Subsequences of Automatic Sequences with Polynomial Growth

Jean-Marc Deshouillers
Université de Bordeaux

(with M. Drmota and J. Morgenbesser)

Newcastle NSW March 16th

In memory of Alf van der Poorten

DENSITIES

Let $\mathcal{A} \subset \mathbb{N}$

NATURAL (or ASYMPTOTIC) DENSITY

$$d(\mathcal{A}) = \lim_{N\to\infty} \frac{1}{N} \#\{n \leq N \,/\, n \in \mathcal{A}\} \quad \text{(if it exists)}$$

LOGARITHMIC DENSITY

$$d_{\log}(\mathcal{A}) = \lim_{N\to\infty} \frac{1}{\log(N)} \sum_{\substack{n \leq N \\ n \in \mathcal{A}}} \frac{1}{n} \quad \text{(if it exists)}$$

Let E denote a finite set and let $\mathbf{v} : \mathbb{N} \longrightarrow E$. For $a \in E$, we shall consider the existence and the value of

$$\delta(\mathbf{v}=a) = \delta(\{n / \mathbf{v}(n) = a\}), \text{ for } \delta = d \text{ or } d_{\log}.$$

Let $c > 1$. We denote by $\mathbf{v}_c$ the sequence defined by $\forall n : \mathbf{v}_c(n) = \mathbf{v}(\lfloor n^c \rfloor)$.

¿ CAN WE COMPARE $\delta(\mathbf{v}=a)$ AND $\delta(\mathbf{v}_c=a)$?

THEOREM (Harman - Rivat, 1995)

If $d(\mathbf{v} = a)$ exists, then

$\tilde{\forall} c \in [1,2]$: $d(\mathbf{v}_c = a)$ exists and is equal to $d(\mathbf{v} = a)$

THEOREM (Mauduit - Rivat, 2005)

Let $1 < c < 7/5$, $q \geqslant 2$, $m \geqslant 2$. Then

$\forall a$: $d(\{n \in \mathbb{N} \,/\, s_q(\lfloor n^c \rfloor) \equiv a \pmod m\}) = \frac{1}{m}$,

where $s_q(m)$ denotes the sum of the digits of m in base q.

FACT The function s_q is q-additive, i.e.

$\forall r \geq 1$, $\forall a \geq 0$, $\forall b \in [0, q^r - 1]$:

$$s_q(aq^r + b) = s_q(aq^r) + s_q(b),$$

and $s_q \big|_{\bmod m}$ is q-additive with values in $\mathbb{Z}/m\mathbb{Z}$.

CONSEQUENCE The function $f(n) = e_m(h\, s_q(n))$ is q-multiplicative. This permits to "factorize" $\sum_{n \leq x} f(n)$ and show that it is $o(x^{1-\delta})$ for some $\delta > 0$.

¿ HOW TO TREAT $\sum_{m \le y} f(\lfloor m^c \rfloor)$?

We have to select those n's which have the shape $\lfloor m^c \rfloor$, for some m.

This is equivalent to $m^c \le n < m^c + 1$

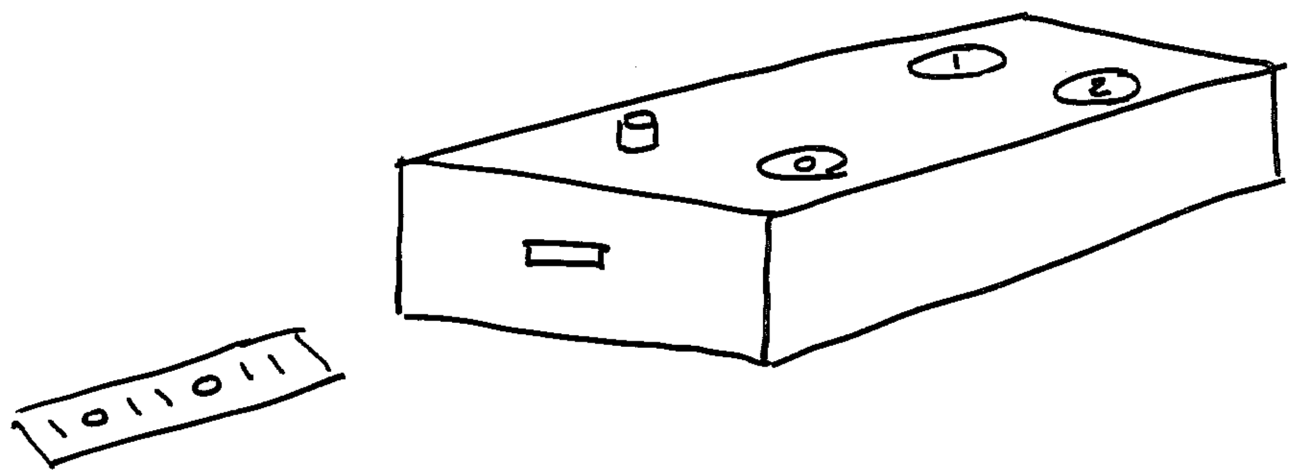
or $m \le n^\gamma < (m^c+1)^\gamma = m + \gamma m^{1-c} + \cdots$, with $\gamma = 1/c$.

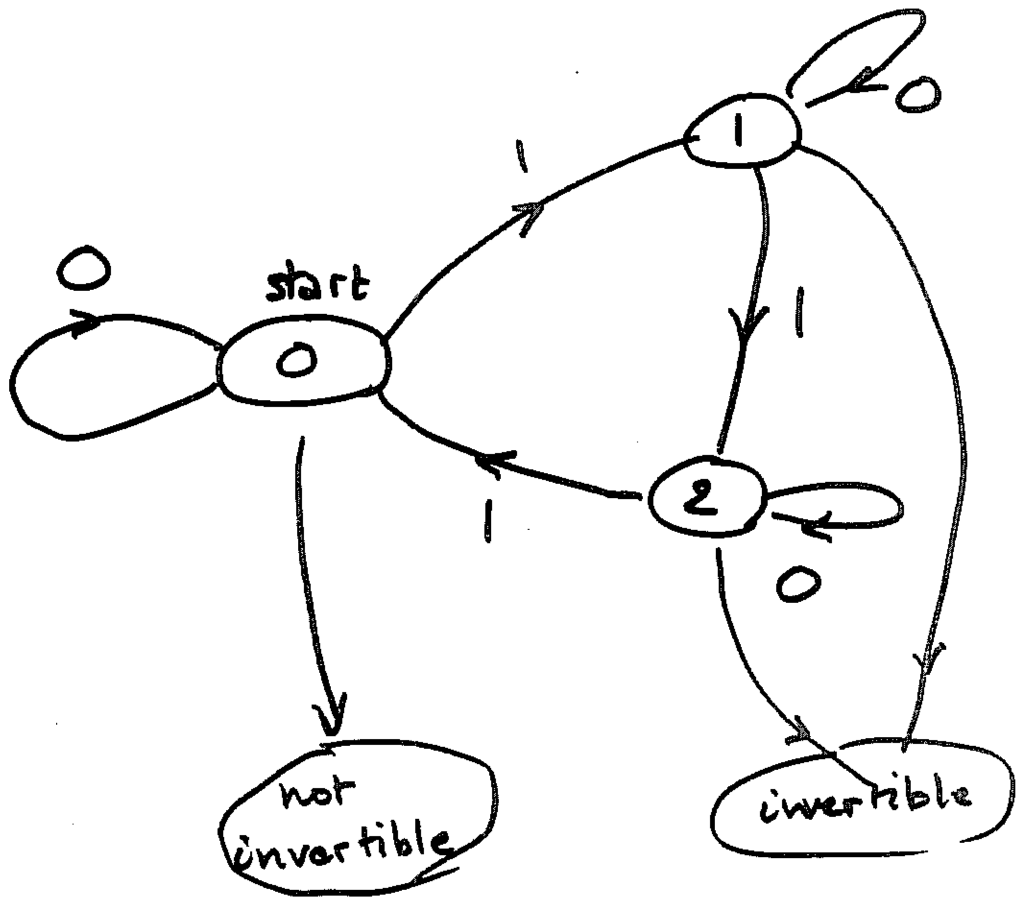
We have (essentially)

$$\sum_{m \le x^\gamma} f(\lfloor m^c \rfloor) = \sum_{\substack{n \le x \\ \{n^\gamma\} \le \gamma n^{\gamma - 1}}} f(n)$$

Build a machine which reads n written in base 2 and produces $s_2(n)$ modulo 3.

Example $91 = \overline{1011011}^{\,2} \longmapsto 2$.





Finite set R (states)

Initial state $r_1 \in R$

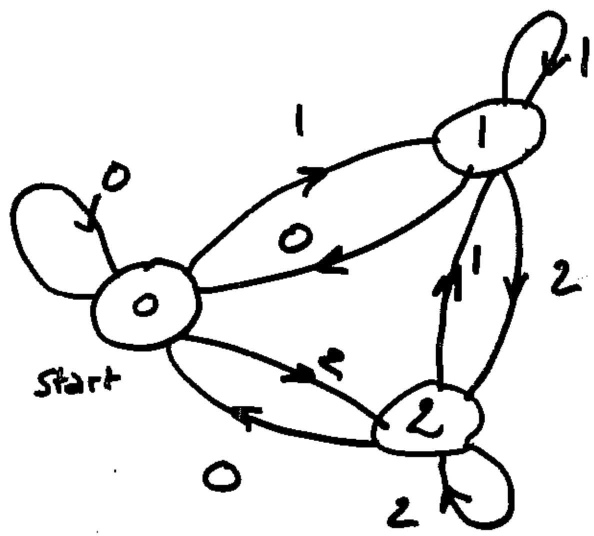
$\Sigma = \{0, 1, \dots, q-1\}$

$R \times \Sigma \xrightarrow{\delta} R$

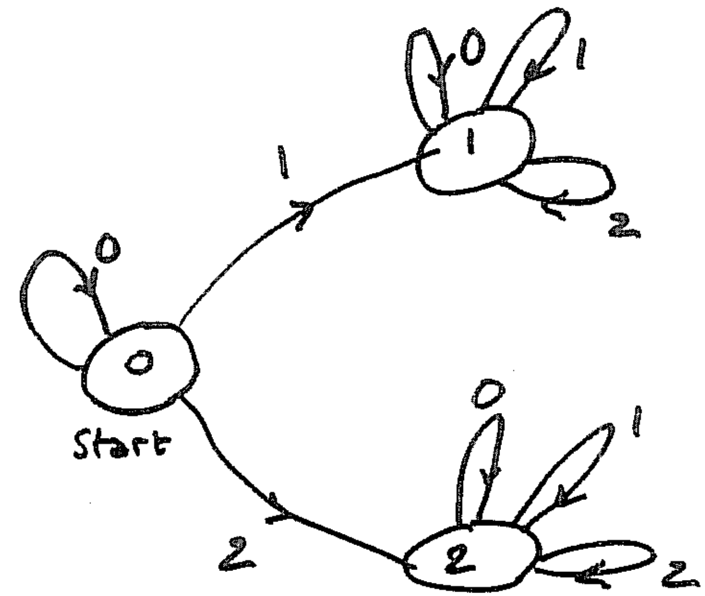
$R \xrightarrow{\tau} E$

Reading the most significant digit of n in base 3

$$92 = \overline{10102}^{\,3}$$



from the right



from the left

THEOREM (M. DRMOTA - J. MORGENBESSER - JRD)

Let $c \in (1, 7/5)$, $q \geq 2$ and $\mathbf{u}$ a q-automatic sequence with value in a (finite) set E.

For any $a \in E$, the logarithmic density $d_{\log}(\mathbf{u}_c = a)$ exists and is equal to $d_{\log}(\mathbf{u} = a)$.

Moreover, $d(\mathbf{u}_c = a)$ exists if and only if $d(\mathbf{u} = a)$ exists (and then there are equal).

DEFINITION $F : \mathbb{N} \to M_d(\mathbb{C})$ is q-multiplicative if there exist $G_k^{(1)}, G_k^{(2)} : \mathbb{N} \to M_d(\mathbb{C})$: $F(q^k a + b) = G_k^{(1)}(b)\, G_k^{(2)}(a)$ (for $k \geq 1$, $a \geq 0$, $0 \leq b < q^k$).

PROPOSITION Let $\|\cdot\|_s$ be a Banach norm on $M_d(\mathbb{C})$, F q-multiplicative with $\| G_k^{(j)}(n) \|_s \leq 1$. Let $c \in (1, 7/5)$, $\delta \in (0, (7-5c)/9)$. Then

$$\Big\| \sum_{1 \leq n \leq x} F(\lfloor n^c \rfloor) - \sum_{1 \leq m \leq x^c} \gamma m^{\gamma - 1} F(m) \Big\|_s = O(x^{1-\delta}).$$

Mahler (1927) proved that for the Thue-Morse sequence $t(n) \equiv s_2(n) \pmod 2$, that for $k>0$ and $(\varepsilon_1,\varepsilon_2) \in \{0,1\}^2$:

$$\ell_k(\varepsilon_1,\varepsilon_2) = \lim_{x\to\infty} \frac{1}{x} \#\{1 \le n \le x \,/\, (t(n), t(n+k)) = (\varepsilon_1,\varepsilon_2)\}$$

exists and is $\neq 1/4$ for infinitely many k's.

THEOREM (M. DRMOTA, J. MORGENBESSER, JOD)

Let $k>0$, $\underline{\underline{\varepsilon}} = (\varepsilon_1,\varepsilon_2) \in \{0,1\}^2$.

For $c \in (1, 7/5)$: $\displaystyle \lim_{x\to\infty} \frac{1}{x} \#\{1 \le n \le x \,/\, (t(\lfloor n^c \rfloor), t(\lfloor n^c \rfloor + k)) = \underline{\underline{\varepsilon}}\} = \ell_k(\underline{\underline{\varepsilon}})$

For $c \in (1, 10/9)$: $\displaystyle \lim_{x\to\infty} \frac{1}{x} \#\{1 \le n \le x \,/\, (t(\lfloor n^c \rfloor), t(\lfloor (n+k)^c \rfloor)) = \underline{\underline{\varepsilon}}\} = \frac{1}{4}$

$$x \longmapsto \frac{1}{x} \#\{1 \le n \le x \,/\, (t(\lfloor n \log n \rfloor), t(\lfloor (n+1) \log (n+1) \rfloor)) = \underline{\underline{\varepsilon}}\}$$

has no limit.