Computation and structure of character polylogarithms

with applications to character Mordell–Tornheim–Witten sums

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THIS TALK: http://carma.newcastle.edu.au/jon/charpoly-talk.pdf

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Companion paper (*Math of Comp*) : http://www.carma.newcastle.edu.au/jon/MTWIII.pdf

Abstract

This work builds on tools developed in

1. D. H. Bailey, J. M. Borwein, and **R. E. Crandall**. Computation and theory of extended Mordell-Tornheim-Witten sums. *Mathematics of Computation*n, 83:1795–1821, 2014. See http://carma.newcastle.edu.au/jon/MTW1.pdf
2. D. H. Bailey and J. M. Borwein. Computation and theory of extended Mordell-Tornheim-Witten sums II. *Journal of Approximation Theory*, In press. at http://carma.newcastle.edu.au/jon/MTW2.pdf

to study *character polylogarithms*.

- These objects are then used to compute MTW character sums and to explore connections with *multiple-zeta values* (MZVs) and their character analogues.
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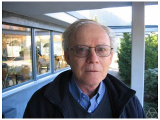

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It is fitting that Alf van der Poorten [1942–2010] wrote the foreword to Lewin's "bible" and highlights the [log-sine] evaluation

$$-\int_0^{\pi/3} \theta \log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}\theta = -\operatorname{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = \frac{17}{6480}\pi^4$$

and its relation with inverse central binomial sums.

This work would be impossible without very extensive symbolic and numeric computations, and makes frequent use of the NIST Handbook of Mathematical Functions (DLMF).

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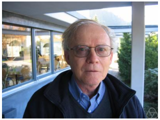

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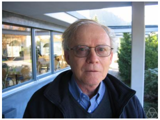

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Contents. We will cover some of the following:

1. 4. Preliminaries
   - 6. Multiple polylogarithms
   - 7. Mordell–Tornheim–Witten sums
   - 11. Generalized MTW sums
   - 12. Character L-series and polylogarithms
2. 21. Character polylogarithms
   - 22. Character polylogarithms and Lerch's formula
   - 27. L-series derivatives at negative integers
   - 33. Multisectioning character polylogarithms
3. 45. Applications to character MTW sums
   - 45. Basics of character MTW sums
   - 48. First order sum computations
   - 54. Second order sum computations
4. 57. Values of character sums including order zero
   - 59. Efficient computation
   - 62. Integral free evaluation
   - 65. Alternating character sums
   - 67. Character sums with $3 \le |d| \le 5$
   - 68. Character sums with $d = -4$
5. 74. Conclusion

Other References

1. Joint with: David Bailey (LBNL) Armin Straub (Tulane) and James Wan (UofN)
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My younger Collaborators

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$$\mathrm{Li}_{a_1,\ldots,a_k}(z) := \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{a_1}\cdots n_k^{a_k}}.$$

Thus, $\mathrm{Li}_{2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \sum_{j=1}^{k-1} \frac{1}{j}$. Specializing produces:

- The *polylogarithm of order k*: $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$.
- *Multiple zeta values*:

$$\zeta(a_1,\ldots,a_k) := \mathrm{Li}_{a_1,\ldots,a_k}(1).$$

- *Multiple Clausen* (Cl) and *Glaisher functions* (Gl) of depth k and weight $w := \sum a_j$:

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MTW Sums

We first recall the definitions of Mordell–Tornheim–Witten (MTW) sums also called Mordell–Tornheim–Witten zeta function values. The multidimensional *Mordell–Tornheim–Witten (MTW) zeta function* is

$$\omega(s_1, \ldots, s_{K+1}) \;:=\; \sum_{m_1,\ldots,m_K > 0} \frac{1}{m_1^{s_1} \cdots m_K^{s_K} (m_1 + \cdots + m_K)^{s_{K+1}}} \tag{1}$$

- ω enjoys known relations, but remains mysterious with respect to many combinatorial phenomena, especially when we contemplate derivatives with respect to the s_i parameters
- $K+1$ is the *depth* and $\sum_{j=1}^{k+1} s_j$ is the *weight* of ω.

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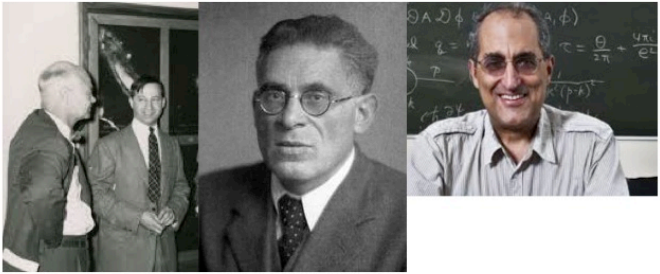
Tornheim, Mordell and Witten



1. Leonard Tornheim (1915–??). Paper in *J. Amer. Math. Soc* (1950).
2. Louis J. Mordell (1888–1972). Two papers in *J. London Math. Soc.* (1958).
3. Edward Witten (1951–). Paper in *Comm. in Math. Phys.* (1991).

MTW Sums

The paper [1] introduced and discussed a novel *generalized MTW zeta function* for positive integers M, N ($M \geq N \geq 1$), nonnegative integers s_i, t_j—with a polylogarithmic-integral representation (on the torus):

$$\omega(s_1,\ldots,s_M \mid t_1,\ldots,t_N) := \sum_{\substack{m_1,\ldots,m_M,n_1,\ldots,n_N>0 \\ \sum_{j=1}^M m_j=\sum_{k=1}^N n_k}} \prod_{j=1}^M \frac{1}{{m_j}^{s_j}} \prod_{k=1}^N \frac{1}{{n_k}^{t_k}} \tag{2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \mathrm{Li}_{s_j}\left(e^{i\theta}\right) \prod_{k=1}^N \mathrm{Li}_{t_k}\left(e^{-i\theta}\right) \mathrm{d}\theta. \tag{3}$$

MTW Sums

- When some s-parameters are zero, there are convergence issues with this integral representation.
- One may, however, use principal-value calculus, or alternative representations given in [1] and expanded upon herein.

When $N = 1$ the representation (3) devolves to the classic MTW form, in that

$$\omega(s_1, \ldots, s_{M+1}) = \omega(s_1, \ldots, s_M \mid s_{M+1}). \tag{4}$$

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Generalized MTW Sums

We then explored a wider *MTW ensemble* involving outer derivatives—introduced to resolve log Gamma integrals—via:

$$\omega\left(\begin{array}{c|c} s_1,\ldots,s_M & t_1,\ldots,t_N \\ d_1,\ldots,d_M & e_1,\ldots e_N \end{array}\right) := \sum_{\substack{m_1,\ldots,m_M,n_1,\ldots,n_N>0 \\ \sum_{j=1}^M m_j=\sum_{k=1}^N n_k}} \prod_{j=1}^M \frac{(-\log m_j)^{d_j}}{m_j{}^{s_j}} \prod_{k=1}^N \frac{(-\log n_k)^{e_k}}{n_k{}^{t_k}} \tag{5}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \mathrm{Li}_{s_j}^{(d_j)}\left(e^{i\theta}\right) \prod_{k=1}^N \mathrm{Li}_{t_k}^{(e_k)}\left(e^{-i\theta}\right) \mathrm{d}\theta, \tag{6}$$

$$= \frac{1}{\pi}\,\mathrm{Re}\int_0^{\pi} \prod_{j=1}^M \mathrm{Li}_{s_j}^{(d_j)}\left(e^{i\theta}\right) \prod_{k=1}^N \mathrm{Li}_{t_k}^{(e_k)}\left(e^{-i\theta}\right) \mathrm{d}\theta.$$

Here $\mathrm{Li}_s^{(d)}(z) := \left(\frac{\partial}{\partial s}\right)^d \mathrm{Li}_s(z)$. Thus, effective computation of (6) requires robust and efficient methods for computing $\mathrm{Li}_s^{(d)}(e^{i\theta})$ [1,2].

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Character L-series and polylogs

We use real character L-series (§27.8 of the DLMF), $\mathrm{L}_{\pm d}$, for $d \geq 1$. These are based on real multiplicative characters χ *modulo* d, which we denote $\chi_{\pm d}$ for $\chi(d-1) = \pm 1$. Then, $\chi_{\pm d}(k) = \pm 1$ when $(k,d) = 1$, zero otherwise (d without sign, denotes $|d|$).
For integer $d \geq 3$:

$$\mathrm{L}_{\pm d}(s) := \sum_{n>0} \frac{\chi_{\pm d}(n)}{n^s}. \qquad (7)$$

Hence, for $m = 0, 1, 2, \ldots$, and $s \neq 1$ we have

$$\mathrm{L}_{\pm d}^{(m)}(s) = \frac{1}{d^s} \sum_{k=1}^{d-1} \chi_{\pm d}(k) \sum_{j=0}^{m} \binom{m}{j} (-\log d)^j \zeta^{(m-j)}\left(s, \frac{k}{d}\right). \qquad (8)$$

Here $\zeta(s,\nu) := \sum_{n\geq 0} 1/(n+\nu)^s$ is *Hurwitz zeta*; $\zeta(s,1) = \zeta(s)$.

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Character L-series and polylogs

- This allows access to numerical methods for derivatives of the Hurwitz zeta function for evaluation of quantities like $\mathrm{L}_{\pm d}^{(m)}(s)$, say with $s > 1$.
- Packages such as *Maple* have a good implementation of $\zeta^{(m)}(s, \nu)$ with respect to arbitrary complex s. [Mathematica is less reliable.]
- For later use we set $\chi_1(n) := 1, \chi_{-2}(n) := (-1)^{n-1}$. Then $\mathrm{L}_1 := \zeta$, while $\mathrm{L}_{-2} := \eta$, the alternating zeta function.

Character L-series

- A character and corresponding series are *principal* if $\chi(k) = 1$ for all k relatively prime to d. For all other characters $\sum_{k=1}^{d-1} \chi(k) = 0$, and we say the character is *balanced*. The character and series are *primitive* if not induced by a character for a proper divisor of d.
- We focus on $d = P$, $4P$ or $8P$, where P is a *product* of distinct odd primes, since only such d admit primitive characters.
- There are unique *primitive* series for 1 and each odd prime p, such as $\mathrm{L}_{-3}, \mathrm{L}_{+5}, \mathrm{L}_{-7}, \mathrm{L}_{-11}, \mathrm{L}_{+13}, \ldots$, with sign determined by remainder modulo 4, and at 4, L_{-4}, four times primes, while two occur at $8p$, e.g., $\mathrm{L}_{\pm 24}$. [Pi&AGM, BGLMW13].

Character L-series

We then obtain primitive sums for products of distinct odd primes P or $4P$, and again two at $8P$. E.g., $\mathrm{L}_{-4}, \mathrm{L}_{+12}, \mathrm{L}_{-20}, \mathrm{L}_{+60}, \mathrm{L}_{-84}$.

- In primitive cases $\chi_{\pm d}(n) := \left(\frac{\pm d}{n}\right)$, where $\left(\frac{\pm d}{n}\right)$ the generalized *Legendre-Jacobi symbol*.
- L_{-2} is a an *imprimitive* series, reducible to L_1 via η.
- $\mathrm{L}_{+6}(s) = \sum_{n>0}(1/(6n+1)^s + 1/(6n+5)^s)$ is imprimitive with all positive coefficients, while $\mathrm{L}_{-6}(s) = \sum_{n>0}(1/(6n+1)^s - 1/(6n+5)^s) = (1-1/2^s)\,\mathrm{L}_{-3}(s)$ is imprimitive but balanced, as is $\mathrm{L}_{-12}(s) = \sum_{n>0}(1/(12n+1)^s + 1/(12n+5)^s - 1/(12n+7)^s - 1/(12n+11)^s)$, which, being non-principal, has $\sum_{k=1}^{11} \chi_{-12}(k) = 0$.
- Recall that the sign determines that $\chi_{\pm d}(d-1) = \pm 1$. So $\chi_{+5}(n) = 1$ for $n = 1, 4$, and $\chi_{+5}(n) = -1$ for $n = 2, 3$.

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Character L-series

A useful integral formula (25.11.27) in [DLMF] is

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + \frac{1}{\Gamma\ (s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{x^{s-1}}{e^{ax}}\, \mathrm{d}x, \tag{9}$$

valid for $\mathrm{Re}\ s > -1, s \neq 1, \mathrm{Re}\ a > 0$; (9) implies for $d \geq 3$ that

$$\mathrm{L}_{\pm d}(s) := \frac{1}{d} \sum_{k=1}^{d-1} \chi_{\pm d}(k) \frac{k^{1-s} - 1}{s-1} + \frac{1}{2} \sum_{k=1}^{d-1} \frac{\chi_{\pm d}(k)}{k^s} \tag{10}$$

$$+ \int_0^\infty \left(\frac{x^{s-1}}{\Gamma\ (s)} \right) \left(\frac{1}{e^{dx} - 1} - \frac{1}{dx} + \frac{1}{2} \right) \sum_{k=1}^{d-1} \frac{\chi_{\pm d}(k)}{e^{kx}}\, \mathrm{d}x.$$

Character L-series

A useful integral formula (25.11.27) in [DLMF] is

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{1}{2}a^{-s} + \frac{1}{\Gamma\ (s)}\int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)\frac{x^{s-1}}{e^{ax}}\,\mathrm{d}x, \tag{9}$$

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CARMA

Character L-series

- For non-principal characters, the singularity in (9) at $s = 1$ is removable, and (10) can be used to confirm values of $\mathrm{L}_{\pm d}^{(m)}(1)$.

For $d = -3$ we have

$$\mathrm{L}_{-3}(s) = \frac{2^{1-s}-1}{3\,(1-s)} + \frac{1}{2}\left(1 - \frac{1}{2^s}\right) \tag{11}$$

$$+ \frac{2}{\Gamma\,(s)} \int_0^\infty x^{s-1} \mathrm{e}^{-3x/2} \left(\frac{1}{\mathrm{e}^{3\,x}-1} - \frac{2}{3x} + \frac{1}{2}\right) \sinh\left(\frac{x}{2}\right) \mathrm{d}x. \tag{12}$$

For $d = +5$ this simplifies to

$$\mathrm{L}_{+5}(s) = \frac{1-2^{1-s}-3^{1-s}+4^{1-s}}{5(s-1)} + \frac{\left(1-2^{-s}-3^{-s}+4^{-s}\right)}{2} \tag{13}$$

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Character L-series

Example (Primitive L-series and their derivatives at zero)

It helps to know $\zeta(0,a) = 1/2 - a, \zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$. With moment $\mu_{\pm d}(1) := \sum_{k=1}^{d-1} \chi_{\pm d}(k)k$, it then follows that $\mathrm{L}_{\pm d}(0) = \sum_{k=1}^{d-1} \binom{\pm d}{k} \zeta\left(0, \frac{k}{d}\right) = -\frac{\mu_{\pm d}(1)}{d}$, which is zero for $+d$. So

$$\mathrm{L}_{-d}(0) = \sum_{k=1}^{d-1} \binom{-d}{k} \zeta\left(0, \frac{k}{d}\right) = -\frac{\mu_{-d}(1)}{d} \text{ and } \mathrm{L}_{+d}(0) = 0, \quad (14)$$

since $\sum_{k=1}^{d-1} \chi_{\pm d}(k) = 0$ and $\sum_{k=1}^{d-1} \chi_{+d}(k)k = 0$ for primitive characters. On differentiating in (7) we have

$$\mathrm{L}_{\pm d}^{(1)}(0) = \mathrm{L}_{\pm d}(0)\log d + \sum_{k=1}^{d-1} \binom{\pm d}{k} \log\Gamma\left(\frac{k}{d}\right). \quad (15)$$

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Character L-series

- Recall, for $d > 4$, Dirichlet showed the *class number formula* for imaginary quadratic fields $-\frac{\mu_{pd}(1)}{d} = h(-d)$.

Each such primitive L-series obeys a simple functional equation of the kind known for ζ:

$$\mathrm{L}_{\pm d}(s) = C(s) \left\{ \begin{array}{c} \sin\left(s\pi/2\right) \\ \cos\left(s\pi/2\right) \end{array} \right\} \mathrm{L}_{\pm d}(1-s), \tag{16}$$

where

$$C(s) := 2^s \pi^{s-1} d^{-s+1/2} \Gamma(1-s).$$

Indeed, this is true exactly for primitive series.

CARMA

Character L-series and polylogs

Primitive series can be summed at various integer values:

$$\begin{aligned} \mathrm{L}_{\pm d}(1-2m) &= \begin{cases} (-1)^m R(2m-1)!/(2d)^{2m-1} \\ 0 \end{cases} \\ \mathrm{L}_{\pm d}(-2m) &= \begin{cases} 0 \\ (-1)^m R'(2m)!/(2d)^{2m} \end{cases} \\ \mathrm{L}_{+d}(2m) &= R d^{-1/2}\pi^{2m}, \qquad \mathrm{L}_{-d}(2m-1) = R' d^{-1/2}\pi^{2m-1}, \end{aligned} \tag{17}$$

for m a positive integer and R, R' are rationals which depend on m, d. For $d=1$ these engage Bernoulli numbers, while for $d=-4$ Euler numbers appear. Also, famously,

$$\mathrm{L}_{+p}(1) = 2\frac{h(p)}{\sqrt{p}}\log \epsilon_0, \tag{18}$$

where $h(p)$ is the class number of the quadratic form with discriminant p and ε_0 is the fundamental unit in the real quadratic field $Q(\sqrt{p})$.

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Character Polylogarithms

We now introduce *character polylogarithms*, namely,

$$\mathrm{L}_{\pm d}(s; z) := \sum_{n=1}^{\infty} \left(\frac{\pm d}{n}\right) \frac{z^n}{n^s} \tag{19}$$

$$\mathrm{L}_{\pm d}^{(m)}(s; z) := \frac{\partial^m}{\partial s^m} \mathrm{L}_{\pm d}(s; z). \tag{20}$$

These are well defined for all characters, but of primary interest for primitive ones.

- While such objects have been used before, most of the computational tools we provide appear to be new or inaccessible.
- In the sequel, one will lose very little on assuming all characters are primitive.

Character Polylogarithms

The following parametric formula holds:

$$\sum_{n=0}^{\infty} \frac{z^{(n+\nu)}}{(n+\nu)^s} = \Gamma(1-s)(-\log z)^{s-1} + \sum_{r=0}^{\infty} \zeta(s-r,\nu)\,\frac{(\log z)^r}{r!}. \tag{21}$$

Here $\zeta(s,\nu)$ is again the *Hurwitz zeta* function, $s \neq 1,2,3,\ldots$, $\nu \neq 0.-1,-2,\ldots$, and, as before, $|\log z| < 2\pi$.

- Using (21) it is possible to substantially extend our prior formulae.

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Character Polylogarithms

We derive

$$\sum_{n=0}^{\infty} \frac{z^{(dn+k+\varepsilon)}}{(dn+k+\varepsilon)^s} = \frac{1}{d}\Gamma(1-s)(-\log z)^{s-1} + \sum_{r=0}^{\infty} \zeta\left(s-r, \frac{k+\varepsilon}{d}\right) \frac{d^{r-s}(\log z)^r}{r!}. \qquad (22)$$

For $1 \le k \le d-1$, $s \ne 1, 2, 3, \ldots$, $0 < \varepsilon < 1$, *if* $\sum_{m=1}^{d-1} \binom{\pm d}{m} = 0$,

$$\sum_{n=1}^{\infty} \binom{\pm d}{n} \frac{z^{(n+\varepsilon)}}{(n+\varepsilon)^s} = \sum_{r=0}^{\infty} \left(\frac{1}{d^{s-r}} \sum_{k=1}^{d-1} \binom{\pm d}{k} \zeta\left(s-r, \frac{k+\varepsilon}{d}\right)\right) \frac{(\log z)^r}{r!}. \qquad (23)$$

This holds for all primitive and other balanced characters such as -12; then any term independent of m vanishes.

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This holds for all primitive and other balanced characters such as -12; then any term independent of m vanishes.

Character Polylogarithms

- We then obtain a tractable formula for differentiation wrt the order.

For $m = 0, 1, 2, \ldots$, we can write

$$\mathrm{L}_{\pm d}^{(m)}(s;z) := \sum_{n=1}^{\infty} \left(\frac{\pm d}{n}\right) \frac{(\log n)^m}{n^s} z^n$$

$$= \sum_{r=0}^{\infty} \frac{\partial^m}{\partial s^m} \left(\frac{1}{d^{s-r}} \sum_{k=1}^{d-1} \left(\frac{\pm d}{k}\right) \zeta\left(s-r, \frac{k}{d}\right)\right) \frac{(\log z)^r}{r!} \tag{24}$$

We can now derive the following:

CARMA

Character Polylogarithms

Theorem (L-series sums for primitive character polylogarithms)

For primitive $\pm d = -3, -4, 5, \ldots$ *and all* s *we have*

$$\mathrm{L}_{\pm d}^{(m)}(s; z) = \sum_{r=0}^{\infty} \mathrm{L}_{\pm d}^{(m)}(s - r) \frac{(\log z)^r}{r!} \tag{25}$$

when $|\log z| < 2\pi/d$.

- Now, however, unlike the case for ζ, this is also applicable at $s = 1, 2, 3, \ldots$ (since the poles at $s = 1, 2, \ldots$ cancel).

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- By contrast, integral (6), is less attractive since it cannot be applied (to the real part) on the full range $[0, \pi]$.

It does, however, lead to two attractive Clausen-like Fourier series

$$\sum_{n=1}^{\infty} \chi_{\pm d}(n) \frac{\cos n\theta}{n^s} = \sum_{r=0}^{\infty} \mathrm{L}_{\pm d}^{(m)}(s-2r) \frac{(-1)^r \theta^{2r}}{(2r)!} \tag{26a}$$

$$\sum_{n=1}^{\infty} \chi_{\pm d}(n) \frac{\sin n\theta}{n^s} = \sum_{r=0}^{\infty} \mathrm{L}_{\pm d}^{(m)}(s-2r+1) \frac{(-1)^r \theta^{2r-1}}{(2r-1)!} \tag{26b}$$

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Character Polylogarithms

- To employ (25) for non-negative integer order s, we must evaluate $\mathrm{L}_{\pm d}^{(m)}(-n)$ at negative integers.
- This can be achieved from the functional equation (16) by methods of Apostol.

We begin for primitive $d = 1, 2, \ldots,$ with (16), which we write as:

$$\sqrt{d}\,\mathrm{L}_{\pm d}(1-s) = \Psi_{\pm d}(s)\,\mathrm{L}_{\pm d}(s), \qquad \Psi_{\pm d}(s) := \left(\frac{d}{2\pi}\right)^s \left\{ \begin{array}{c} 2\,\mathrm{Re}\,e^{i\pi s/2} \\ 2\,\mathrm{Im}\,e^{i\pi s/2} \end{array} \right\}$$

Then for real s and $\kappa_d := -\log\frac{2\pi}{d} + \frac{1}{2}\pi i$ write:

$$\sqrt{d}\,\mathrm{L}_{+d}(1-s) = \left(\mathrm{Re}\ 2e^{s\kappa_d}\right)\Gamma(s)\,\mathrm{L}_{+d}(s), \qquad (27a)$$

$$\sqrt{d}\,\mathrm{L}_{-d}(1-s) = \left(\mathrm{Im}\ 2e^{s\kappa_d}\right)\Gamma(s)\,\mathrm{L}_{-d}(s). \qquad (27b)$$

Leibnitz' formula for n-fold differentiation wrt to s leads to:

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Leibnitz' formula for n-fold differentiation wrt to s leads to:

Character Polylogarithms

Theorem (L-series derivatives at negative integers)

Let $\mathrm{L}_{\pm d}$ *be a primitive non-principal* L*-series. For all* $n \geq 1$,

$$\mathrm{L}_{+d}^{(m)}(1-2n) = \frac{(-1)^{m+n} d^{2n-1/2}}{2^{2n-1}\pi^{2n}} \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (\mathrm{Re}\ \kappa_d^j) \Gamma^{(k-j)}(2n)\, \mathrm{L}_{+d}^{(m-k)}(2n) \quad (28a)$$

$$\mathrm{L}_{+d}^{(m)}(2-2n) = \frac{(-1)^{m+n} d^{2n-3/2}}{2^{2n-2}\pi^{2n-1}} \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (\mathrm{Im}\ \kappa_d^j) \Gamma^{(k-j)}(2n-1)\, \mathrm{L}_{+d}^{(m-k)}(2n-1) \quad (28b)$$

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$$(\kappa_d = -\log\frac{2\pi}{d} + \frac{1}{2}\pi i)$$

Character Polylogarithms

Since $j > 0$ is integer, $\mathrm{Re}\ \kappa_d^j$ and $\mathrm{Im}\ \kappa_d^j$ can be expanded. As $\Gamma^{(m)}(n) \approx \log^m(n)\Gamma(n)$, for $\mathrm{L}_{\pm d}$ a primitive non-principal L-series:

Corollary (Positive L-series derivative asymptotics)

For all integers $m \geq 0$, as $n \to +\infty$ we have

$$\frac{\mathrm{L}_{+d}^{(m)}(1-2n)}{(2n-1)!} \approx 2\frac{(-1)^{m+n}d^{2n-1/2}}{(2\pi)^{2n}}\,\mathrm{Re}\left(\frac{\pi i}{2}+\log\left(\frac{(2n)d}{2\pi}\right)\right)^m \tag{29a}$$

$$\frac{\mathrm{L}_{+d}^{(m)}(2-2n)}{(2n-2)!} \approx 2\frac{(-1)^{m+n}d^{2n-3/2}}{(2\pi)^{2n-1}}\,\mathrm{Im}\left(\frac{\pi i}{2}+\log\left(\frac{(2n-1)d}{2\pi}\right)\right)^m \tag{29b}$$

Character Polylogarithms

Corollary (Negative L-series derivative asymptotics)

$$\frac{\mathrm{L}_{-d}^{(m)}(1-2n)}{(2n-1)!} \approx 2\frac{(-1)^{m+n}d^{2n-1/2}}{(2\pi)^{2n}}\,\mathrm{Im}\left(\frac{\pi i}{2}+\log\left(\frac{(2n)d}{2\pi}\right)\right)^{m} \tag{30a}$$

$$\frac{\mathrm{L}_{-d}^{(m)}(2-2n)}{(2n-2)!} \approx 2\frac{(-1)^{m+n+1}d^{2n-3/2}}{(2\pi)^{2n-1}}\,\mathrm{Re}\left(\frac{\pi i}{2}+\log\left(\frac{(2n-1)d}{2\pi}\right)\right)^{m}. \tag{30b}$$

Character Polylogarithms

- One may use Stirling's approximation to remove the factorial.
  - For modest n this asymptotic allows an excellent estimate of the size of derivative. For instance,

$$\frac{\mathrm{L}_5^{(3)}(-98)}{98!} = -1.157053952 \cdot 10^{-8}\ldots$$

— while the asymptotic gives $-1.159214401 \cdot 10^{-8}\ldots$.
Similarly

$$\frac{\mathrm{L}_{-3}^{(5)}(-38)}{38!} - 1.078874094 \cdot 10^{-10}\ldots,$$

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  - These are the type of terms we need to compute below.
- Taking n-th roots in Corollary 4 shows that the radius of convergence in Theorem 2 is as given.

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- Taking n-th roots in Corollary 4 shows that the radius of convergence in Theorem 2 is as given.

Character Polylogarithms

- We also observe that $\left(\frac{\pi^2}{4}+\log^2\left(\frac{nd}{\pi}\right)\right)^{m/2}$ provides a useful upper bound for each real and imaginary part in Corollary 4.
  - For example,

$$\sqrt{\left(\frac{\mathrm{L}_{-d}^{(m)}(1-2n)}{(2n-1)!}\right)^2+\left(\frac{\mathrm{L}_{+d}^{(m)}(1-2n)}{(2n-1)!}\right)^2} \approx \frac{2}{\sqrt{d}}\left(\frac{\pi^2}{4}+\log^2\left(\frac{nd}{\pi}\right)\right)^{m/2}\left(\frac{d}{2\pi}\right)^{2n}.$$

All physicists and a good many quite respectable mathematicians are contemptuous about proof. (G. H. Hardy, 1877-1947)

Character Polylogarithms

- We also observe that $\left(\frac{\pi^2}{4}+\log^2\left(\frac{nd}{\pi}\right)\right)^{m/2}$ provides a useful upper bound for each real and imaginary part in Corollary 4.
  - For example,

$$\sqrt{\left(\frac{\mathrm{L}_{-d}^{(m)}(1-2n)}{(2n-1)!}\right)^2+\left(\frac{\mathrm{L}_{+d}^{(m)}(1-2n)}{(2n-1)!}\right)^2} \approx \frac{2}{\sqrt{d}}\left(\frac{\pi^2}{4}+\log^2\left(\frac{nd}{\pi}\right)\right)^{m/2}\left(\frac{d}{2\pi}\right)^{2n}.$$

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Multisectioning Character Polylogarithms

- All character polylogarithms obey the general rule

$$\mathrm{L}_{\pm d}(s;x) = \int_0^x \frac{\mathrm{L}_{\pm d}(s-1;y)}{y}\,\mathrm{d}y,$$

and, in particular, $Li_n(1) = \zeta(n)$, $\mathrm{Li}_n(-1) = -\eta(n)$, and $\mathrm{Ti}_n(1) = \beta(n)$.

Moreover, as is sometimes advantageous, 'multi-sectioning' (demideation) allows us to write all of our character polylogarithms in terms of the classical one.

Multisectioning Character Polylogarithms

Recall that for integer $d > 0$, given a formal power series

$$g(z) = \sum_{n \geq 0} a_n z^n,$$

one may algebraically extract the function

$$g_{d,q}(z) := \sum_{n \geq 0} a_{nd+q} z^{nd+q},$$

for $0 \leq q \leq d-1$ by by the *multi-sectioning* formula

$$g_{d,q}(z) = \frac{1}{d} \sum_{m=0}^{d-1} \omega_d^{-mq} g(\omega_d^m z), \qquad \omega_d = e^{2\pi i/d}.$$

Applying this to the polylogarithm of order t, we arrive at:

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Multisectioning Character Polylogarithms

Theorem (Multi-sectioning for Hurwitz zeta and char. polylog)

For order t *and integers* q, d *with* $0 \leq q \leq d-1$, *set* $\omega_d = e^{2\pi i/d}$. *Then*

$$\sum_{k=1}^{\infty} \frac{x^{dk+q}}{(dk+q)^t} = \frac{1}{d} \sum_{m=0}^{d-1} \omega_d^{-mq} \operatorname{Li}_t(\omega_d^m x), \tag{31}$$

and so with a Gauss sum $\gamma_{\pm d}(m) := \frac{1}{d} \sum_{q=1}^{d-1} \chi_{\pm d}(q) \omega_d^{-mq}$,

$$\operatorname{L}_{\pm d}(t; x) = \sum_{m=0}^{d-1} \gamma_{\pm d}(m) \operatorname{Li}_t(\omega_d^m x). \tag{32}$$

Multisectioning Character Polylogarithms

Example (Examples of multi-sectioning)

We directly computed $\gamma_{\pm d}$, in (32), for $d = -3, d = +5, d = \pm 8$, and $d = +12$.

We get $\sqrt{-3}\gamma_3(m) = \chi_{-3}(m)$, $\sqrt{5}\gamma_{+5}(m) = \chi_{+5}(m)$.

For $d = +8$, we have $\sqrt{8}\gamma_{+8}(m) = \chi_{+8}(m)$ and for $d = -8$ we obtain $\sqrt{-8}\gamma_{-8}(m) = \chi_{+8}(m)$. Finally for $d = +12$ we again have $\sqrt{12}\gamma_{+12}(m) = \chi_{+12}(m)$.

- From this we rediscover the closed form $\gamma_{\pm d}(m) = \frac{\chi_{\pm d}(m)}{\sqrt{\pm d}}$ for primitive characters. In Apostol—explicitly for primes and implicitly generally—we find the requisite proof.
- Of course, for any given small $\pm d$ we can verify it directly. The formula fails for imprimitive forms.

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Multisectioning Character Polylogarithms

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Multisectioning Character Polylogarithms

Thus,with $\omega_d = e^{2\pi i/d}$ we have:

Corollary (Primitive character polylogarithms)

For a primitive character $\chi_{\pm d}$, non-negative m, and all s, we have

$$\mathrm{L}_{\pm d}^{(m)}(s;x) = \frac{\sqrt{\pm d}}{d} \sum_{k=1}^{d-1} \chi_{\pm d}(k)\, \mathrm{Li}_s^{(m)}(\omega_d^k x), \tag{33}$$

valid for $\max_k |\log(x\omega_d^k)| < 2\pi$. On the unit disk we obtain

$$\mathrm{L}_{\pm d}^{(m)}\left(s; e^{i\theta}\right) = \frac{\sqrt{\pm d}}{d} \sum_{k=1}^{d-1} \chi_{\pm d}(k)\, \mathrm{Li}_s^{(m)}\left(e^{i(\theta+2k\pi/d)}\right), \tag{34}$$

valid for all θ not equal to $2k\pi/d$ for any $k = 1, \ldots, d-1$.

Multisectioning Character Polylogarithms

Example (Explicit polylogarithms for small d ($d=-2$))

For $d=-2$, and t arbitrary we write

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}x^m}{m^t} =: \mathrm{L}_{-2}(t;x) := \eta(t;x) = -\,\mathrm{L}_{+1}(t;-x) = -\,\mathrm{Li}_t(-x), \tag{35}$$

since (25) of Theorem 2 holds for any balanced character.

More significantly:

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Multisectioning Character Polylogarithms

Example (Explicit polylogarithms for small d ($d = -3, -4$))

For $d = -3$ with $\tau := (-1 + i\sqrt{3})/2$, we have

$$\sum_{m=1}^{\infty} \frac{x^{3m-2}}{(3m-2)^t} - \sum_{m=1}^{\infty} \frac{x^{3m-1}}{(3m-1)^t} = \mathrm{L}_{-3}(t;x) = \frac{2}{\sqrt{3}} \,\mathrm{Im}\; \mathrm{Li}_t(\tau x), \tag{36}$$

while for $d = -4$,

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^{2m-1}}{(2m-1)^t} =: \beta(t;x) = \mathrm{L}_{-4}(t;x) = \mathrm{Ti}_t(x). \tag{37}$$

Multisectioning Character Polylogarithms

- It is useful to know [DLMF] that for $\mathrm{Re}\ s > 0$, we have

$$-\mathrm{L}_{-4}(s) = \beta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{2\cosh(x)}\, \mathrm{d}x, \qquad (38)$$

  which may be repeatedly differentiated to obtain numerical values of $\beta^{(n)}(s)$ for integers $n \geq 1$.
- Herein, $\mathrm{Ti}_t(x)$ is the inverse tangent integral of Lewin
  - that he relates to Legendre's chi-function, confusingly also denoted as $\chi_t(x)$.
- Note that $\mathrm{Li}_1(x) = -\log(1-x)$, while $\mathrm{Ti}_1(x) = \arctan(x)$.

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Multisectioning Character Polylogarithms

- The equation (34) may be used to exploit character generalizations of (5) and (6).
- Note that (28b) and (28d) for $n = 1$, express the derivatives at zero in terms of the derivative and values at one.
  - While the quantities are all finite, recall that the Hurwitz form in (7) involves a cancellation of singularities, and so is hard to use directly, while the definition is very slowly convergent at $s = 1$ or near one.
- We do, however, have recourse to a useful special case of the last Corollary.

Multisectioning Character Polylogarithms

Now we may usefully employ the Corollary at roots of unity.

Example (L-series at unity ($\omega_d = e^{2\pi i/d}$))

For any primitive character $\chi_{\pm d}$ and non-negative m we have

$$\mathrm{L}_{\pm d}^{(m)}(s) = \frac{\sqrt{\pm d}}{d} \sum_{k=1}^{d-1} \chi_{\pm d}(k)\, \mathrm{Li}_s^{(m)}(\omega_d^k). \tag{39}$$

- Polylogarithms, and order derivatives $\mathrm{Li}_s^{(m)}(\exp(i\theta))$, were studied [1], as they resolve Eulerian log Gamma integrals.

Multisectioning Character Polylogarithms

Example (Symbolic recovery of values)

The Hurwitz L-series derivative with local notation $\lambda(m, \pm d, s) := \mathrm{L}_{\pm d}^{(m)}(s)$ in (7) implements neatly in *Maple*. We use the 'identify' function and—after a little prettification—have evaluations given in (17):

$$\left[\lambda\left(0,-4,-3\right)=\frac{1}{32}\,\pi^3,\ \lambda\left(0,-3,5\right)=\frac{4\sqrt{3}}{2187}\pi^5,\lambda\left(0,-4,5\right)=\frac{5}{1536}\,\pi^5\right];$$

and first-derivative (algebraic unit) values at zero:

$$\left[\lambda\left(1,5,0\right)=\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right),\ \lambda\left(1,13,0\right)=\log\left(\frac{3}{2}+\frac{1}{2}\sqrt{13}\right),\ \lambda\left(1,17,0\right)=\log\left(4+\sqrt{17}\right)\right],$$

(as per Denninger and Berndt).

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Multisectioning Character Polylogarithms

- The ease of such manipulations highlights the value of modern numeric-symbolic experimentation.
- One may similarly use (39) when $s = 1$.
- Interestingly using 'sum' rather than 'add' in *Maple* led to some problems with larger values of $\pm 8P$ such as ± 120.

> *I have myself always thought of a mathematician as in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. (G.H. Hardy)*

Applications to Character Sums

- On this foundation, one may then analyse extended *character MTW sums*, in which more general character polylogarithms replace the classical one defined earlier in (6).

That is, we may consider, for real $q, r, s \geq 1$,

$$\mu_{\pm d_1, \pm d_2}(q, r, s) := \sum_{n,m>0} \frac{\chi_{\pm d_1}(m)}{m^q} \frac{\chi_{\pm d_2}(n)}{n^r} \frac{1}{(m+n)^s} \tag{40}$$

$$= \frac{1}{\Gamma(s)} \int_0^1 \mathrm{L}_{\pm d_1}(q;x)\, \mathrm{L}_{\pm d_2}(r;x) (-\log x)^{s-1} \frac{\mathrm{d}x}{x}, \tag{41}$$

where as before for $d > 2$, $\chi_{\pm d}(n) := \left(\frac{\pm d}{n}\right)$, while

$\chi_{-2}(n) := (-1)^{n-1}$ and $\chi_{+1}(n) := 1$.

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Logarithmic Character Sums

- We may now also take derivatives in (40) and (41) and so doing is the source of much of our computational interest.

Explicitly, we write $\left(\mu_{\pm d_1,\pm d_2}\right)_{a,b,c}(q,r,s)$

$$:= \sum_{n,m>0} \frac{(-\log m)^a \chi_{\pm d_1}(m)}{m^q} \frac{(-\log n)^b \chi_{\pm d_2}(n)}{n^r} \frac{(-\log(m+n))^c}{(m+n)^s} \tag{42}$$

$$= \int_0^1 \mathrm{L}^{(a)}_{\pm d_1}(q;x)\, \mathrm{L}^{(b)}_{\pm d_2}(r;x) \left(\frac{(-\log x)^{s-1}}{\Gamma(s)}\right)^{(c)} \frac{\mathrm{d}x}{x}. \tag{43}$$

- Such sums do not appear to have been studied in detail, and never with derivatives.
  - The case of $\chi_{-2}(n)$ or $\chi_{-2}(m+n)$ was studied *ab initio* by Tsimura.

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Examples of Character Sums

- As explained in [BZB], for Euler sums, there is an impediment to a general integral representation if one adds a non-trivial character to the $m+n$ variable other than $(\pm 1)^{n-1}$.
- In the context of MTWs, this asymmetry is better explained.
  - The change of variables $m \mapsto m+n$ does not respect the mutiplicative structure

> *Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.* (W.S. (Bill) Angelin)

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Examples of Character Polylogarithms

Example (Some explicit character polylogs of order one)

$$\mathrm{L}_{+1}(1;x) = -\log(1-x) \tag{44}$$

$$\mathrm{L}_{-3}(1;x) = \frac{2}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}\,x}{x+2}\right), \tag{45}$$

$$\sqrt{5}\;\mathrm{L}_{5}(1;x) = \log(x^2+\omega x+1) - \log(x^2 - x/\omega + 1), \tag{46}$$

$$\omega := \frac{\sqrt{5}+1}{2}$$

$$\sqrt{12}\;\mathrm{L}_{12}(1;x) = \log(x^2+\sqrt{3}\,x+1) - \log(x^2-\sqrt{3}\,x+1). \tag{47}$$

Further Examples of Character Polylogarithms

In general for primitive $\pm d$, (39) of Corollary 8 implies that

$$\mathrm{L}_{\pm d}(1;x) = -\frac{\sqrt{\pm d}}{d}\log\left(\frac{\prod_j\left(1-\omega_d^j x\right) : \chi_{\pm d}(j)=+1}{\prod_k\left(1-\omega_d^k x\right) : \chi_{\pm d}(k)=-1}\right). \quad (48)$$

It is instructive to verify that

$$\sqrt{8}\,\mathrm{L}_{+8}(1;x) = -\log\left(\frac{1-\sqrt{2}x+x^2}{1+\sqrt{2}x+x^2}\right), \quad (49a)$$

$$\sqrt{8}\,\mathrm{L}_{-8}(1;x) = \arctan\left(\sqrt{8}x\left(1-x^2\right), 1-4x^2+x^4\right). \quad (49b)$$

Here $\arctan(y,x) := -i\log\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right)$, so as to assure we return a value in $(\pi,\pi]$.

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Further Examples of Character Polylogarithms

Correspondingly

$$\sqrt{20}\,\mathrm{L}_{-20}(1;x) = i\log\left(\frac{1-i\sqrt{5}x-3\,x^2+i\sqrt{5}x^3+x^4}{1+i\sqrt{5}x-3\,x^2-i\sqrt{5}x^3+x^4}\right). \quad (50)$$

> *Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it (Sir Alec Cairncross, in the Economist, April 20, 1996)*

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Further Examples of Character Polylogarithms

Recall the character sum definition

$$\left(\mu_{\pm d_1,\pm d_2}\right)_{a,b,c}(q,r,s) :=$$

$$\sum_{n,m>0} \chi_{\pm d_1}(m)\frac{(-\log m)^a}{m^q}\,\chi_{\pm d_2}(n)\frac{(-\log n)^b}{n^r}\,\frac{(-\log(m+n))^c}{(m+n)^s} \qquad (51)$$

- q, r, s are the powers of the denominator requested.
- a, b, c are the powers of the logarithm requested.

Further Examples of Character Polylogarithms

From various formulas above, integrals for μ sums in the notation of (40) or (51) follow. Thence, $\mu_{-3,1}(1,1,s)$

$$= \frac{2/\sqrt{3}}{\Gamma(s)} \int_0^1 \arctan\left(\frac{\sqrt{3}\,x}{x+2}\right) \left(-\log(1-x)\right) \left(-\log x\right)^{s-1} \frac{\mathrm{d}x}{x}, \tag{52}$$

and

$$\mu_{-3,-3}(1,1,s) = \frac{4/3}{\Gamma(s)} \int_0^1 \arctan^2\left(\frac{\sqrt{3}\,x}{x+2}\right) \left(-\log x\right)^{s-1} \frac{\mathrm{d}x}{x}. \tag{53}$$

For example,

$$\mu_{-3,-3}(1,1,1) \approx 0.259589$$
$$\mu_{-3,-3}(1,1,3) \approx 0.0936667862.$$

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Further Examples of Character Polylogarithms

Similarly,

$$\mu_{-12,-12}(1,1,3) = -\frac{1}{72}\int_0^1 \log^2\left(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}\right)\log^3(x)\,\frac{\mathrm{d}x}{x}$$
$$= 0.062139235322359770447911814351\ldots \quad \textbf{(54)}$$

and, with $\omega = \frac{\sqrt{5}+1}{2}$ as above, we have

$$\mu_{+5,+5}(1,1,5) = \frac{1}{120}\int_0^1 \log^2\left(\frac{x^2+\omega x+1}{x^2-x/\omega+1}\right)\log^4(x)\,\frac{\mathrm{d}x}{x}$$
$$= 0.026975379493214862581276332615\ldots \quad \textbf{(55)}$$

- Polylogarithms and Euler sums based primarily on mixes of the characters χ_{-4} and χ_1 are studied at length in [BZB]. For higher order, less can be hoped for explicitly. That said:

Further Examples of Character Polylogarithms

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Further Examples of Character Polylogarithms

Example (Some explicit character polylogarithms of order two)

Lewin shows in terms of the *Clausen function*, $\mathrm{Cl}_2(\theta) := \sum_{n>0} \sin(n\theta)/n^2$, we have:

$$\mathrm{L}_{-3}(2;x) = \frac{1}{2}\,\mathrm{Cl}_2(2w) + \frac{1}{2}\,\mathrm{Cl}_2\left(\frac{4\pi}{3}\right) - \frac{1}{2}\,\mathrm{Cl}_2\left(2w + \frac{4\pi}{3}\right) + w\log x, \quad \text{where } w := \arctan\left(\frac{\sqrt{3}x}{x+2}\right). \tag{56}$$

- The paucity of results for L_{-4} shows the terrain we enter is rocky.

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Further Examples of Character Polylogarithms

There is, nonetheless, a functional equation for $\mathrm{Ti}_2(x) = \mathrm{L}_{-4}(2; x) = \mathrm{Im}\ \mathrm{Li}_2(ix)$:

Example (Some explicit character polylogarithms of order two)

$$\frac{1}{3}\mathrm{Ti}_2(\tan 3\theta) = \mathrm{Ti}_2(\tan\theta) + \mathrm{Ti}_2(\tan(\pi/6 - \theta)) - \mathrm{Ti}_2(\tan(\pi/6+\theta)) + \frac{\pi}{6}\log\left(\frac{\tan(\pi/6+\theta)}{\tan(\pi/6-\theta)}\right). \quad (57)$$

Since $\mathrm{Ti}_2(\pi/4) = \mathrm{G}$, Catalan's constant, (57) gives Ramanujan's:

$$\mathrm{Ti}_2\left(\frac{\pi}{12}\right) = \frac{2}{3}\mathrm{G} + \frac{\pi}{12}\log\tan\left(\frac{\pi}{12}\right)$$

(used for computation); $\theta = \pi/24$ yields an interesting relation.

Further Examples of Character Polylogarithms

Example (Some explicit character polylogarithms of order two)

For $d = +5$ we obtain

$$\sqrt{5}\,\mathrm{L}_{+5}(2;x) = \int_0^x \log\left(\frac{1 + r\left(\frac{1+\sqrt{5}}{2}\right) + r^2}{1 + r\left(\frac{1-\sqrt{5}}{2}\right) + r^2}\right)\frac{\mathrm{d}r}{r}, \qquad (58)$$

by integration or by exploiting

$$\mathrm{Re}\ \mathrm{Li}_2(re^{i\theta}) = -\frac{1}{2}\int_0^r \log(1 - 2w\cos\theta + w^2)\frac{\mathrm{d}w}{w}.$$

For larger $\pm d$, more cumbersome versions of some of the above formulas can still be given.

Applications to Character MTW Sums

- Integral representation (6) is valid only when $d \le 2$, and all s_j, t_k numerator (non-logarithmic) parameters are non-zero; so we must attend to such more general degenerate cases.

For our current three-variable sums, we may freely use formulas such as: $\omega_{a,b,c}(q, r, s) =$

$$\omega\left(\begin{array}{cc|c} q\ , & r & s \\ a\ , & b & c \end{array}\right) = \int_0^\infty \left(\frac{x^{s-1}}{\Gamma(s)}\right)^{(c)} \mathrm{Li}_q^{(a)}(e^{-x})\,\mathrm{Li}_r^{(b)}(e^{-x})\,\mathrm{d}x. \tag{59}$$

This is valid when $q \ge 0, r \ge 0, s > 0$, with $q + r + s > 2$, and $a \ge 0, b \ge 0, c \ge 0$. Here the notation $(\cdot)^{(c)}$ denotes the c-th partial derivative of the expression in parentheses with respect to s.

Applications to Character MTW Sums

Split the integral in two, and set $u = e^{-x}$ in the second integral:

$$\omega_{a,b,c}(q,r,s) = \int_0^1 \left(\frac{x^{s-1}}{\Gamma(s)}\right)^{(c)} \mathrm{Li}_q^{(a)}(e^{-x})\,\mathrm{Li}_r^{(b)}(e^{-x})\,\mathrm{d}x + \int_0^{1/e} \left(\frac{(-\log u)^{s-1}}{\Gamma(s)}\right)^{(c)} \mathrm{Li}_q^{(a)}(u)\,\mathrm{Li}_r^{(b)}(u)\,\frac{\mathrm{d}u}{u}. \qquad (60)$$

We were able to use formula (60) and its predecessors—with related machinery described in [1,2] to produce high-precision numerical values of all the degenerate omega constants needed in this and our earlier studies.

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Applications to Character MTW Sums

Alternatively, for ω or μ, one may directly substitute $u = e^{-x}$ in the analogue of formula (59) and obtain the following result, which provides an efficient evaluation method.

For this we require the *incomplete Gamma function*

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t}\, \mathrm{d}t, \tag{61}$$

so that $\Gamma(s, 0) = \Gamma(s)$. Since the size of d determines the domain of validity of (25), we replace e by a general parameter $\sigma > 1$.

Applications to Character MTW Sums

Fix character series $L_1 := \mathrm{L}_{\pm d_1}$ and $L_2 := \mathrm{L}_{\pm d_2}$.

Proposition (Depth three character sum computation)

For $q \geq 0, r \geq 0, s > 0$, with $q + r + s > 2$, and $a \geq 0, b \geq 0, c \geq 0$, in notation of (42) *we have, for $\sigma > 1$ that* $(\mu_{d_1,d_2})_{a,b,c}(q,r,s)$

$$= \int_0^{1/\sigma} \left(\frac{(-\log u)^{s-1}}{\Gamma(s)}\right)^{(c)} {L_1}^{(a)}(q;u)\, {L_2}^{(b)}(r;u) \frac{\mathrm{d}u}{u}$$
$$+ \int_{1/\sigma}^{1} \left(\frac{(-\log u)^{s-1}}{\Gamma(s)}\right)^{(c)} {L_1}^{(a)}(q;u)\, {L_2}^{(b)}(r;u) \frac{\mathrm{d}u}{u}. \qquad (62)$$

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Applications to Character MTW Sums

Proposition (Depth three character sum computation)

Thence, $(\mu_{d_1,d_2})_{a,b,c}(q,r,s)$

$$= \sum_{m,n>0} \left(\frac{\Gamma(s,(m+n)\log\sigma)}{\Gamma(s)(m+n)^s} \right)^{(c)} \tag{63}$$

$$\times \frac{\chi_{\pm d_1}(m)(-\log m)^a}{m^q} \frac{\chi_{\pm d_2}(n)(-\log n)^b}{n^r}$$

$$+ \int_{1/\sigma}^{1} \left(\frac{(-\log u)^{s-1}}{\Gamma(s)} \right)^{(c)} L_1{}^{(a)}(q;u)\, L_2{}^{(b)}(r;u) \frac{\mathrm{d}u}{u}. \tag{64}$$

In (64) we express the result in terms of the incomplete Gamma function of (61):

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Applications to Character MTW Sums

When this applies to both L_1, L_2 and $1/\sigma \leq \exp(-2\pi/d)$ for each character, we arrive at effective integral free summations.

Theorem (Explicit character sum computation)

Suppose L_1 and L_2 satisfy Theorem 2. For $q \geq 0, r \geq 0, s > 0$, with $q + r + s > 2$, and $a \geq 0, b \geq 0, c \geq 0$ we have,

$$(\mu_{\pm d_1, \pm d_2})_{a,b,c}(q, r, s)$$

$$= \sum_{m,n>0} \left(\frac{\Gamma(s, (m+n)\log\sigma)}{\Gamma(s)(m+n)^s} \right)^{(c)} \chi_{\pm d_1}(m)\chi_{\pm d_2}(n) \frac{(-\log m)^a}{m^q} \frac{(-\log n)^b}{n^r}$$

$$+ \sum_{j,k \geq 0} \frac{L_1^{(a)}(q-j)}{j!} \frac{{L_2}^{(b)}(r-k)}{k!} \int_{1/\sigma}^{1} \left(\frac{(-\log u)^{s-1}}{\Gamma(s)} \right)^{(c)} (\log u)^{j+k} \frac{\mathrm{d}u}{u}, \qquad (65)$$

where the final integral may now be evaluated symbolically, since

$$\int_{1/\sigma}^{1} \frac{\log^{n-1} u}{u} du = -\frac{(-\log)^n \sigma}{n}.$$

Applications to Character MTW Sums

- Note that $\sigma = e$ may be used when neither of d_1, d_2 exceeds six.
- In general, to determine the truncation needed in the final term (65), we have proceeded by precomputing the needed L-series and using only those summands which are larger than the desired error.
- Corollary 4 provides excellent estimates for these L-series terms.
- For truncation of the first term on the right of (65), the next remark yields an effective *a priori* estimate (when $c = 0$) which decays exponentially in z.

Applications to Character MTW Sums

Remark (Error estimates for $\Gamma(s,z)$)

For fixed positive integer n and real s, with

$$u_k = (-1)^k (1-a)_k = (a-1)(a-2)\cdots(a-k),$$

we have [DLMF] that

$$\Gamma(s,z) = z^{s-1}e^{-z}\left(\sum_{k=0}^{n-1} \frac{u_k}{z^k} + R_n(s,z)\right), \qquad (66)$$

where for real z the error $R_n(s,z) = O(z^{-n})$, is is bounded in absolute value by the first neglected term u_n/z^n and has the same sign provided only that $n \geq s-1$.

Applications to Character MTW Sums

As Crandall observed, for $L_1 = L_2 = \mathrm{L}_{-2}$, some seemingly more difficult character sums can be computed easily:

Example (Alternating MTWs)

For example, $\mathrm{L}_{-2}(z,s) = \sum_{m \geq 0} \eta(s-m) \frac{\log^m z}{m!}$ and we may write $(\mu_{-2,-2})_{1,1,0}(q,r,s)$

$$= \sum_{n,m>0} \left(\frac{\Gamma(s, n+m)}{\Gamma(s)(n+m)^s} \right) \frac{(-1)^n \log n}{n^q} \frac{(-1)^m \log m}{m^r}$$

$$+ \frac{1}{\Gamma(s)} \sum_{j,k \geq 0} \frac{\eta^{(1)}(q-j)}{j!} \frac{\eta^{(1)}(r-k)}{k!} \frac{(-1)^{j+k}}{j+k+s}. \quad (67)$$

For positive integer s, the incomplete Gamma function value above is elementary. Using (65) of Theorem 17 with $q = r = s = 1$ and summing say $m, n, j, k \leq 240$, yields

$$(\mu_{-2,-2})_{0,0,0}\,(1,1,1) := \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{mn(m+n)} \tag{68}$$

$$= 0.30051422578989857134993454037786249769124657308512 47\ldots$$

agreeing with $(\mu_{-2,-2,0})_{0,0,0}\,(1,1,1) = \frac{1}{4}\zeta(3)$, a known evaluation. Likewise, using the first derivative of the η function,

$$(\mu_{-2,-2})_{1,1,0}\,(1,1,1) := \sum_{m,n\geq 1} (-1)^{m+n} \frac{\log m\, \log n}{m\,n(m+n)} \tag{69}$$

$$= 0.0084654591832435660002204654836228807098258834876951\ldots$$

Both evaluations are correct to the precision shown.

Applications to Character MTW Sums

For primitive characters with $3 \leq d_1, d_2 \leq 5$, we have
$\left(\mu_{\pm d_1, \pm d_2}\right)_{a,b,0}(q, r, s)$

$$= \sum_{m,n \geq 1} \chi_{\pm d_1}(m) \chi_{\pm d_2}(n) \frac{(-\log m)^a \, (-\log n)^b}{m^r \, n^q (m+n)^s}$$

$$= \sum_{n,m>0} \left(\frac{\Gamma(s, n+m)}{\Gamma(s)\,(n+m)^s} \right) \frac{\chi_{\pm d_1}(m)(-\log m)^a}{m^q} \frac{\chi_{\pm d_2} n(-\log n)^b}{n^r}$$

$$+ \frac{1}{\Gamma(s)} \sum_{j,k \geq 0} \frac{\mathrm{L}^{(a)}_{\pm d_1}(q-j)}{j!} \frac{\mathrm{L}^{(b)}_{\pm d_2}(r-k)}{k!} \frac{(-1)^{j+k}}{j+k+s}, \qquad (70)$$

in analogy with the previous Example.

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Applications to Character MTW Sums

Example (Character MTWs)

For $d = -4$ with $\beta := \mathrm{L}_{-4}$ replacing $\eta := \mathrm{L}_{-2}$ we get:

$(\mu_{-4,-4})_{1,1,0}\,(q,r,s)$

$$= \sum_{n,m>0} \left(\frac{\Gamma\left(s, n+m\right)}{\Gamma\left(s\right)\left(n+m\right)^{s}} \right) \frac{\chi_{-4}(n) \log n}{n^{q}} \frac{\chi_{-4}(m) \log m}{m^{r}} + \frac{1}{\Gamma(s)} \sum_{j,k \geq 0} \frac{\beta^{(1)}(q-j)}{j!} \frac{\beta^{(1)}(r-k)}{k!} \frac{(-1)^{j+k}}{j+k+s}. \qquad (71)$$

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Applications to Character MTW Sums

Hence

$$(\mu_{-4,-4})_{1,1,0}(1,1,1) := \sum_{m,n\geq 1} \chi_{-4}(n)\chi_{-4}(m)\frac{\log m \, \log n}{m\,n(m+n)} \quad (72)$$

$$= 0.0083251207501535752106219744827 1 \ldots.$$

- To compute the requisite value of

$$\beta^{(1)}(1) = 0.1929013167969124293...,$$

 we may use (38), and for $\beta^{(1)}(-n)$ with $n \geq 0$, we can use many methods including (8).
- We also computed the same value to the precision shown directly from the sum expressed in terms of Psi functions.

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Applications to Character MTW Sums

In like vein, from Theorem 17 or (70), we compute various sums:

$$-\sum_{m,n\geq 1} \chi_{-4}(m)\chi_{-4}(n)\frac{\log^2 m \log n}{m\,n(m+n)^5} = -0.0000123714496 6467 \ldots . \tag{73}$$

$$-\sum_{m,n\geq 1} \chi_{-4}(m)\chi_{-4}(n)\frac{\log^2 m \log n}{m\,n(m+n)^8} = -7.23894004469 9712819 \cdot 10^{-8} \ldots . \tag{74}$$

$$\sum_{m,n\geq 1} \chi_{-4}(m)\chi_{-3}(n)\frac{\log^2 m}{m\,n(m+n)^7} = -0.150314175 \cdot 10^{-5} \ldots . \tag{75}$$

(76)

Applications to Character MTW Sums

and higher-order variants such as

$$\sum_{m,n\geq 1} \chi_{-4}(m)\chi_{-4}(n)\frac{\log^2 m \log^2 n}{m^2\, n^2 (m+n)^4} \tag{77}$$

$$= 0.921829712836 \cdot 10^{-5} \ \ldots .$$

$$\sum_{m,n\geq 1} \chi_{-4}(m)\chi_{-4}(n)\frac{\log^3 m \log^3 n}{m^3\, n^3 (m+n)^3} \tag{78}$$

$$= 0.69071031171 \cdot 10^{-5} \ \ldots .$$

and so on.

- In each case the precision shown has been confirmed directly from the definitional sum.
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A Character Sums Ladder

We illustrate for $\sigma = e$, and $d_1, d_2 = -2, -3, -4, +5$. We have

$$\frac{1}{\Gamma(c)} \int_{1/\sigma}^{1} \frac{\left(-\log\left(-\log x\right)\right)^c \left(-\log x\right)^{n-1}}{x} dx = \frac{c}{n^{c+1}}.$$

We adduce $\left(\mu_{\pm d_1, \pm d_2}\right)_{a,b,c}(q, r, s)$

$$= -\sum_{k=0}^{c-1} \binom{c}{k} \frac{\Gamma^{(c-k)}(c)}{\Gamma(c)} \left(\mu_{\pm d_1, \pm d_2}\right)_{a,b,k}(q, r, s)$$
$$+ c \sum_{j,k>0} \frac{\mathrm{L}_1^{(a)}(q-j)}{j!} \frac{\mathrm{L}_2^{(b)}(r-k)}{k!} \frac{(-1)^{j+k}}{(j+k+s)^{c+1}}$$
$$+ \sum_{m,n>0} \chi_{\pm d_1}(m) \chi_{\pm d_2}(n) \frac{(-\log m)^a}{m^q} \frac{(-\log n)^b}{n^r} \mathcal{I}_{s,c}(m+n). \qquad (79)$$

Here $\mathcal{I}_{s,c}(k) := \int_0^{1/e} \log^c(-\log x)(-\log x)^{s-1} x^{k-1}\, \mathrm{d}x$.

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The $c = 0$ case which 'ignites' the ladder is also covered by the simplest case of (65). Also,

$$\mathcal{I}_{s,0}(k) = \frac{1}{k^s} \int_k^\infty z^{s-1} \mathrm{e}^{-z} \, \mathrm{d}z = \frac{\Gamma(s,k)}{k^s} \tag{80}$$

and $\mathcal{I}_{s,c}(k) = \mathcal{I}_{s,0}^{(c)}(k)$. By (8.7.3) of [DLMF] we have

$$\frac{\Gamma(s,z)}{z^s} = \frac{\Gamma(s)}{z^s} - \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!(s+j)}, \tag{81}$$

which can easily be symbolically differentiated.

Conclusion

1. We also undertook various studies of relations between such sums
   - computing various sums to much higher precision;
   - using character sum ladders and ‘PSLQ‘;
   - uncovering and proving unexpected relations;
   - and ruling out many more [1,2].
   - Different methods star in different settings.
2. Thank you!

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