A new characterization of the Fibonacci sequence

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van der Corput 1939: there are infinitely many arithmetic progressions of primes of length 3 van der Corput 1939: there are infinitely many arithmetic progressions of primes of length 3

Heath-Brown 1981: there are infinitely many four-term progressions consisting of three primes and a number that is either a prime or product of two (possibly equal) primes

A linear recurrence f_n of order d is a complex sequence satisfying the recurrence

$$f_{n+d} = a_{d-1} f_{n+d-1} + \cdots + a_0 f_n$$

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Let $\alpha_1, \ldots, \alpha_r$ be the zeros of the companion polynomial P and assume that α_i is a zero of multiplicity σ_i. Then we can write

$$f_n = \sum_{i=1}^{r} p_i(n) \alpha_i^n,$$

where $p_i(n)$ are polynomials of degree $\sigma_i - 1$.

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- Can there be arbitrary many arithmetic progressions?
  Answer: Yes! But the recurrences with this property are very special (see the talk).

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- A sequence is called unitary if α_i is a root of unity for some i.

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- We call f_n exceptional if there exists an integer $N > 0$ such that each α_i is an rational power of N, each $|\alpha_i| > 1$ or each $|\alpha_i| < 1$ and $p_i(n) = \gamma_i(n - \gamma)$ with $\gamma \in \mathbb{Q}$.

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Note that a recurrence cannot be both symmetric and exceptional.

If (f_k, f_m, f_n) is a three-term arithmetic progression, then

$$2f_m - f_n - f_k = 0.$$

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Schmidt and Schlickewei (1993) considered linear equations in recurrences in detail:

Consider the equation

$$Af_n + Bf_m + Cf_k = 0, \quad f_n f_m f_k \neq 0,$$

then all but finitely many solutions

- are contained in finitely many families $\mathcal{F}_j$ of the form

$$m = n + a_j, \quad k = n + b_j$$

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- if f_n is exceptional, solutions may also be contained in finitely many families, e.g. of type $\mathcal{E}_j^{(n)}$ of the form

$$n = c_j N^s + \gamma, \quad m = c_j N^s + as + b_j \quad k = c_j N^s + a's + b'_j$$

Theorem (Ziegler, P 2012)

Let (f_n) *be a non-degenerate and non-unitary recurrence sequence with companion polynomial* P*. Then there is a finite set* $S_0 \subset \mathbb{N}^3$ *such that all three-term arithmetic progressions* (f_m, f_n, f_k) *with* $f_n \neq 0$ *satisfy* $(m, n, k) \in S_0$ *(isolated solutions) or one of the following three cases occurs:*

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- *The indizes for which (f_m, f_n, f_k) is an arithmetic progression are of the form $m = k + a, n = k + b$, with $a, b \in \mathbb{Z}$ such that $P(X) | (X^a - 2X^b + 1)X^{-\min\{a,b,0\}}$.*

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- *The recurrence is of the form*

$$f_n = \sum_{i=1}^{r} c_i \left(\alpha_{2i-1}^n + \alpha_{2i}^n \frac{\alpha_{2i-1}^{a+c} + \alpha_{2i-1}^{b+c}}{2} \zeta_i^c \right), \quad \text{with}$$

$$0 = (\zeta_i^a + \zeta_i^b - 4\zeta_i^c) + \zeta_i^a \alpha_j^{b-a} + \zeta_i^b \alpha_j^{a-b} \qquad \textit{or}$$

$$f_n = \sum_{i=1}^{r} c_i \left(\alpha_{2i-1}^n + \alpha_{2i}^n (\alpha_{2i-1}^{a+c} + 2\alpha_{2i-1}^{b+c}) \zeta_i^c \right), \quad \text{with}$$

$$0 = (\zeta_i^a + 4\zeta_i^b - \zeta_i^c) - 2\zeta_i^a \alpha_j^{b-a} - 2\zeta_i^b \alpha_j^{a-b} \qquad \textit{or}$$

$$f_n = \sum_{i=1}^{r} c_i \left(\alpha_{2i-1}^n + \alpha_{2i}^n (2\alpha_{2i-1}^{a+c} + \alpha_{2i-1}^{b+c}) \zeta_i^c \right), \quad \text{with}$$

$$0 = (4\zeta_i^a + \zeta_i^b - \zeta_i^c) - 2\zeta_i^a \alpha_j^{b-a} - 2\zeta_i^b \alpha_j^{a-b}$$

where $j = 2i-1, 2i$, $c_i \in \mathbb{C}$, $\alpha_{2i-1}\alpha_{2i} = \zeta_i$ is an M-th root of unity with M minimal for all $i = 1, \ldots, r/2$ and the indices are of the form $m = Mt + a, n = Mt + b, k = -Mt + c$.

- The recurrence is of the form

$$f_n = C(n-\gamma)2^{n/K}\zeta_K^n$$

where ζ_K is a K-th root of unity, with $\gamma, K \in \mathbb{Z}$ and $C \in \mathbb{C}$. Then f_n, f_m and f_k form an arithmetic progression (arranged in some order) if
$n = c2^s + \gamma, m = c2^s + as + b, k = c2^s + a's + b'$ with a, a', b, b', c integers. Moreover K and c can only be both positive if K is even and ζ_K is a root of -1.

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- The sequence is symmetric and of a very special form.
  Example: Let $f_{n+2} = 4f_{n+1} + f_n$, $f_0 = -3$, $f_1 = 2$, then f_n is defined over the integers and contains infinitely many arithmetic three-term progressions $(f_{2n+2}, f_{-2n+1}, f_{2n+1})$.

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The first and the second case can occur in one sequence! Note the Fibonacci sequence with $f_0 = 0$ and $f_1 = 1$ also contains the infinite family $(f_{2n+2}, f_{-2n-1}, f_{2n-1})$ of arithmetic three term progressions.

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Corollary

Let f_n be defined over the integers and assume f_n contains infinitely many three-term arithmetic progressions (f_m, f_n, f_k) with $n, m, k \geq 0$, then $P(X)$ is a factor of $\frac{X^a - 2X^b + 1}{X^d - 1}$ or $\frac{X^a + X^b - 2}{X^d - 1}$ with $a > b > 0$ and $d = \gcd(a, b)$.

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Lemma (Schinzel 1963)

The polynomial

$$\frac{X^a - 2X^b + 1}{X^{\gcd(a,b)} - 1}$$

is irreducible over $\mathbb{Q}$*, except* $a = 7k$ *and* $b = 5k$ *or* $b = 2k$ *and in this case the polynomial factors into*
$(X^{3k} + X^{2k} - 1)(X^{3k} + X^k - 1)$ *or*
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Lemma (Schinzel in Ziegler, P 2012)

The polynomial

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Theorem (Ziegler, P 2012)

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- *The sequence is listed in some table (symmetric cases). Moreover, the zeros of the companion polynomial are $\pm 2 \pm \sqrt{5}$, $\frac{\pm 1 \pm \sqrt{5}}{2}$ or $\pm 1 \pm \sqrt{2}$*

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- *The sequence is listed in some table (symmetric cases). Moreover, the zeros of the companion polynomial are $\pm 2 \pm \sqrt{5}$, $\frac{\pm 1 \pm \sqrt{5}}{2}$ or $\pm 1 \pm \sqrt{2}$*
- *The companion polynomial of the recurrence f_n is listed in a finite table.*

Table: Companion polynomials for which f_{n+a}, f_{n+b} and f_n are in arithmetic progression.

a	b	$P(X)$
3	1	X^2+X-1
		X^2+X+2
		$2X^2+2X+1$
3	2	X^2-X-1
		X^2+2X+2
		$2X^2+X+1$

Corollary (Ziegler, P 2012)

The only increasing, simple, non-degenerate, non-unitary, binary recursion f_n for $n \geq 0$ defined over the rationals that contains infinitely many three-term arithmetic progressions (f_m, f_n, f_k) with $m, n, k \geq 0$, which additionally satisfies $f_0 = 0$ and $f_1 = 1$ is the Fibonacci sequence.

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Omitting the condition non-unitary but assuming to be defined over the integers we have $f_n = \frac{2^n-(-1)^n}{3}$ or f_n is the Fibonacci sequence.

Theorem (Ziegler, P 2012)

Let f_n be a non-degenerate, non-unitary, ternary recurrence, which is defined over the rationals and contains infinitely many arithmetic progressions. Then f_n has companion polynomial listed in the table next slide.

Table: Companion polynomials for which f_{n+a}, f_{n+b} and f_n are in arithmetic progression.

a	b	$P(X)$
4	1	X^3+X^2+X-1
		X^3+X^2+X+2
		$2X^3+2X^2+2X+1$
4	3	X^3-X^2-X-1
		X^3+2X^2+2X+2
		$2X^3+X^2+X+1$
7	2	X^3+X^2+1
		X^3-X-1
7	5	X^3+X^2-1
		X^3+X-1