\ &\le \log\bigl(c_2([K:F](q^r-1)h_G(\phi))^{10(r+1)^7}\bigr) + \left(\frac{q}{q-1} - \frac{q^r}{q^r-1}\right).\end{aligned}$$

The result follows. ■

Applying Proposition 2.2, we recover the following result, which was originally proved by Taguchi [Tag99].

Corollary 3.4. *Each $\bar{K}$-isogeny class of Drinfeld modules defined over K contains only finitely many $\bar{K}$-isomorphism classes of Drinfeld modules.*

Note that our approach would lead the interested reader to an explicit bound on the number of $\bar{K}$-isomorphism classes within a $\bar{K}$-isogeny class.

4. Lattices and Taguchi's height

4.1. Lattices. Let $F_\infty = \mathbb{F}_q((1/t))$ be the completion of F at the place ∞, and $\mathbb{C}_\infty = \hat{\bar{F}}_\infty$ the completion of an algebraic closure of F_∞; it is complete and algebraically closed and plays the role of the complex numbers in characteristic $p > 0$. Recall that $A = \mathbb{F}_q[t]$.

A *lattice* of rank $r \ge 1$ is an A-submodule $\Lambda \subset \mathbb{C}_\infty$ of the form $\Lambda = \omega_1 A + \dots + \omega_r A$, where the $\omega_1, \dots, \omega_r \in \mathbb{C}_\infty$ are F_∞-linearly independent.

A *successive minimum basis* for a lattice Λ is an A-basis $(\omega_1, \dots, \omega_r)$ for Λ satisfying the properties

$$|\omega_1| \ge \dots \ge |\omega_r|$$

and, for $k = 1, \dots, r$,

$$|\omega_k| = \inf\Bigl(\Bigl\{\lambda - \sum_{i=k+1}^{r} a_i\omega_i \Bigm| a_{k+1}, \dots, a_r \in A,\ \lambda \in \Lambda\Bigr\} \smallsetminus \{0\}\Bigr).$$

In other words, ω_r is a minimal non-zero element of Λ and each ω_k is minimal among the non-zero elements of Λ not spanned by the $\omega_{k+1}, \dots, \omega_r$. We can think of such a basis as being an "orthogonal" basis. Every lattice has a successive minimum basis, and we define the *covolume* of Λ by

$$D(\Lambda) := |\omega_1| \cdots |\omega_r|, \tag{4.1}$$

where $(\omega_1, \ldots, \omega_r)$ is any successive minimum basis of Λ. By [Tag93, (4.1)] or [Gek19, Prop. 3.1], this is independent of the choice of successive minimum basis.

The covolume of a lattice satisfies the following desirable properties.

LEMMA 4.1. *Let $\Lambda \subset \mathbb{C}_\infty$ be a lattice of rank r.*

(1) *Choose an $\mathbb{F}_q[t]$-basis $(\omega_1, \ldots, \omega_r)$ of Λ. Let $\gamma \in \mathrm{GL}_r(F_\infty)$ and denote by $\gamma\Lambda$ the lattice spanned by $(\omega_1, \ldots, \omega_r)\gamma^{\mathrm{T}}$. Then*

$$D(\gamma\Lambda) = |\det \gamma| D(\Lambda).$$

(2) *Let $c \in \mathbb{C}_\infty$. Then*

$$D(c\Lambda) = |c|^r D(\Lambda).$$

(3) *Let Λ' be a lattice of rank r such that $\Lambda \subset \Lambda' \subset \mathbb{C}_\infty$. Then*

$$(\Lambda' : \Lambda) = D(\Lambda)/D(\Lambda').$$

Proof. Part (1) is [Tag93, Prop. 4.4] applied to the A-lattice Λ inside the F_∞-vector space $V \subset \mathbb{C}_\infty$ spanned by $(\omega_1, \ldots, \omega_r)$.

Part (2) follows from the definition. Part (3) follows from (1) as $\Lambda = \gamma\Lambda'$ for a suitable $\gamma \in \mathrm{GL}_r(F)$ with coefficients in A and $|\det \gamma| = (\Lambda' : \Lambda)$. ∎

A lattice $\Lambda \subset \mathbb{C}_\infty$ is said to be *reduced* if it has a successive minimum basis $(\omega_1, \ldots, \omega_r)$ with $\omega_r = 1$. Equivalently,

> Λ is reduced if and only if $1 \in \Lambda$ and every non-zero $\lambda \in \Lambda$ satisfies $|\lambda| \geq 1$.

Every Drinfeld module ϕ over $\mathbb{C}_\infty$ is associated to a rank r lattice $\Lambda \subset \mathbb{C}_\infty$ and vice versa. We call a Drinfeld module ϕ *reduced* if its associated lattice is reduced. Every Drinfeld module is isomorphic over $\mathbb{C}_\infty$ to a reduced Drinfeld module. (The analogous condition on an elliptic curve is to correspond to a point in the fundamental domain of the upper half-plane.)

LEMMA 4.2 (Analytic Isogeny Lemma). *Let $f : \phi \to \phi'$ be an isogeny of reduced Drinfeld modules over $\mathbb{C}_\infty$ with associated reduced lattices $\Lambda, \Lambda' \subset \mathbb{C}_\infty$, respectively. Then*

$$-\log \deg \hat{f} \leq \log D(\Lambda) - \log D(\Lambda') \leq \log \deg f. \tag{4.2}$$

Proof. Analytically, the isogeny $f : \phi \to \phi'$ is given by multiplication by $\alpha \in \mathbb{C}_\infty$ for which $\alpha\Lambda \subset \Lambda'$ and $\ker f \cong \Lambda'/\alpha\Lambda$. Thus

$$\deg f = (\Lambda' : \alpha\Lambda) = D(\alpha\Lambda)/D(\Lambda') = |\alpha|^r D(\Lambda)/D(\Lambda'),$$

so $D(\Lambda)/D(\Lambda') = |\alpha|^{-r} \deg f$. Since Λ is reduced, $1 \in \Lambda$ and thus $\alpha \cdot 1 \in \Lambda'$. Since Λ' is reduced, we must have $|\alpha| \geq 1$, giving

$$D(\Lambda)/D(\Lambda') \leq \deg f.$$

This proves the upper bound, and the lower bound follows by applying this to $\hat{f} : \phi' \to \phi$. ■

4.2. Taguchi's height. Let ϕ be a Drinfeld module defined over a finite extension K/F. We recall that ϕ is said to have *stable reduction* at a place $v \in M_K^f$ if it is isomorphic over K to a Drinfeld module $\tilde{\phi}$ defined over the valuation ring $\mathcal{O}_v \subset K$ of v whose reduction modulo the maximal ideal $\mathfrak{m}_v$ of $\mathcal{O}_v$ is a Drinfeld module of positive rank over the residue field $\mathcal{O}_v/\mathfrak{m}_v$. Equivalently, $h_G^v(\phi) \in \mathbb{Z}$ (see [Tag93, p. 301]). We say that ϕ has *everywhere stable reduction* if it has stable reduction at every finite place $v \in M_K^f$, equivalently if $h_G^v(\phi) \in \mathbb{Z}$ for every $v \in M_K^f$. By [DD99, Lemme 2.10], every Drinfeld module over K acquires everywhere stable reduction after replacing K by a finite extension thereof.

In [Tag93] Taguchi defines the *differential height* of ϕ as the degree of the metrized conormal line bundle along the unit section associated to a minimal model of ϕ. It serves as the analogue of the Faltings height. All we need here is the identity (5.9.1) of [Tag93], valid for Drinfeld modules with everywhere stable reduction, which we adopt as our definition:

$$
\begin{aligned}
h_{\mathrm{Tag}}(\phi) &:= \frac{1}{[K:F]}\Big[\sum_{v\in M_K^f} h_G^v(\phi) - \sum_{v\in M_K^\infty} n_v \log D(\Lambda_v)^{1/r}\Big] \\
&= h_{\mathrm{Tag}}^f(\phi) + h_{\mathrm{Tag}}^\infty(\phi),
\end{aligned}
\tag{4.3}
$$

where we set

$$
h_{\mathrm{Tag}}^f(\phi) = \frac{1}{[K:F]}\sum_{v\in M_K^f} h_G^v(\phi),
$$

$$
h_{\mathrm{Tag}}^\infty(\phi) = -\frac{1}{[K:F]}\sum_{v\in M_K^\infty} n_v \log D(\Lambda_v)^{1/r}.
$$

Notice that the sign is included in the definition of $h_{\mathrm{Tag}}^\infty(\phi)$—the reader should keep this in mind when reading the remaining calculations in this paper.

Here $\Lambda_v \subset \mathbb{C}_\infty$ is the lattice associated to the Drinfeld module ϕ^σ over $\mathbb{C}_\infty$ obtained by embedding the coefficients g_k into $\mathbb{C}_\infty$ via the embedding $\sigma : K \hookrightarrow \mathbb{C}_\infty$ associated to the infinite place v.

We see that the finite part coincides with the finite part of our graded height:

$$
h_{\mathrm{Tag}}^f(\phi) = h_G^f(\phi).
$$

Our definition of $h_{\mathrm{Tag}}(\phi)$ only coincides with Taguchi's differential height when ϕ has stable reduction at every finite place. For the general case, the reader will find an excellent treatment of Taguchi's height in [Wei20, §5.1].

We add a proof that $h_{\mathrm{Tag}}(\phi)$ as defined above satisfies the following desirable properties.

LEMMA 4.3. *Let ϕ be a Drinfeld module with everywhere stable reduction, defined over a global function field K.*

(1) *$h_{\mathrm{Tag}}(\phi)$, $h^f_{\mathrm{Tag}}(\phi)$ and $h^\infty_{\mathrm{Tag}}(\phi)$ do not depend on the choice of the field K.*
(2) *$h_{\mathrm{Tag}}(\phi)$ is invariant under $\bar{K}$-isomorphism.*

Proof. Suppose that L/K is a finite extension. Suppose $v_1, v_2 \in M_L$ lie above the same place of K. Then $|g_i|_{v_1} = |g_i|_{v_2}$ for each i and also $D(\Lambda_{v_1}) = D(\Lambda_{v_2})$. The first item follows as usual from $[L:K] = \sum_{v|w}[L_v : K_w]$.

To prove the second item, let $c \in \bar{K}^*$ and replace K by $K(c)$, which we may by the first item. Now

$$h^f_{\mathrm{Tag}}(c^{-1}\phi c) = h^f_{\mathrm{Tag}}(\phi) + \frac{1}{[K:F]} \sum_{v \in M_K^f} n_v \log |c|_v$$

and

$$\begin{aligned} h^\infty_{\mathrm{Tag}}(c^{-1}\phi c) &= -\frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v \log D(c^{-1}\Lambda_v)^{1/r} \\ &= -\frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v \log[|c|^{-r} D(\Lambda_v)]^{1/r} \\ &= h^\infty_{\mathrm{Tag}}(\phi) + \frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v \log |c|_v. \end{aligned}$$

The result now follows from the product formula $\sum_{v \in M_K} n_v \log |c|_v = 0$. ■

The advantage of h_{Tag} is that it behaves well under isogenies.

LEMMA 4.4 (Taguchi's Isogeny Lemma). *Let $f : \phi \to \phi'$ be a $\bar{K}$-isogeny between two rank r Drinfeld modules over K with everywhere stable reduction. Then*

$$-\frac{1}{r} \log \deg \hat{f} \le h_{\mathrm{Tag}}(\phi') - h_{\mathrm{Tag}}(\phi) \le \frac{1}{r} \log \deg f.$$

Proof. We start with [Tag93, Lemma 5.5]; it states that

$$h_{\mathrm{Tag}}(\phi') - h_{\mathrm{Tag}}(\phi) = \frac{1}{r} \log \deg f - \frac{1}{[K:F]} \log \#(R/D_f).$$

Here, R is the integral closure of A in F, and the ideal $D_f \subset R$ is the *different* of f. We do not need the exact definition of this, merely the fact that $\#(R/D_f)$ is a positive integer, so $\log \#(R/D_f) \ge 0$. This gives us the upper bound, and the lower bound is obtained by applying the upper bound to the dual isogeny $\hat{f} : \phi' \to \phi$. ■

5. Analytic estimates. The proof of Theorem 3.1 involves breaking up the difference in heights, using the identity $h^f_{\mathrm{Tag}}(\phi) = h^f_G(\phi)$, as follows:

$$\begin{aligned}(5.1)\qquad & h_G(\phi') - h_G(\phi) \\ &= \underbrace{[h_{\mathrm{Tag}}(\phi') - h_{\mathrm{Tag}}(\phi)]}_{\text{(A)}} + \underbrace{[h^\infty_G(\phi') - h^\infty_G(\phi)]}_{\text{(B)}} + \underbrace{[h^\infty_{\mathrm{Tag}}(\phi) - h^\infty_{\mathrm{Tag}}(\phi')]}_{\text{(C)}}.\end{aligned}$$

Part (A) is bounded using Taguchi's Isogeny Lemma 4.4.

Bounding the terms (B) and (C) will require some analytic estimates, which we outline next.

5.1. Proof of Theorem 3.1(1). We start by [DD99, Lemme 2.10]: we may replace K by a finite extension so that ϕ and ϕ' have everywhere stable reduction. From now on, our Drinfeld modules are all assumed to have everywhere stable reduction.

Lemma 5.1. *Let ϕ be a Drinfeld module of rank r over $\mathbb{C}_\infty$ with associated lattice Λ. Then the quantity*

$$\log\max\{|g_1|^{1/(q-1)}, |g_2|^{1/(q^2-1)}, \dots, |g_r|^{1/(q^r-1)}\} + \log D(\Lambda)^{1/r} \in \mathbb{R}$$

is invariant under isomorphisms of ϕ.

Proof. Let $\phi' = c^{-1}\phi c$ with $c \in \mathbb{C}_\infty^*$ be another Drinfeld module isomorphic to ϕ. Then

$$\begin{aligned}&\log\max\{|g_1'|^{1/(q-1)}, |g_2'|^{1/(q^2-1)}, \dots, |g_r'|^{1/(q^r-1)}\} + \log D(\Lambda')^{1/r} \\ &\qquad = \log\max\{|c^{q-1}g_1|^{1/(q-1)}, |c^{q^2-1}g_2|^{1/(q^2-1)}, \dots, |c^{q^r-1}g_r|^{1/(q^r-1)}\} \\ &\qquad\quad + \log D(c^{-1}\Lambda)^{1/r} \\ &\qquad = \log\max\{|g_1|^{1/(q-1)}, |g_2|^{1/(q^2-1)}, \dots, |g_r|^{1/(q^r-1)}\} + \log D(\Lambda)^{1/r}. \ ■\end{aligned}$$

Let ϕ/K be a rank r Drinfeld module with coefficients $g_1, \dots, g_r \in K$. To each infinite place $v \in M_K^\infty$ we associate an embedding $\sigma : K \hookrightarrow \mathbb{C}_\infty$ for which $|x|_v = |x^\sigma|$ for any $x \in K$. Then $\Lambda_v \subset \mathbb{C}_\infty$ is the lattice associated to the Drinfeld module $\phi^\sigma/\mathbb{C}_\infty$ defined by the coefficients $g_1^\sigma, \dots, g_r^\sigma \in \mathbb{C}_\infty$.

We rewrite (B) + (C) of (5.1) as follows:

$$\begin{aligned}(5.2)\qquad & \text{(B)} + \text{(C)} \\ &= \frac{1}{[K:F]} \sum_{\sigma: K\hookrightarrow\mathbb{C}_\infty} n_\sigma\Bigg(\Big[\log\max_{1\le i\le r}|g_i'^\sigma|^{1/(q^i-1)} - \log\max_{1\le i\le r}|g_i^\sigma|^{1/(q^i-1)}\Big] \\ &\qquad + \Big[\frac{1}{r}\log D(\Lambda_\sigma') - \frac{1}{r}\log D(\Lambda_\sigma)\Big]\Bigg).\end{aligned}$$

By Lemma 5.1, the term for each $\sigma : K \hookrightarrow \mathbb{C}_\infty$ in (5.2) depends only on the isomorphism classes of ϕ^σ and ϕ'^σ. Therefore, in the remainder of this section, we will frequently make the following reduction:

REDUCTION 5.2. *Whenever the Drinfeld module ϕ^σ arises in the context of* (5.2), *we replace it by an isomorphic reduced Drinfeld module, which we may by Lemma* 5.1, *and which by abuse of notation we again denote by ϕ^σ.*

Under Reduction 5.2, (C) is bounded by Lemma 4.2:

$$h^\infty_{\mathrm{Tag}}(\phi) - h^\infty_{\mathrm{Tag}}(\phi') \le \frac{1}{[K:F]} \sum_{v\in M_K^\infty} n_v \frac{1}{r} \log\deg \hat{f} = \frac{1}{r}\log\deg\hat{f}. \tag{5.3}$$

Next, we obtain an absolute bound on part (B).

LEMMA 5.3. *Let $\phi/\mathbb{C}_\infty$ be a reduced Drinfeld module of rank r. Then*

$$\frac{q^r}{q^r-1} \le \log \max_{1\le i\le r} |g_i|^{1/(q^i-1)} \le \frac{q}{q-1}. \tag{5.4}$$

Proof. For this we must recall some concepts introduced in [Gek17]. Define

$$\begin{aligned}\mathcal{F} := \{(\omega_1,\dots,\omega_r)\in\mathbb{C}_\infty^r \mid \omega_r = 1 \text{ and } (\omega_1,\dots,\omega_r) \text{ forms a}\\ \text{successive minimum basis for the lattice } \omega_1 A+\dots+\omega_r A\}.\end{aligned}$$

This set is a fundamental domain (in a suitable sense) for the action of $\mathrm{GL}_r(A)$ on the Drinfeld period domain

$$\begin{aligned}\Omega^r = \{(\omega_1,\dots,\omega_r)\in\mathbb{C}_\infty^r \mid \omega_r = 1 \text{ and } \omega_1,\dots,\omega_r \text{ are linearly}\\ \text{independent over } F_\infty\}.\end{aligned}$$

Every reduced Drinfeld module $\phi/\mathbb{C}_\infty$ corresponds to a reduced lattice of the form $\Lambda = \omega_1 A+\dots+\omega_r A$ for some $(\omega_1,\dots,\omega_r)\in\mathcal{F}$.

Denote by $\mathcal{BT}$ the Bruhat–Tits building of $\mathrm{PGL}_r(F_\infty)$ and by $\mathcal{BT}(\mathbb{Q})$ the points in the realization of $\mathcal{BT}$ with rational barycentric coordinates. The image of $\mathcal{F}$ under the building map $\lambda : \Omega^r \to \mathcal{BT}(\mathbb{Q})$ (see [Gek17, §2.3]) is an $(r-1)$-dimensional simplicial complex W whose vertices correspond to integer r-tuples $(k_1,\dots,k_r)\in\mathbb{Z}^r_{\ge 0}$ with $k_1\ge\dots\ge k_r = 0$. The preimage of such a vertex consists of lattice bases $(\omega_1,\dots,\omega_r)$ satisfying $|\omega_i| = q^{k_i}$ for each i.

The origin of W is denoted $\mathbf{o} = (0,\dots,0)$. By [Gek17, §4.6], for $\omega\in\lambda^{-1}(\mathbf{o})$ we have $\log|g_r(\omega)| = q^r$, and for each $i = 1,\dots,r-1$, $\log|g_i(\omega)|\le q^i$, with equality achieved somewhere on the set $\lambda^{-1}(\mathbf{o})$ by [Gek17, Cor. 4.16].

By [Gek17, Cor. 4.11 and 4.16] it follows that each $\log|g_i(\omega)|$ is non-increasing as $\lambda(\omega)$ moves away from $\mathbf{o}$ in $W(\mathbb{Q})$, and so, for every $i = 1,\dots,r$,

$$\log|g_i(\omega)| \le q^i \quad \text{for all } \omega\in\mathcal{F}. \tag{5.5}$$

This implies the upper bound in (5.4).

For each $i = 1,\dots,r-1$ we define the *ith wall* of W to be the subcomplex $W_i\subset W$ spanned by vertices satisfying $k_i = k_{i+1}$. Its preimage under the building map is denoted

$$\mathcal{F}_i = \lambda^{-1}(W_i(\mathbb{Q})) = \{(\omega_1, \ldots, \omega_r) \in \mathcal{F} \mid |\omega_i| = |\omega_{i+1}|\}.$$

To prove the lower bound, first note that by [Gek17, Cor. 4.16],

$$\log|g_1(\omega)| = q \quad \text{for all } \omega \in \mathcal{F} \smallsetminus \mathcal{F}_{r-1}; \tag{5.6}$$

we claim that, for $i = 2, 3, \ldots, r-1$,

$$\log|g_i(\omega)| = q^i \quad \text{for all } \omega \in (\mathcal{F}_{r-1} \cap \mathcal{F}_{r-2} \cap \cdots \cap \mathcal{F}_{r-i+1}) \smallsetminus \mathcal{F}_{r-i}. \tag{5.7}$$

Indeed, by [Gek17, Cor. 4.16], since $\omega \notin \mathcal{F}_{r-i}$, $\log|g_i(\omega)|$ is constant on the fibres of λ, we may consider $\log|g_i|$ as a function on $W(\mathbb{Q})$. Since $1 = |\omega_r| = |\omega_{r-1}| = \cdots = |\omega_{r-i+1}|$, the point $\lambda(\omega) \in W(\mathbb{Q})$ lies in a simplex all of whose edges can be reached from the vertex $\mathbf{o}$ by paths consisting entirely of edges of the form $\mathbf{k} \to \mathbf{k} + \mathbf{k}_\ell$ for $\ell \le r - i$ (here $\mathbf{k}_\ell = (1, \ldots, 1, 0, \ldots, 0)$ contains ℓ ones). By [Gek17, Prop. 4.10 and Cor. 4.16], $\log|g_i|$ is constant on these edges, and it interpolates linearly within each simplex, hence $\log|g_i(\omega)| = \log\|g_i(\lambda^{-1}(\mathbf{o}))\| = q^i$, by [Gek17, §4.6]. This proves the claim.

Every $\omega \in \mathcal{F}$ lies in one of the subsets

$$\mathcal{F} \smallsetminus \mathcal{F}_{r-1},\ \mathcal{F}_{r-1} \smallsetminus \mathcal{F}_{r-1},\ (\mathcal{F}_{r-1} \cap \mathcal{F}_{r-2}) \smallsetminus \mathcal{F}_{r-3},\ \ldots,\ (\mathcal{F}_{r-1} \cap \cdots \cap \mathcal{F}_1) = \{\mathbf{o}\}.$$

Hence, by (5.6), (5.7) and $\log|g_r(\mathbf{o})| = q^r$, we have $\log|g_i(\omega)| = q^i$ for some $i = 1, \ldots, r$, and the lower bound in (5.4) follows. ∎

In particular, we find that in (5.1), after Reduction 5.2,

$$\begin{aligned} |h_G^\infty(\phi') - h_G^\infty(\phi)| &\le \frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v \left(\frac{q}{q-1} - \frac{q^r}{q^r - 1} \right) \\ &= \frac{q}{q-1} - \frac{q^r}{q^r-1}. \end{aligned} \tag{5.8}$$

Now Lemma 4.4 together with (5.3) and (5.8) and the fact that $\deg N = \frac{1}{r}(\log\deg f + \log\deg \hat{f})$ imply Theorem 3.1(1). ∎

5.2. Proof of Theorem 3.1(2)

Lemma 5.4. *Let $\phi/\mathbb{C}_\infty$ be a reduced rank 2 Drinfeld module with associated reduced lattice $\Lambda \subset \mathbb{C}_\infty$. Then*

$$1 \le D(\Lambda) \le \max\left\{ \frac{1}{q} \log|j(\phi)|, 1 \right\}.$$

Proof. We use an estimate of $|j(\phi)|$ obtained by Gekeler [Gek97]. The reduced rank 2 lattice Λ has a successive minimum basis $(\omega_1, 1)$, where $\omega_1 \in \mathbb{C}_\infty$ satisfies

$$1 \le |\omega_1| = |\omega_1|_i := \inf_{x \in F_\infty} |\omega_1 - x| = D(\Lambda).$$

Suppose first that $k = \log D(\Lambda) \in \mathbb{Z}_{\geq 0}$. Then [Gek97, Theorem 2.17] gives

$$\begin{aligned} \log|j(\phi)| &= q^{k+1} = qD(\Lambda) && \text{if } k \geq 1, \\ \log|j(\phi)| &\leq q = qD(\Lambda) && \text{if } k = 0. \end{aligned}$$

Furthermore, $\log|j(\phi)|$ interpolates linearly between integral values of k [Gek97, Rem. 2.14], in other words, if $k = \lfloor \log D(\Lambda) \rfloor$ and $s = \log D(\Lambda) - k > 0$, then

$$\log|j(\phi)| = sq^{k+1} + (1-s)q^{k+2}.$$

Since $x \mapsto q^{x+1}$ is convex, it follows that for $D(\Lambda) > 1$,

$$\log|j(\phi)| \geq qD(\Lambda).$$

The lemma follows. ∎

We now use this to get another estimate of (C) in the case $r = 2$ and $j = j_1(\phi)$ and $j' = j_1(\phi')$. Since we are assuming that each Λ_v is reduced, $D(\Lambda_v) \geq 1$. We obtain

$$\begin{aligned} &\frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v [\log D(\Lambda_v')^{1/2} - \log D(\Lambda_v)^{1/2}] \\ &\leq \frac{1}{[K:F]} \sum_{v \in M_K^\infty} n_v \log D(\Lambda_v')^{1/2} \leq \frac{1}{2[K:F]} \sum_{v \in M_K^\infty} n_v \log \max\left\{\frac{1}{q}\log|j'|_v, 1\right\} \\ &= \frac{1}{2[K:F]} \sum_{\sigma: K \hookrightarrow \mathbb{C}_\infty} \log\max\left\{\frac{1}{q}\log|j'^\sigma|^{n_v}, 1\right\} \\ &= \frac{1}{2}\log\left[\prod_{\sigma: K \hookrightarrow \mathbb{C}_\infty} \max\left\{\frac{1}{q}\log|j'^\sigma|^{n_v}, 1\right\}^{1/[K:F]}\right] \\ &\leq \frac{1}{2}\log\left[\frac{1}{[K:F]} \sum_{\sigma: K \hookrightarrow \mathbb{C}_\infty} \max\left\{\frac{1}{q}\log|j'^\sigma|^{n_v}, 1\right\}\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the AM-GM inequality)} \\ &\leq \frac{1}{2}\log\left[1 + \frac{1}{q}\frac{1}{[K:F]} \sum_{\sigma: K \hookrightarrow \mathbb{C}_\infty} n_v \max\{\log|j'^\sigma|, 0\}\right] \leq \frac{1}{2}\log\left[1 + \frac{1}{q}h(j')\right]. \end{aligned}$$

Plugging this, Lemma 4.4 and (5.8) into (5.1), we obtain

$$h_G(\phi') - h_G(\phi) \leq \frac{1}{2}\log\deg f + \left(\frac{q}{q-1} - \frac{q^2}{q^2-1}\right) + \frac{1}{2}\log\left[1 + \frac{1}{q}h(j')\right].$$

Finally, since $h(j) = (q^2-1)h_G(\phi)$, we obtain Theorem 3.1(2), after multiplying by $q^2 - 1$. ∎

6. Drinfeld modular polynomials. Let $m \in \mathbb{F}_q[t]$ be monic. We define

$$\psi(m) = |m| \prod_{P|m} \left(1 + \frac{1}{|P|}\right) \quad \text{and} \quad \kappa(m) = \sum_{P|m} \frac{\deg P}{|P|},$$

where P ranges over all monic irreducible factors of m.

In analogy to classical modular polynomials, Bae [Bae92] constructed polynomials $\Phi_m(X,Y) \in \mathbb{F}_q[t][X,Y]$ for each monic $m \in \mathbb{F}_q[t]$, called *Drinfeld modular polynomials*, with the following properties:

- Degree: $\Phi_m(X,Y)$ is monic of degree $\psi(m)$ in each variable.
- Symmetry: $\Phi_m(X,Y) = \Phi_m(Y,X)$.
- Irreducibility: $\Phi_m(X,Y)$ is irreducible in $\mathbb{C}_\infty[X,Y]$.
- Isogeny: $\Phi_m(j,j') = 0$ if and only if $j = j(\phi)$ and $j' = j(\phi')$ are the j-invariants of rank 2 Drinfeld modules ϕ and ϕ' linked by an isogeny of kernel A/mA.

To study the coefficients of $\Phi_m(X,Y)$, we introduce yet another height. To a polynomial f in several variables with coefficients in $\mathbb{C}_\infty$, we associate its naïve height:

$$h(f) = \log \max_c |c|_\infty,$$

where c ranges over all the coefficients of f.

Hsia proved the following asymptotic result [Hsi98, p. 237]:

THEOREM 6.1 (Hsia). *For any monic, non-constant polynomial $m \in \mathbb{F}_q[t]$ we have*

$$h(\Phi_m) = \frac{q^2-1}{2}\psi(m)\big(\deg m - 2\kappa(m) + O(1)\big) \quad \textit{as } |m| \to \infty.$$

Our goal in this last section is to give a completely explicit upper bound for $h(\Phi_m)$. We start by preparing an interpolation lemma with the following set of interpolation points.

LEMMA 6.2. *Let $n \geq 0$ be an integer. Consider the set*

$$S_n = \big\{\alpha_n t^n + \cdots + \alpha_0 + \cdots + \alpha_{-n} t^{-n} \bigm| \forall i \in \{-n,\ldots,n\},\ \alpha_i \in \mathbb{F}_q\big\}.$$

It has cardinality q^{2n+1}. Let $d \leq q^{2n+1}-1$, and consider $d+1$ distinct points $y_0, y_1, \ldots, y_d \in S_n$. For any $k \in \{0,\ldots,d\}$, denote

$$T_k(Y) = \prod_{\substack{s=0\\ s\neq k}}^{d} (Y - y_s) = \sum_{j=0}^{d} a_j Y^j.$$

Then:

(1) $\max\big\{|a_j|_\infty \bigm| j \in \{0,\ldots,d\}\big\} \leq q^{nd}$,

(2) $\displaystyle\prod_{\substack{s=0\\ s\neq k}}^{d} |y_k - y_s|_\infty \geq q^{-nd}$.

Proof. The maximum degree in t of elements in S_n is n, the upper bound then comes from the explicit computation of the coefficients of T_k in terms of elements of S_n, and the degree of T_k is d. The minimum degree in t of a non-zero difference of elements in S_n is $-n$, the lower bound is direct as well. Note that in [Hsi98, Lemma 5.1], the interpolation set of points is chosen with the extra property $|y_k - y_s|_\infty = \max\{|y_k|_\infty, |y_s|_\infty\}$, which is not assumed here. ∎

Lemma 6.3. *Let $P \in \mathbb{C}_\infty[X,Y]$ be a non-zero polynomial of degree at most $d \geq 1$ in each variable. Suppose there exists a real number $B > 0$ such that $h(P(X, y_k)) \leq B$ for each y_k in the set S_n defined in Lemma* 6.2*. Then*

$$h(P) \leq B + 2nd. \tag{6.1}$$

Proof. We may write $P(X,Y) = \sum_{0\leq r\leq d} Q_r(Y)X^r$ for some polynomials $Q_r(Y) \in \mathbb{C}_\infty[Y]$. For any degree $0 \leq r \leq d$ and any of the above points y_k, let $c_{k,r} = Q_r(y_k)$ be the coefficient of X^r of the polynomial $P(X, y_k)$. By Lagrange interpolation, one has

$$Q_r(Y) = \sum_{k=0}^{d} c_{k,r} \prod_{\substack{s=0\\ s\neq k}}^{d} \frac{Y - y_s}{y_k - y_s}. \tag{6.2}$$

We write

$$T_k(Y) = \prod_{\substack{s=0\\ s\neq k}}^{d} (Y - y_s);$$

by Lemma 6.2 we have $h(T_k) \leq q^{nd}$ and

$$\prod_{\substack{s=0\\ s\neq k}}^{d} |y_k - y_s|_\infty \geq q^{-nd} \qquad \text{for any } k \in \{0,\dots,d\},$$

and by assumption $|c_{k,r}|_\infty \leq B$. The result follows. ∎

We add a small technical lemma.

Lemma 6.4. *Let a be a positive real number. Let $q \geq 2$ be a prime power. Assume $x \geq q^3$. Then $\ln(1 + x/q) \leq x/q^2$ and the inequality*

$$x \leq a + \frac{q^2-1}{2}\log\left(1 + \frac{x}{q}\right) \tag{6.3}$$

implies

$$x \leq a + \frac{q^2-1}{2}\log\left(1 + \frac{a}{q}\left(1 - \frac{q^2-1}{2q^2\ln q}\right)^{-1}\right), \tag{6.4}$$

where ln *is the natural logarithm and* log *is the logarithm to base q.*

Proof. Direct computation. ∎

We are now ready to prove the following.

Proposition 6.5. *For any monic, non-constant polynomial $m \in \mathbb{F}_q[t]$, the height $h(\Phi_m)$ is bounded above by*

$$\psi(m)\max\left\{q^3, \left[\frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1) + \frac{q^2-1}{2}\log\left(1+\frac{a}{q}\left(1-\frac{q^2-1}{2q^2\ln q}\right)^{-1}\right)\right]\right\} + 2\psi(m)\log\psi(m),$$

where $a = \frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1)$.

Proof. Fix $j_0 \in S_n$, where n is chosen so that $q^{2n+1} \geq \psi(m)+1 \geq q^{2n-1}$, and S_n is the set of Lemma 6.2. The relation between roots and coefficients for the polynomial $\Phi_m(X, j_0)$ gives in particular the inequality

$$h(\Phi_m(X, j_0)) \leq \psi(m)\max_j h(j),$$

where the maximum is taken over all the roots j of $\Phi_m(X, j_0)$. Each of these roots corresponds to a Drinfeld module isogenous to the fixed one corresponding to j_0, hence by Theorem 3.1 we get

$$h(j) - h(j_0) \leq \frac{q^2-1}{2}\deg m + \frac{q^2-1}{2}\log\left(1+\frac{1}{q}h(j)\right) + q. \tag{6.5}$$

Now for any $j_0 \in S_n$, we have $h(j_0) \leq n \leq \frac{1}{2}(\log\psi(m)+1)$. This leads to

$$h(j) \leq \frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1) + \frac{q^2-1}{2}\log\left(1+\frac{1}{q}h(j)\right). \tag{6.6}$$

Assume $h(j) \geq q^3$. Then by Lemma 6.4 we get

$$h(j) \leq \frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1) + \frac{q^2-1}{2}\log\left(1+\frac{a}{q}\left(1-\frac{q^2-1}{2q^2\ln q}\right)^{-1}\right), \tag{6.7}$$

where $a = \frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1)$, and hence $h(\Phi_m(X, j_0))$ is bounded above by a quantity B equal to

$$\psi(m)\max\left\{q^3, \left[\frac{q^2-1}{2}\deg m + q + \frac{1}{2}(\log\psi(m)+1) + \frac{q^2-1}{2}\log\left(1+\frac{a}{q}\left(1-\frac{q^2-1}{2q^2\ln q}\right)^{-1}\right)\right]\right\}, \tag{6.8}$$

and by Lemma 6.3 we obtain the result. ∎

Asymptotically, this gives

$$h(\Phi_m) < \left(\frac{q^2+4}{2} + \epsilon\right)\psi(m)\deg m \tag{6.9}$$

for $\deg m$ sufficiently large compared to $\epsilon > 0$. This is only slightly weaker than Hsia's exact asymptotic in Theorem 6.1.

Another completely explicit upper bound on $h(\Phi_m)$, of order

$$\frac{q}{2}|m|\psi(m)^2,$$

was obtained by Bae and Lee [BL97, Theorem 3.7].

Acknowledgements. The authors are grateful to Fu-Tsun Wei for helpful discussions, and the anonymous referee for constructive feedback.

The authors thank the CNRS and the IRN GANDA for the support. The first author thanks the Alexander-von-Humboldt Foundation for partial support. The second author thanks the projects ANR-14-CE25-0015 Gardio and ANR-17-CE40-0012 Flair.

References

[Bae92] S. Bae, *On the modular equation for Drinfeld modules of rank 2*, J. Number Theory 42 (1992), 123–133.

[BL97] S. Bae and S. Lee, *On the coefficients of the Drinfeld modular equation*, J. Number Theory 66 (1997), 85–101.

[DD99] S. David et L. Denis, *Isogénie minimale entre modules de Drinfel'd*, Math. Ann. 315 (1999), 97–140.

[Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73 (1983), 349–366.

[Gek97] E.-U. Gekeler, *On the Drinfeld discriminant function*, Compos. Math. 106 (1997), 181–202.

[Gek17] E.-U. Gekeler, *On Drinfeld modular forms of higher rank*, J. Théor. Nombres Bordeaux 29 (2017), 875–902.

[Gek19] E.-U. Gekeler, *Towers of* $\mathrm{GL}(r)$*-type of modular curves*, J. Reine Angew. Math. 754 (2019), 87–141.

[Hsi98] L.-C. Hsia, *On the coefficients of modular polynomials for Drinfeld modules*, J. Number Theory 72 (1998), 236–256.

[Paz19] F. Pazuki, *Modular invariants and isogenies*, Int. J. Number Theory 15 (2019), 569–584.

[Pot98] I. Yu. Potemine, *Minimal terminal* $\mathbb{Q}$*-factorial models of Drinfeld coarse moduli schemes*, Math. Phys. Anal. Geom. 1 (1998), 171–191.

[Ray85] M. Raynaud, *Hauteurs et isogénies*, Astérisque 127 (1985), 199–234.

[Tag93] Y. Taguchi, *Semi-simplicity of the Galois representations attached to Drinfeld modules over fields of "infinite characteristics"*, J. Number Theory 44 (1993), 292–314.

[Tag99] Y. Taguchi, *Finiteness of an isogeny class of Drinfeld modules, correction to a previous paper*, J. Number Theory 74 (1999), 337–348.

[Wei20] F.-T. Wei, *On Kronecker terms over global function fields*, Invent. Math. 220 (2020), 847–907.

Florian Breuer
School of Mathematical
and Physical Sciences
University of Newcastle
University Drive
Callaghan 2308 NSW, Australia
E-mail: florian.breuer@newcastle.edu.au

Mahefason Heriniaina Razafinjatovo
Département de Mathématiques
et Informatique
Université d'Antananarivo
BP 906, Antananarivo 101, Madagascar
E-mail: razafpouf@univ-antananarivo.mg

Fabien Pazuki
Institute of Mathematics
University of Copenhagen
Universitetsparken 5
2100 København, Denmark
and
Institut de Mathématiques de Bordeaux
Université de Bordeaux
Cours de la Libération
33400 Talence, France
E-mail: fpazuki@math.ku.dk

Abstract (will appear on the journal's web site only)

We provide explicit bounds on the difference of heights of isogenous Drinfeld modules. We derive a finiteness result in isogeny classes. In the rank 2 case, we also obtain an explicit upper bound on the size of the coefficients of modular polynomials attached to Drinfeld modules.