Convex analysis on groups and semigroups

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CARMA, University of Newcastle

AustMS, Adelaide, Sept 30, 2015
Noumea, February 18, 2016 Carthagena, June 9–10, 2016
MODU, Melbourne, July 18–22, 2016
Last Revised March 28, 2016

https://www.carma.newcastle.edu.au/jon/ConvOnGroups.pdf
(Math Programming 2016)
https://www.carma.newcastle.edu.au/jon/LCGroups.pdf
(J Convex Analysis 2016)

Outline

1. Convex analysis on groups: Part I
   - Convex sets and functions
   - Divisible monoids & groups
   - Selected examples
   - Convex analysis on groups and monoids
     - Interpolation of convex functions
     - Maximum formula
     - Optimisation

2. Convex analysis on groups: Part II
   - Topological groups
   - Hahn-Banach in topological groups
   - Krein-Milman theorem and Milman's converse
   - Minimax theorem on monoids

MoCaO: New AustMS Special Interest Group

Proposal for a
"Mathematics of computation and optimisation"
AustMS Special Interest Group

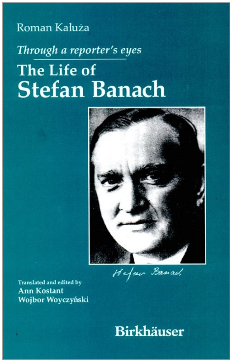
Using computational techniques to approximate solutions to physical models is common, but these techniques are not worth much if not backed up by serious mathematical analysis – it is not rare to see algorithms that are ill-designed or ill-used, and thus lead to unrealistic solutions. Modern opportunities test the boundaries of old algorithms, and ask for new numerical or symbolic techniques and analysis to be performed. This must be founded on solid mathematical theories if we want to go beyond a trial-and-error process.

The AustMS SIG we propose revolves around the design and analysis, using mathematical rigour, of numerical algorithms for models based on differential equations and mathematical optimisation.

At the 2006 annual meeting, a special session of the AustMS annual meeting was held around computational mathematics and optimisation. Since then, special sessions involving these areas have been regularly held at the AustMS annual meetings. In the last few years, a regular *Computational Maths* session has been organised by B. P. Lamichhane and Q. T. Le Gia, with 20+ participants each year on average. Similarly, since 2009 the organisers of the ANZIAM SigmaOpt group have run successful Optimisation/Control sessions at the annual AustMS meeting, with 15-30 participants each year; in 2015 the session had to refuse talks due to the lack of available slots. Several other sessions at AustMS meetings also indicate an interest in mathematical topics strongly

General themes

- Define a canonical convexity not an axiomatic one (a family closed under $\cap$ and directed $\cup$)
  - Should yield known results in a real vector space.
- Many known results hold assuming only an additive structure.
- Integer programming is harder than convex programming.
  - More reasons why?



General themes

A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.
(Stefan Banach, 1892–1945)

See www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html

Research started in 1982

1993 Status

CONVEXITY OVER GROUPS:
(OR SEMI-GROUPS + MODULES)

(1) MOTIVATION:
(i) Integer programming.
(ii) Non linear programming analogues.
(iii) "Natural" vs "axiomatic generalizations" of convexity. (ALIGNMENTS)
(iv) Explain Blatter + Seever's "outrider".

(2) Basic DFns:

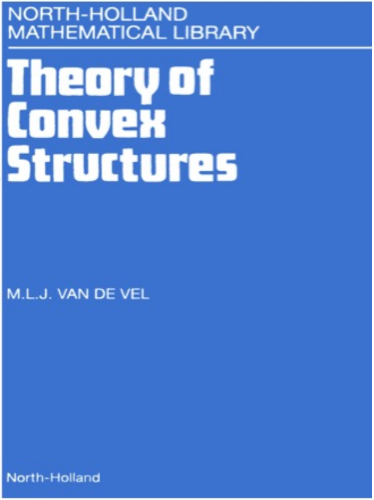
(3) Sandwich results...

(4)...and NLP (ILP)

(5) The operator case...and a proof!

(6) Recovering standard results and extensions.

J. M. Borwein, Dalhousie (Feb 1983)



Definition (Convex sets in vector spaces)

If X is a vector space, $A \subseteq X$ is convex if $x_1, \ldots, x_n \in A$, $\alpha_i > 0$,

$$\sum_{i=1}^{n} \alpha_i = 1 \Longrightarrow \sum_{i=1}^{n} \alpha_i x_i \in A.$$

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Definition (Convex hull)

For $A \subseteq X$, $\mathrm{conv}(A)$ is the smallest convex set that contains A.

Convex functions

Definition (Convex functions on vector spaces)

If X is a vector space, $f\colon X \to \mathbb{R}$ is convex if

$$f(x) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$$

whenever $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$;
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If X is a monoid, $f: X \to \mathbb{R}$ is convex if
$$mf(x) \le \textstyle\sum_{i=1}^n m_i f(x_i)$$
whenever $mx = \sum_{i=1}^n m_i x_i$, $m = \sum_{i=1}^n m_i$;
f is concave if $-f$ is convex.

This can be done more generally for X a (semi-)module, range $[-\infty, +\infty]$ or an ordered group ...

Some basic properties

Many properties extend. Some do not without ‘divisibility’ or other restrictions.

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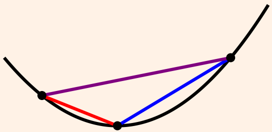
Example: three slopes lemma

If X is a monoid, $f : X \to \mathbb{R}$ is convex, $m, m_1, m_2 \in \mathbb{N}$, $x, x_1, x_2 \in X$ are such that $mx = m_1x_1 + m_2x_2$, then

$$\frac{f(x) - f(x_1)}{m_2} \leq \frac{f(x_2) - f(x_1)}{m_1 + m_2} \leq \frac{f(x_2) - f(x)}{m_1}$$



Semidivisible monoids & groups

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Equivalently, p-semidivisible iff for each $x \in X$ there is $y \in X$ so that

$$x = py$$

.

Examples

Finite groups

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The only convex sets are X is and $\emptyset$.

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If $X = \mathbb{R}/\mathbb{Z}$, $\mathrm{conv}(\{x\}) = \{x + y \mid y \in \mathbb{Q}\} \implies$ no convex singletons in X.

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Integer lattice $\mathbb{Z}^d$

For $A \subseteq \mathbb{Z}^d$ then $\mathrm{conv}_{\mathbb{Z}^d}(A) = \mathrm{conv}_{\mathbb{R}^d}(A) \cap \mathbb{Z}^d$. Likewise for other integer lattices.

Examples

Arctan semigroup

Let $X = [0, \infty)$ with the addition

$$a \oplus b = \frac{a+b}{1+ab} \qquad (so\ 0 \oplus b = b, 1 \oplus b = 1).$$

If $a, b \neq 0$ then

$$\frac{1}{a} \oplus \frac{1}{b} = a \oplus b.$$

Thus, if $a \neq 1$, then $\frac{1}{a} \in \mathrm{conv}(\{a\})$, and so $\{0\}$, $\{1\}$ is are the only convex singletons. Also X is 3-semidivisible but not 2-semidivisible.

Examples

Hyperbolic group

Let X be the (commutative) matrices of the form

$$e^{it}M(\theta) = e^{it}\begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}, \quad t, \theta \in \mathbb{R}$$

with matrix multiplication. Then X is divisible since

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Let X_p be the subgroup of matrices of the form $e^{\frac{2\pi\ell}{p}}M(\theta)$ for $\theta \in \mathbb{R}$, $0 \leq \ell \leq p-1$. Then $nX = X \iff p \nmid n$.

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If $f : \mathbb{R} \to \mathbb{R}$ is convex then $F : X \to \mathbb{R}$ given by $F\left(e^{it}M(\theta)\right) = f(\theta)$ is convex $\implies$ can produce many convex functions on X, X_p.

Interpolation of convex functions

$f\colon X \to \mathbb{R}$ is subadditive if $f(x+y) \leq f(x) + f(y)$.

Theorem (Kaufman, 1966)

Suppose X is a monoid, $f, -g\colon X \to \mathbb{R}$ are subadditive, and $g \leq f$. Then there exists an additive $a\colon X \to \mathbb{R}$ such that $g \leq a \leq f$.

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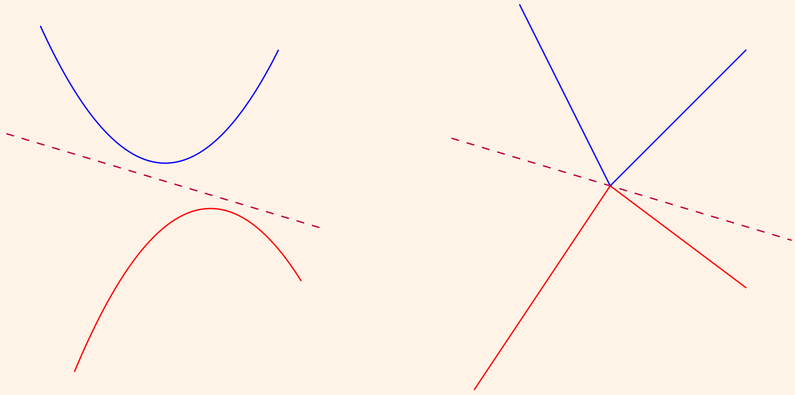
Theorem

Suppose X is a semidivisible monoid, $f, -g\colon X \to \mathbb{R}$ convex, and $g \leq f$. Then there exists an affine $a\colon X \to \mathbb{R}$ such that $g \leq a \leq f$.

Picture: interpolation of subadditive/convex functions

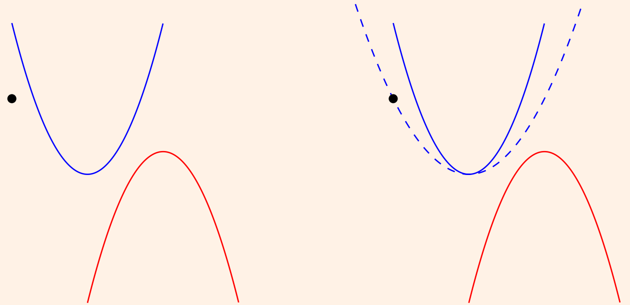


Convex case

Subadditive case

Interpolation theorem: idea of proof

If $f = g$ we are done. If $g(x_0) < f(x_0)$ we can replace one of the two by a ‘better’ function.



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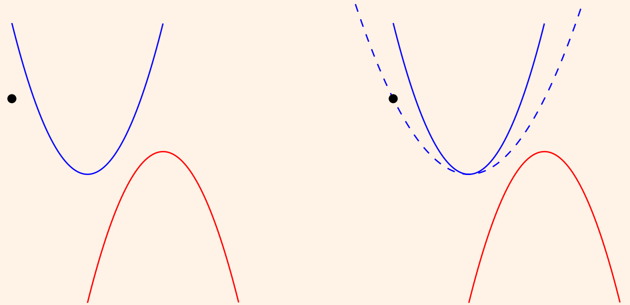
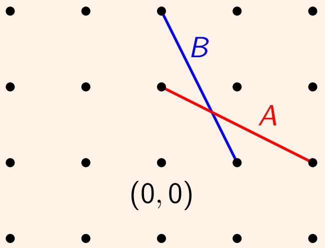


We continue the process until $f = g$ (might be transfinite).

Example: nondivisible case

Failure in the nondivisible case

$X = \mathbb{Z}^2$, $f(x) = 5d_A(x) - 1$ is and $g = -5d_B(x) + 1$.



$f, -g$ is are convex, $g \leq f$, but there is no affine a s.t. $g \leq a \leq f$.

Extended real-valued functions

Sandwich holds if $f, g \colon X \to [-\infty, \infty]$ with a generalised affine if mX is divisible for some $m \in \mathbb{N}$ or one of the functions is finite.

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But not every group satisfies it. For example $X = X_2 \times X_3 \times X_5 \times \ldots$, X_p the pth hyperbolic group.

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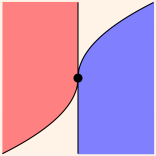
If X is a group and either f or g is everywhere finite and the other is somewhere finite, then the affine separator is finite.

Extended real-valued functions

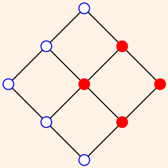
In general there is no finite affine separation

In $X = \mathbb{R}$, take $g(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\infty & x < 0 \end{cases}$, $f(x) = -g(-x)$. Then any affine separator must be 0 at 0 and $\pm\infty$ elsewhere.



Example: separation in meet semilattice

- $(X, \wedge)$ is a meet semilattice: X is divisible since $x \wedge x = x$.
  - This monoid does not embed in any group.
- If $C \subseteq X$, $\mathrm{conv}(C)$ is the semilattice generated by C.
- The above interpolation theorem, or Kaufman's result, (via Stone's lemma) implies that disjoint sub-semilattices lie in partitioning sub-semilattices.



Directional derivative and subgradient

Definition (Directional derivative)

$$f_x(h) = \inf \left\{ n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h \right\}$$

If f is convex : $n\big(f(x+g) - f(x)\big)$ is decreasing in n.

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Recall, in a VS: $f_x(h) = \inf \left\{ \frac{1}{t}\big(f(x+th) - f(x)\big) \;\middle|\; t > 0 \right\}$.

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Theorem (Max formula)

If X is a semidivisible group and $f\colon X \to \mathbb{R}$ is convex

$$f_x(h) = \max\left\{ a(h) \;\middle|\; a \in \partial f(x) \right\}$$

Max formula for sublinear functions and consequences

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We say $f : X \to \mathbb{R}$ is sublinear if f is subadditive and

$$f(nx) = nf(x) \text{ for all } n \in \mathbb{N}.$$

Note: sublinear $\Longrightarrow$ convex ('additive' does not).

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If f is sublinear, then the max formula holds without any semidivisibility assumption.

Using the extended version of our sandwich theorem we arrive at:

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Theorem (Hahn-Banach for groups)

Suppose X is a group and $Y \subseteq X$ is a subgroup, $f\colon X \to \mathbb{R}$ is sublinear and $h\colon Y \to \mathbb{R}$ is additive such that $h \le f$ on Y.

Then there exists $\bar{h}\colon X \to \mathbb{R}$ is additive such that $\bar{h} \le f$ is and $\bar{h} = h$ on Y.

This extends when $\mathbb{R}$ is replaced by an order complete ordered group.

Additive dual group and conjugate function

Define the additive dual of a group:

$$X^* = \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is additive } \}.$$

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Definition (Conjugate function)

Given $f : X \to \mathbb{R}$, define $f^\star : X^* \to \mathbb{R}$ be

$$f^\star(\varphi) = \sup_{x \in X} \{\varphi(x) - f(x)\}.$$

Can define a conjugate function if we replace $\mathbb{R}$ by another group Y, assuming that Y has some partial ordering.

Weak and strong Fenchel-Rockafellar duality

Theorem (Fenchel-Young inequality)

Let $f : X \to \mathbb{R}$. Then for every $x \in X$ and $\varphi \in X^*$,

$$f(x) + f^\star(\varphi) \geq \varphi(x).$$

Equality holds if and only if $\varphi \in \partial f(x)$.

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Theorem (Weak and strong Fenchel duality)

Let $f : X_1 \to \mathbb{R}$, $g : X_2 \to \mathbb{R}$ and $T : X_1 \to X_2$ additive. Let

$$P = \inf_{x \in X_1} \{f(x) + g(Tx)\}, \quad D = \sup_{\varphi \in X^*} \{-f^\star(T^*\varphi) - g^\star(\varphi)\}.$$

Then $P \geq D$ (weak duality). If X is semidivisible and f and g are convex, then $P = D$ (strong duality).

Weak and strong Fenchel duality – remarks

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$$P = \inf_{x \in X_1} \{f(x) + g(Tx)\}, \quad D = \sup_{\varphi \in X^*} \{-f^\star(T^*\varphi) - g^\star(\varphi)\}.$$

Then $P \geq D$ (weak duality). If X is semidivisible and f and g are convex, then $P = D$ (strong duality).

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Weak duality follows from Fenchel-Young inequality. Strong duality follows from the max principle.

In general, can replace $\mathbb{R}$ by a group with some partial ordering. Can also add a maximal element in the range. In such case, need to deal with the core of the domain (as in the vector space setting).

Convex optimisation, the value function

Consider the constrained problem

$$v(b) = \inf \left\{ f(x) \;\middle|\; g_1(x) \leq b_1, \ldots, g_k(x) \leq b_k \right\}.$$

The function v is the value function.

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Proposition (Subadditive/sublinear value function)

If $f, g_1, \dots, g_k$ are subadditive, then v is subadditive. If X is p-semidivisible and $\forall x$, $f(px) = pf(x)$, $g_j(px) = pg_j(x) = g_j(px)$, $1 \leq j \leq k$ then $v(pb) = pv(b)$. This also implies that v is convex.

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No homogeneity in the nondivisible case $X = \mathbb{Z}$

$$v(b) = \inf\left\{ -x \;\middle|\; 2x \le b,\; x \in \mathbb{Z} \right\} = -\left\lceil \frac{b}{2} \right\rceil, \quad b \in \mathbb{R}.$$

$f(x) = -x$, $g(x) = 2x$ are homogeneous, but v is not.

Subgradient of the maximum function

Value function

$$v(b) = \inf\left\{f(x) \;\middle|\; g_1(x) \le b_1, \ldots, g_k(x) \le b_k\right\}.$$

An important special case yields:

Theorem (Subgradient of max function)

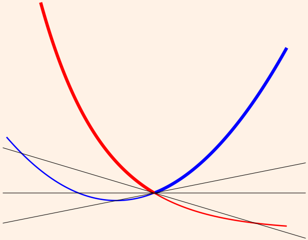
Suppose X is a semidivisible group and $f_1, \ldots, f_k : X \to \mathbb{R}$ are convex. Let $g(x) = \max_{1 \le i \le k} f_i(x)$. Then

$$\partial g(x) = \mathrm{conv}\left(\bigcup_{f_i(x)=g(x)} \partial f_i(x) \right)$$

Convex optimisation on groups

Subgradient of max function

$$\partial g(x) = \operatorname{conv}\Big(\bigcup_{f_i(x)=g(x)} \partial f_i(x) \Big), \quad g = \max_{1 \le i \le k} f_i$$



Future Directions

- Convexity on non-commutative groups.
- More constructions of convex functions on groups.

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- Convexity on non-commutative groups.
- More constructions of convex functions on groups.
- Applications in integer (non-divisible) programming.
  - One of our original goals.
  - Generalise role of $\lfloor \cdot \rfloor$ in non-divisible case.

Thank you

IT'S WEIRD HOW PROUD PEOPLE ARE OF NOT LEARNING MATH WHEN THE SAME ARGUMENTS APPLY TO LEARNING TO PLAY MUSIC, COOK, OR SPEAK A FOREIGN LANGUAGE.

Part II

Topological groups

Definition (Topological group)

A group which is also a topological space such that the group operations are continuous.

Topological monoid: enough that the addition is continuous.

Definition (Local convexity)

If there is a basis for the topology that contains only convex sets.

Rational dilation of sets

Sum of a set

$$mA = \left\{a_1 + \cdots + a_m \;\middle|\; a_1, \ldots, a_m \in A\right\}.$$

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Proposition (Monotonicity of dilations)

Suppose X is a monoid, $C \subseteq X$ convex and $0 \in C$. Then

$$q_1 C \subseteq q_2 C$$

whenever $0 \leq q_1 \leq q_2$.

Gauge functional

Using monotonicity of the dilations, we can define a group-theoretic version of the gauge functional.

Definition (Gauge function)

$$\rho_C(x) = \inf\{q \in \mathbb{Q}_+ \mid x \in qC\}.$$

Proposition

If X is a monoid and $C \subseteq X$ is convex, then ρ_C is sublinear.

- So far we did not use topological properties of the group.
- The gauge function will be used as a control function in the proof of the KM theorem.

Hahn-Banach in topological groups

- Previous version of Hahn-Banach: used only algebraic structure.
- In topological groups: we want a continuous separator.

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Proposition (Continuity of ρ_C)

If X is a topological group, $C \subseteq X$ convex, $0 \in \operatorname{int} C$, then ρ_C is continuous (on domain). If X is connected, ρ_C is everywhere finite.

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Theorem (Hahn-Banach strict separation in topological groups)

Let X be a connected locally convex topological group with $C \subseteq X$ closed and convex and $x_0 \notin C$. Then there exists $\varphi \colon X \to \mathbb{R}$ continuous and additive such that

$$\sup_{c \in C} \varphi(c) < \varphi(x_0).$$

No strict separation between sets

Theorem (Hahn-Banach strict separation in topological groups)

Let X be a connected, locally convex topological group with $C \subseteq X$ closed and convex and $x_0 \notin C$. Then there exists $\varphi\colon X \to \mathbb{R}$ continuous and additive such that

$$\sup_{c \in C} \varphi(c) < \varphi(x_0).$$

- In vector spaces, if D is a compact convex set with $C \cap D = \emptyset$, use $0 \notin C - D$ (closed) to obtain separation between C and D.
- Does not work in arbitrary groups. Take $X = \mathbb{Z}^2$ with $C = \{(0,1),(2,0)\}$, $D = \{(0,2),(1,0)\}$, then $C - D$ is not convex (note: additive preimages are convex).

Extreme points

Definition (Extreme point)

When X is a monoid and $C \subseteq X$, $x \in C$ is an extreme point of C if $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n} m_i$, $x_i \in C \implies x = x_1 = \cdots = x_n$.
Let $\mathcal{E}(C)$ be the collection of all extreme points of C.

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Theorem (Krein-Milman in topological groups)

Suppose X is a semidivisible, locally convex, connected topological group, and $C \subseteq X$ is convex and compact. Then

$$C = \overline{\mathrm{conv}(\mathcal{E}(C))}.$$

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A semilattice is a non-connected semigroup, but still KM type results exist (Poncet '14 or via Stone's lemma).

Example

Positive hyperbolic group

Let X be the collection of 2×2 matrices of the form

$$M(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}.$$

Previously studied $e^{it}M(\theta)$. Addition: matrix multiplication. Topology: $\mathbb{R}^4$ topology. X is connected, locally convex and divisible (actually a $\mathbb{Q}$-module).

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If $C \subseteq X$ is compact and convex, $\mathcal{E}(C) = \{M(\alpha), M(\beta)\}$ where,

$$\alpha = \inf\{\theta | M(\theta) \in C\}, \quad \beta = \sup\{\theta | M(\theta) \in C\}.$$

By KM, $C = \overline{\mathrm{conv}(M(\alpha, \beta))} = M([\alpha, \beta])$, a curve in $\mathbb{R}^4$.

Another example

σ-algebra with symmetric difference

$(\Omega, \mathcal{F}, \mu)$ a measure space. For $A, B \in \mathcal{F}$, let

$$A + B = A \triangle B.$$

Under this operation get a $(2n-1)$-semidivisible but not $2n$-semidivisible group ($A \triangle A = \emptyset$, $A \triangle A \triangle A = A$).

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For example, taking μ Lebesgue measure on $[0, 1]$, $\mathcal{F}$ the Borel sets and the psuedo-metric

$$d_\mu(A, B) = \mu(A \triangle B),$$

get a connected topological group which is not locally convex: take many small sets in a neighbourhood of $\emptyset$ and the convex hull can have full measure.

Milman's converse

The converse proof needs the property that the convex hull of finite unions of compact sets are compact.

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This is not always the case:

If $X = \mathbb{Q}$, $A = \{0\}$, $B = \{1\}$, then $\mathrm{conv}\big(A \cup B\big) = [0, 1] \cap \mathbb{Q}$ which is not convex.

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This is not always the case:

If $X = \mathbb{Q}$, $A = \{0\}$, $B = \{1\}$, then $\mathrm{conv}\big(A \cup B\big) = [0, 1] \cap \mathbb{Q}$ which is not convex.

Theorem (Milman's converse)

Suppose X ilocally convex group such that the convex hull of finite unions of compact sets is always compact. Suppose also that $C \subseteq X$ is compact and such that $\overline{\mathrm{conv}(C)}$ is compact. Then

$$\mathcal{E}\left(\overline{\mathrm{conv}(C)}\right) \subseteq C.$$

Minimax theorem in arbitrary spaces

Definition (Convex-like function)

Let X be any set. We say $f : X \to \mathbb{R}$ is convex-like if $\forall x, y \in X$, $\forall \mu \in [0,1]$, $\exists z \in X$ such that

$$f(z) \leq \mu f(x) + (1-\mu) f(y);$$

f is concave-like if $-f$ is convex-like.

$f : X \times Y \to \mathbb{R}$ is convex-concave-like if $f(\cdot, y)$ is convex-like $\forall y \in Y$ is and $f(x, \cdot)$ is concave-like $\forall x \in X$.

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Theorem (Fan '53, Borwein-Zhuang '86)

Let X, Y be non-empty, $f : X \times Y \to \mathbb{R}$ convex-concave-like. Suppose X is compact and $f(\cdot, y)$ is lower semicontinuous $\forall y \in Y$. Then

$$\min_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \min_{x \in X} f(x,y).$$

Proposition (Convex-like on monoids)

Suppose X is a semidivisible topological monoid such that for every $x_1, x_2 \in X$, $\operatorname{conv}(\{x_1, x_2\})$ is precompact. Assume $f : X \to \mathbb{R}$ is convex and lower semicontinuous. Then f is convex-like.

As a result, we immediately get:

Theorem (Minimax theorem for monoids)

Let X be compact and convex in a semidivisible topological monoid, and Y be a subset of semidivisible topological monoid such that $\operatorname{conv}(\{y_1, y_2\})$ is precompact $\forall y_1, y_2 \in Y$.
Suppose $f : X \times Y \to \mathbb{R}$ is such that $f(\cdot, y)$ is convex and lower semicontinuous, $f(x, \cdot)$ is concave and upper semicontinuous $\forall x \in X$, $\forall y \in Y$. Then

$$\min_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \min_{x\in X} f(x,y).$$

Minimax theorem: example

Minimax in the positive hyperbolic group

Choose again the positive hyperbolic group X of 2×2 matrices of the form

$$M(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \cosh(\theta) & \sinh(\theta) \end{bmatrix}, \quad \theta \in \mathbb{R},$$

with matrix multiplication.
If $\Lambda : \theta \mapsto M(\theta)$, then for $\alpha, \beta \in \mathbb{R}$

$$\mathrm{conv}(\{M(\alpha), M(\beta)\} \subseteq \Lambda([\alpha, \beta])$$

which is compact $\Longrightarrow$ if $C \subseteq X$ is convex and compact, every $f : C \times X \to \mathbb{R}$ is as above will satisfy the minimax theorem.

Future directions

- More convex analysis (differentiation, variational principle, monotone operators ...).
- A unified approach to Krein-Milman that includes semilattices.
- Examples of semidivisible connected locally convex topological groups which are not divisible (if any).
- Everything else.

Thank you

Algebra is generous; she often gives more than is asked of her.
(Jean d´Alembert, 1717–1783)