Monotonicity of Riemann Sums

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Abstract

We consider conditions ensuring the monotonicity of right and left Riemann sums of a function $f : [0,1] \to \mathbb{R}$ with respect to uniform partitions. Experimentation suggests that symmetrization may be important and leads us to results such as: *if f is decreasing on $[0,1]$ and its symmetrization, $F(x) := \frac{1}{2}(f(x) + f(1-x))$ is concave then its right Riemann sums increase monotonically with partition size.* Applying our results to functions such as $f(x) = 1/\left(1+x^2\right)$ also leads to a nice application of Descartes' rule of signs.

1 Introduction

For a bounded function $f : [0,1] \to \mathbb{R}$ the *left* and *right Riemann sums* of f with respect to the uniform partition $\mathcal{U}_n$ of $[0,1]$ into n equal intervals are,

$$\sigma_n := \sigma_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \quad \text{and} \quad \tau_n := \tau_n(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right). \tag{1}$$

So, $\sigma_n(f) - \tau_n(f) = \dfrac{1}{n}(f(0) - f(1))$, and both σ_n and τ_n are linear functionals with $\sigma_n(1) = \tau_n(1) = 1$. If f is decreasing (increasing) on $[0,1]$ then σ_n is the upper (lower), and τ_n the lower (upper), Riemann sum of f with respect $\mathcal{U}_n$. If f is symmetric about the midpoint of $[0,1]$; that is, $f(x) = f(1-x)$, then $\tau_n(f) = \sigma_n(f)$ for all n.

And, of course, if f is Riemann integrable (as it is if f is monotonic, concave, or continuous) then both σ_n and τ_n converge to $\int_0^1 f$. (See, for example [1].) Further, if for example f is decreasing then $\tau_{2n} \geq \tau_n$, so τ_{2^n} increases monotonically to $\int_0^1 f$, but how does τ_{n+1} compare to τ_n?

We are thus led to seek conditions which will ensure (σ_n) and (τ_n) or other Riemann sums form decreasing/imcreasing sequences.

In the process of producing [2] one of the current authors gave the following example.

Example 1 (Digital assistance, arctan(1) and a black-box)**.** Consider for integer $n > 0$ the sum

$$\sigma_n := \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2}.$$

The definition of the Riemann sum means that

$$\begin{aligned}
\lim_{n\to\infty} \sigma_n &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{1 + (k/n)^2} \frac{1}{n} \\
&= \int_0^1 \frac{1}{1 + x^2} \mathrm{d}x \\
&= \arctan(1). \qquad (2)
\end{aligned}$$

Even without being able to do this *Maple* will quickly tell you that

$$\sigma_{10^{14}} = 0.78539816339746\ldots$$

Now if you ask for 100 billion terms of most slowly convergent series, a computer will take a long time.

So this is only possible because *Maple* knows

$$\sigma_N = -\frac{i}{2}\Psi\left(N - iN\right) + \frac{i}{2}\Psi\left(N + iN\right) + \frac{i}{2}\Psi\left(-iN\right) - \frac{i}{2}\Psi\left(iN\right)$$

using the imaginary i, and it has a fast algorithm for our new friend the *psi* function. Now `identify`(0.78539816339746) yields $\frac{\pi}{4}$.

We can also note that

$$\tau_n := \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$$

is another Riemann sum converging to $\int_0^1 \frac{1}{1+x^2}\mathrm{d}x$. Indeed, $\sigma_n - \tau_n = \frac{1}{2n} > 0$. Moreover, experimentation *suggests* that σ_n decreases, and τ_n increases, to $\pi/4$. $\Diamond$

If we enter "monotonicity of Riemann sums" into Google, one of the first entries is `http://elib.mi.sanu.ac.rs/files/journals/tm/29/tm1523.pdf` which is a 2012 article [4] that purports to show the monotonicity of the two sums for the function

$$f(x) =: \frac{1}{1+x^2}. \tag{3}$$

The paper goes on to prove that *if* $f\colon [0,1] \to R$ *is continuous, concave, or convex, and decreasing then* $\tau_n := \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ *increases and* $\sigma_n := \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ *decreases to* $\int_0^1 f(x)\,\mathrm{d}x$, *as* $n \to \infty$. Related results for a continuous concave, or convex, and increasing function follow by applying these results to $-f$.

That a condition such as concavity (or convexity) is necessary is readily seen by considering a function such as $\chi_{[0,\frac{1}{2}]}$,the characteristic function for the interval $[0,\frac{1}{2}]$, for which $\tau_{2m-1}+\frac{1}{2(m-1)}=\tau_{2m}=\tau_{2m+1}+\frac{1}{2m}$.

All proofs in [4] are based on looking at the rectangles which comprise the difference between τ_{n+1} and τ_n as in Figure **??** (or the corresponding sums for σ_n). This is

$$\sum_{k=1}^{n}\left\{\frac{(n+1-k)}{n+1}f\left(\frac{k}{n+1}\right)+\frac{k}{n+1}f\left(\frac{k+1}{n+1}\right)-f\left(\frac{k}{n}\right)\right\}. \tag{4}$$

In the easiest case, each bracketed term

$$\delta_n(k) := \frac{(n+1-k)}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n}\right)$$

has the same sign for all n and $1 \leq k \leq n$ as happens for a function which is concave, or convex, and decreasing.

But in [4] the author mistakenly asserts this applies for $1/(1+x^2)$ which has an inflection point at $1/\sqrt{3}$. Indeed, the proffered proof flounders at the inequality in the last line of page 115 which fails for instance when $n = 5$ and $k = 1$.

It appears, however, on checking in a computer algebra system (CAS), that $\delta_n(k) + \delta_n(n-k) \geq 0$ which if established would repair the hole in the proof, it also suggests that symmetry may have a role to play.

Accordingly, we define the *symmetrization* of $f : [0,1] \to \mathbb{R}$ about $x = \frac{1}{2}$ to be

$$F(x) := F_f(x) = \frac{1}{2}\left(f(x) + f(1-x)\right). \tag{5}$$

We will make use of F_f throughout the rest of this note and start by observing that such a symmetrization never destroys convexity or concavity and often improves it.

For example, for $f(x) = 1/(1+x^2)$ we have

$$F_f(x) = \frac{x^2 - x + 3/2}{(x^2+1)\,(x^2 - 2\,x + 2)}.$$

That F_f is concave on $[0,1]$ can be checked by computing $F''(x)$. (See theorem 2 for more details.) Graphs of f and F_f are shown in in Figure 1, together with the graph of a symmetric concave function. $\Diamond$

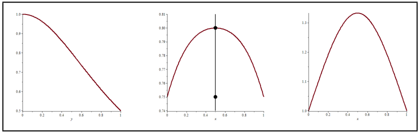

Figure 1: $\frac{1}{1+x^2}$ (L), its symmetrization around $\frac{1}{2}$ (M), and $\frac{1}{1-x+x^2}$ (R)

Another example of functions with a concave symmetrization is $x \mapsto e^{-ax^2}$, for $a > 0$.

What a fine instance of digital assistance in action all this provides.

2 Monotonicity and symmetrization

Numerical experiments suggest it is very common for f to be such that τ_n and σ_n exhibit monotonicity but it is harder to find applicable conditions that assure this. Thus, we seek verifiable conditions that in particular will apply to (3). Motivated by example 1 and the preceding discussion, we exploit symmetry around 1/2. This is a very simple case of Schwartz or *Steiner symmetrization* [3, §3.4.3] used in some proofs of the isoperimetric problem. As will soon become apparent, calculations involving symmetric (concave) functions lead us naturally to the introduction of the following *symmetric Riemann sum*.

For $f : [0,1] \to \mathbb{R}$ we define:

$$\lambda_n := \lambda_n(f) = \frac{1}{n}\left(\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\right) \; - \; \frac{1}{n} f\left(\frac{1}{2}\right). \tag{6}$$

For all $n \in \mathbb{N}$, $\lambda_n(f)$ is linear and symmetric in that $\lambda_n(f) = \lambda_n(f(1-\cdot))$ and so $\lambda_n(f) = \lambda_n(F)$ where F is the symmetrization of f; namely, $F(x) := \frac{1}{2}\left(f(x) + f(1-x)\right)$. The term involving $f\left(\frac{1}{2}\right)$ ensures that $\lambda_n(1) = 1$ by making a correction to the central term(s) of $\frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)$; if n is even we simply omit the central term, $\frac{1}{n}f(1/2)$, while if n is odd we replace the two central terms by $\frac{1}{n}(f(1/2 - 1/(2n)) - f(1/2) + f(1/2 + 1/(2n)))$.

Further,

$$\lambda_n(f) = \frac{\tau_n + \sigma_n}{2} + \frac{1}{2n}\left(f(0) + f(1) - 2f\left(\frac{1}{2}\right)\right) \tag{7}$$

$$= \tau_n + \frac{1}{n}\left(f(0) - f\left(\frac{1}{2}\right)\right) \tag{8}$$

$$= \sigma_n + \frac{1}{n}\left(f(1) - f\left(\frac{1}{2}\right)\right) \tag{9}$$

Theorem 1 (Monotonicity for symmetric concave function). *If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,1]$ and is symmetric about its midpoint, then the sequence $\{\lambda_n\}$ is increasing with n.*

Note that a concave function on $[0,1]$ symmetric around $1/2$ takes its maximum there and is necessarily decreasing on $(1/2, 1)$—and increasing on $(0, 1/2)$.

Corollary 1. *If the function $f : [0,1] \to \mathbb{R}$ is convex on the interval $[0,1]$, and is symmetric about its midpoint, then the sequence $\{\lambda_n\}$ is decreasing with n.*

Proof of Corollary 1. This follows since Theorem 1 applies to $-f$. $\square$

Before proceeding to a proof of Theorem 1 we first give a lemma.

Lemma 1. *If $f : [0,1] \to \mathbb{R}$ is concave and decreasing on the interval $[1/2, 1]$ and*

$$\frac{k}{n} \geq \frac{1}{2}, \tag{10}$$

then

$$f\left(\frac{k+1}{n+1}\right) \geq \frac{n-k}{n} f\left(\frac{k+1}{n}\right) + \frac{k}{n} f\left(\frac{k}{n}\right). \tag{11}$$

Proof. Since f is concave on $[1/2, 1]$ and (10) holds we have

$$\begin{aligned} \frac{n-k}{n} f\left(\frac{k+1}{n}\right) + \frac{k}{n} f\left(\frac{k}{n}\right) &\leq f\left(\frac{n-k}{n} \cdot \frac{k+1}{n+1} + \frac{k}{n} \cdot \frac{k}{n}\right) \\ &= f\left(\frac{nk+n-k}{n^2}\right). \end{aligned} \tag{12}$$

Due to the monotonicity of f on $[1/2, 1]$ and the inequalities

$$\frac{nk+n-k}{n^2} \geq \frac{k+1}{n+1} > \frac{k}{n} \geq \frac{1}{2}, \tag{13}$$

we have

$$f\left(\frac{k+1}{n+1}\right) \geq f\left(\frac{nk+n-k}{n^2}\right). \tag{14}$$

Together, inequalities (12) and (14) imply inequality (11). This completes the proof of the lemma. $\square$

Since for any constant K we have $\lambda_n(f+K) = \lambda_n(f) + K$ (and the same for τ_n and σ_n), we may suppose without loss in generality that $f(0) = f(1) = 0$. Observe that inequality (11) is equivalent to

$$\frac{1}{n+1} f\left(\frac{k+1}{n+1}\right) \geq \frac{1}{n} f\left(\frac{k+1}{n}\right) + \frac{1}{n(n+1)}\left(k f\left(\frac{k}{n}\right) - (k+1) f\left(\frac{k+1}{n}\right)\right) \tag{15}$$

from which it follows that

$$\frac{1}{n+1} \sum_{k=m}^{n-1} f\left(\frac{k+1}{n+1}\right) \geq \frac{1}{n} \sum_{k=m}^{n-1} f\left(\frac{k+1}{n}\right) + \frac{m}{n(n+1)} f\left(\frac{m}{n}\right),$$

or equivalently

$$\frac{1}{n+1} \sum_{k=m+1}^{n+1} f\left(\frac{k}{n+1}\right) \geq \frac{1}{n} \sum_{k=m+1}^{n} f\left(\frac{k}{n}\right) + \frac{m}{n(n+1)} f\left(\frac{m}{n}\right). \tag{16}$$

Because of the symmetry of f we deduce from (16) that

$$\frac{1}{n+1}\sum_{k=0}^{n-m} f\left(\frac{k}{n+1}\right) \geq \frac{1}{n}\sum_{k=0}^{n-m-1} f\left(\frac{k}{n}\right) + \frac{m}{n(n+1)} f\left(\frac{m}{n}\right). \tag{17}$$

We consider the cases n odd and n even separately.
Case i: $n = 2m - 1$. Adding the inequalities (16) and (17) we get

$$\frac{1}{n+1}\sum_{k=0}^{n+1} f\left(\frac{k}{n+1}\right) \geq \frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right) + \frac{2m}{n(n+1)} f\left(\frac{m}{n}\right) + \frac{1}{n+1} f\left(\frac{m}{n+1}\right) \tag{18}$$

$$- \frac{1}{n} f\left(\frac{m-1}{n}\right) - \frac{1}{n} f\left(\frac{m}{n}\right). \tag{19}$$

Since

$$\frac{m-1}{n}=\frac{1}{2}-\frac{1}{2n},\quad \frac{m}{n}=\frac{1}{2}+\frac{1}{2n},\quad \frac{m}{n+1}=\frac{1}{2},$$

it follows from (18) that

$$\tau_{n+1}-\tau_n \geq \frac{1}{n}f\left(\frac{1}{2}+\frac{1}{2n}\right)+\frac{1}{n+1}f\left(\frac{1}{2}\right)-\frac{1}{n}f\left(\frac{1}{2}-\frac{1}{2n}\right)-\frac{1}{n}f\left(\frac{1}{2}+\frac{1}{2n}\right) \tag{20}$$

$$= \frac{1}{n+1}f\left(\frac{1}{2}\right)-\frac{1}{n}f\left(\frac{1}{2}-\frac{1}{2n}\right)$$

$$\geq \frac{1}{n+1}f\left(\frac{1}{2}\right)-\frac{1}{n}f\left(\frac{1}{2}\right) \tag{21}$$

and

$$\lambda_{n+1} = \tau_{n+1} - \frac{1}{n+1} f\left(\frac{1}{2}\right) \geq \tau_n - \frac{1}{n} f\left(\frac{1}{2}\right) = \lambda_n. \tag{22}$$

Case ii: $n = 2m$. In this case adding the inequalities (16) and (17) we get

$$\begin{aligned} \frac{1}{n+1} \sum_{k=0}^{n+1} f\left(\frac{k}{n+1}\right) &\geq \frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) + \frac{1}{n} f\left(\frac{m}{n}\right) + \frac{2m}{n(n+1)} f\left(\frac{m}{n}\right) \\ &= \frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) - \frac{1}{n} f\left(\frac{1}{2}\right) + \frac{1}{n+1} f\left(\frac{1}{2}\right). \end{aligned} \tag{23}$$

It follows from (23) that

$$\lambda_{n+1} = \tau_{n+1} - \frac{1}{n+1} f\left(\frac{1}{2}\right) \geq \tau_n - \frac{1}{n} f\left(\frac{1}{2}\right) = \lambda_n. \tag{24}$$

It now follows from (20) and (24) that $\{\lambda_n\}$ is increasing with n. $\square$

Corollary 2. *If the function* $f : [0,1] \to \mathbb{R}$ *has a concave symmetrization and* $f(0) > f(1/2)$ *then* τ_n *increases with* n*.*

Proof. Theorem 1 applies to F_f to show that $\lambda_n(f) = \lambda_n(F_f)$ is increasing and the conclusion follows from (8). $\square$

In particular we have,

Corollary 3 (Monotonicity for decreasing functions with a concave symmetrization)**.** *If the function* $f : [0,1] \to \mathbb{R}$ *is decreasing on the interval* $[0,1]$ *and its symmetrization;* $F_f(x) = \frac{1}{2}\left(f(x) + f(1-x)\right)$*, is concave, then* τ_n *increases with* n*, necessarily to* $\int_0^1 f$*.*

Example 2 (Monotonicity of τ_n for $1/(1+x^2)$)**.** Consider the function $f(x) =: 1/(1+x^2)$ for which

$$\tau_n := \sum_{k=1}^{n} \frac{n}{n^2+k^2}.$$

Clearly f is decreasing on $[0,1]$ and we already observed in example 1 that its symmetrization $F_f(x) := \frac{1}{2}(f(x) + f(1-x))$ is concave, so corollary 3 applies to show that τ_n is increasing. $\Diamond$

Similarly, for $a > 0$ and $f_a(x) := e^{-ax^2}$, we see by calculating f_a' and F_{f_a}'' that $\tau_n(f_a)$ increases with n.

Remark 1 (Variations on the theme)**.** Let $f : [0,1] \to \mathbb{R}$. Noting from their linearity that $\tau_n(-f) = -\tau(f)$ and similarly for σ_n, and also observing that $\sigma_n(f(x)) = \tau_n(f(1-x))$, we can deduce the following variants of the results above.

(i) If f is is decreasing, symmetric and convex, then λ_n is decreasing. [Apply Theorem 1 to $-f$.]

(ii) If $f(0) < f(1/2)$ (in particular, if f is increasing) and has a convex symmetrization, then τ_n is decreasing. [Apply Corollary 2 to $-f$.]

(iii) If $f(1/2) < f(1)$ (in particular, if f increasing) and has a concave symmetrization, then σ_n is increasing. [Apply Corollary 2 to $f(1-x)$.]

(iv) If $f(1/2) > f(1)$ (in particular if f is decreasing) and has a convex symmetrization, then σ_n is decreasing. [Apply Corollary 2 to $-f(1-x)$.]

Since the symmetrization of f is concave (convex) if f is concave (convex) we observe that Corollary 2 and (iv) extend the final two theorems in [4]. $\lozenge$

3 The function $\frac{1}{1-bx+x^2}$

As a way of highlighting the subtleties in a seemingly innocent question, we finish by analyzing a one-parameter class of functions to which our results sometimes apply.

We consider the the family of functions

$$f_b : [0,1] \to \mathbb{R}, \quad \text{where } f_b(x) := \frac{1}{x^2 - bx + 1} \tag{25}$$

in the parameter range $|b| < 2$ so that each f_b assumes only positive values.

The symmetrization of f_b about $1/2$ is

$$F_b(x) = \frac{x^2 - x + (3-b)/2}{(x^2 - bx + 1)\left(x^2 - (2-b)x + (2-b)\right)}. \qquad (26)$$

Then $f_0(x) = 1/(1+x^2)$ while $f_1(x) = F_1(x) = 1/(x^2 - x + 1)$. Now F_0, F_1 and $F_{3/2}$ are concave on $[0,1]$, while F_{-1} is convex and

$$F_2(x) = \frac{(1-x)x + 1/2}{(1-x)^2\, x^2}$$

is convex as an extended value function from $[0,1]$ into $(-\infty, \infty]$. By contrast $F_{5/4}, F_{7/4}$ are neither convex nor concave on the unit interval (for more details see remark 2 below).

In passing we compute for $|b| < 2$ that

$$\int_0^1 \frac{\mathrm{d}x}{x^2 - bx + 1} = \frac{2}{\sqrt{4-b^2}} \left(\arctan\left(\frac{b}{\sqrt{4-b^2}} \right) + \arctan\left(\frac{2-b}{\sqrt{4-b^2}} \right) \right).$$

When $b \to -2$ we arrive at $\int_0^1 \frac{\mathrm{d}x}{x^2+2x+1} = \frac{1}{2}$.

With a view to applying Corollaries 3 or 2 we begin by noting that $f_b(x)$ is decreasing on $[0,1]$ for $b \leq 0$ and increasing only for $b \geq 2$, however $f_b(0) > f_b(1/2)$ whenever $b < 1/2$.

We next prove that F_b is concave for $0 \leq b \leq 1$. We will employ *Descartes' rule of signs*, see http://mathworld.wolfram.com/DescartesSignRule.html, which says that for a polynomial p, *the number $n(p)$ of zeros on the positive axis does not exceed the number of sign changes $s(p)$ in the nonzero coefficients (in order) and that*

$2|(n(p) - s(p))$.

Theorem 2 (Concavity of F_b)**.** *The function given by* (26) *is concave on* $[0,1]$ *for* $0 \leq b \leq 1$.

Proof. To establish concavity of F_b we show that F_b'' is negative on $[0,1]$, see figure **??** and to do this we need only show its the numerator polynomial, n_b, is negative, as the denominator is always positive.

Further, since F_b and hence F_b'' are symmetric about $\frac{1}{2}$ we need only show this on $[1/2, 1]$. Moreover, using the change of variable $x := (y+1)/2$ allows us to use Descartes' rule of signs to detect roots of $n_b(x)$ for $x \geq 1/2$ (that is, $y \geq 0$).

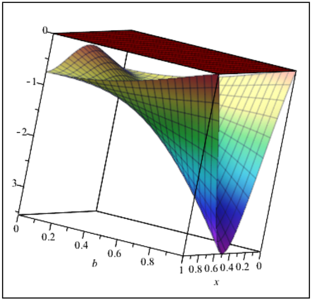

Figure 2: The second derivative of F_b for $0 \leq b, x \leq 1$.. Note the advantage of plotting the plane.

Now, the numerator of $F_b''((y+1)/2)$ is

$$n_b(y) := 24\,y^8 + 32\,\left(b^2 - 6\,b + 11\right) y^6 + 48\,(2\,b - 5)\left(6\,b^2 - 10\,b + 1\right) y^4 \\ - 96\,(2\,b - 5)\left(4\,b^2 - 2\,b - 11\right)(b-1)^2\,y^2 - 8\,\left(4\,b^2 - 6\,b - 1\right)(2\,b - 5 \tag{27}$$

For $0 < b < 1$ the first two terms in (27) are always positive and the final two are negative, so irrespective of the sign of the coefficient of y^4 (it in fact has three zeroes—at $5/2, (5 \pm \sqrt{19})/6$) Descartes' rule of signs applies to show the numerator has one positive real zero (including multiplicity). This zero must lie to the right of 1 except for $b = 1$ when it equals 1, as illustrated in Figure 3. (Note how close to one the inflection point is for $b = 5/4$.)

For $0 \le b < 1$ we have

$$n_b(0) = 8\,\left(4\,b^2 - 6\,b - 1\right)(5 - 2\,b)^3 < 0$$

and

$$n_b(1) = -1024\,(b-2)\,(b-1)\left(b^3 - 3\,b^2 + 3\right) < 0.$$

Thus, when $0 \le b \le 1$ the numerator is non-positive for $y \in [-1, 1)$ and so $F_b(x)$ is concave on $[0, 1]$. $\square$

This proof of concavity for F_b was discovered by examining animations of the behaviour of n_b and then getting a CAS to provide the requisite expressions after shifting the symmetry to zero so that Descartes' rule was applicable. some snapshots of the animation are illustrated in Figure 3.

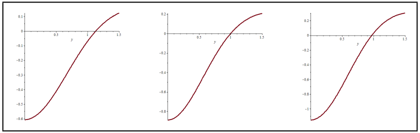

Figure 3: Graph of $n_b(y)$ on $[0, 3/2]$ for $b = 3/4$ (L), $b = 1$ (M), and $b = 5/4$ (R)

Remark 2 (Convexity properties throughout the range $|b| < 2$)**.** In this range the function provides further interesting applications of Descartes' rule of signs.

A careful analysis of the coefficients a_k of y^{2k} for $k = 0, 1, 2, 3$ in (27) and the signs of $n_b(0)$ and $n_b(1)$ [see Figure 3 where we plot $n_b(0)$ and $n_b(1)$ with $n_0(b)$ a dashed line], coupled with reasoning similar to that in the proof of Theorem 2 allows us to extend the results of that theorem to the whole parameter range $|b| < 2$.

The analysis and conclusions are summarized in Table 1, where

$$
\begin{aligned}
\alpha_- &= \text{the negative root of } b^4 - 3b^2 + 3 \approx -0.8794 \\
\alpha &= \text{the smallest positive root of } b^4 - 3b^2 + 3 \\
&= 1 + \sqrt{3}\sin(2\pi/9) \approx 1.3473 \\
\alpha_+ &= \text{the largest root of } b^4 - 3b^2 + 3 \approx 2.5231 \\
\beta_-,\ \beta_+ &= \text{the roots of } 4b^2 - 6b - 1 = (3 \pm \sqrt{13})/3 \approx -0.1539,\ 1.6514 \\
\gamma_-,\ \gamma_+ &= \text{the roots of } 6b^2 - 10b + 1 = (5 \pm \sqrt{19})/6 \approx -0.1069,\ 1.5598 \\
\delta_-,\ \delta_+ &= \text{the roots of } 4b^2 - 2b - 11 = (1 \pm \sqrt{45})/4 \approx -1.4271,\ 1.9271
\end{aligned}
$$

and

$$
\# = \text{the number of positive roots of } n_b((y+1)/2)
$$

Table 1: Table of Signs

b	$[-2,\delta_-]$	$[\delta_-,\alpha_-]$	$[\alpha_-,\beta_-]$	$[\beta_-,\gamma_-]$	$[\gamma_-,1]$	$[1,\alpha]$	$[\alpha,\gamma_+]$	$[\gamma_+,\beta_+]$	$[\beta_+,\delta_+]$
a_4	+	+	+	+	+	+	+	+	+
a_3	+	+	+	+	+	+	+	+	+
a_2	-	-	-	-	+	+	+	-	-
a_1	+	-	-	-	-	-	-	-	-
a_0	+	+	+	-	-	-	-	-	+
$n_b(0)$	+	+	+	-	-	-	-	-	+
$n_b(1)$	+	+	-	-	-	+	-	-	-
#	2	2	2	1	1	1	1	1	2
$F_b(x)$	conv	conv	infl	conc	conc	infl	conc	conc	infl

The conclusion that F_b is convex for $-2 < b \leq \alpha_-$ follows from the observation that in this range $n_b(x)$ is negative for values of $x > 1$, so neither positive root can lie within the interval $[0,1]$.

Putting all this together we are able to conclude that $\tau_n(f_b)$ is increasing for $b \in [\beta_-, 1/2]$ and $\sigma_n(f_b)$ is decreasing for $b \in [-2, \alpha_-]$.

A similar analysis in the cases $|b| > 2$ is left to the interested reader. $\Diamond$

4 Concluding Remarks

The story we have told highlights the many accessible ways that the computer and the internet can enrich mathematical research and instruction. The story would be even more complete if we could also deduce that $\sigma_n(1/(1+x^2))$ was decreasing.

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