Averages of Shifted Convolutions of $d_3(n)$

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16 March 2012

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is of central importance in the analytic number theory.
- For $k = 2$, the work of Ingham gives that

$$D_2(N,h) \sim \frac{6}{\pi^2}\sigma_{-1}(h)N\log^2 N, \text{ as } N \to \infty,$$

for given $h \in \mathbb{N}$, where $\sigma_{-1}(h) := \sum_{j|h} j^{-1}$.

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- In general it is expected that $D_k(N, h)$ should be asymptotic to $c_{k,h} N \log^{2k-2} N$, for a suitable constant $c_{k,h} > 0$, uniformly for h in some range.
- However such a description has not yet been proved for any $k \geq 3$, even when h is fixed.

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- $D_k(N, h)$ has a deep connection with

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  for a suitable constant $c_k > 0$.

- Just as for the sums $D_k(N, h)$, we have only succeeded in producing an asymptotic formula for $I_k(T)$ when $k = 1$ (Hardy and Littlewood) or $k = 2$ (Ingham).

Statements of the Results

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- The work of Conrey and Gonek predicts that

$$D(N, h) = \int_N^{2N} \mathfrak{S}(x, h) \mathrm{d}x + O(N^{1/2+\varepsilon}), \tag{1}$$

uniformly for $1 \le h \le N^{1/2}$, where $\mathfrak{S}(x, h)$ is a singular series involving the residue of $\zeta^3(s)$ at $s = 1$.

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  uniformly for $1 \leq h \leq N^{1/2}$, where $\mathfrak{S}(x, h)$ is a singular series involving the residue of $\zeta^3(s)$ at $s = 1$.
- Let

$$\Delta(N, h) := D(N, h) - \int_N^{2N} \mathfrak{S}(x, h)\mathrm{d}x.$$

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Theorem

Assume that $1 \leq H \leq N$. *Then*

$$\sum_{h \leq H} \Delta(N,h) \ll \left(H^2 + H^{1/2} N^{13/12}\right) N^{\varepsilon}.$$

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- The exponents appearing in this estimate can be improved slightly for certain ranges of H. This has been recently done by A. Ivić and J. Wu.

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- The work of Ivić leads to the upper bound

$$I_3(T) \ll T^{1+\varepsilon} + T^{(\alpha+3\beta-1)/2+\varepsilon}$$

for the sixth moment of the Riemann zeta function on the critical line, where $\alpha, \beta \in [0, 1]$ are constants such that $\alpha + \beta \geq 1$ and an asymptotic formula of the shape

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- The theorem gives the choices $\alpha = 1/2$ and $\beta = 13/12$, which yields $I_3(T) \ll T^{11/8+\varepsilon}$. But this does not give any improvement over the well-known bound for $I_3(T)$.

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- The above theorem gives that the asymptotic formula

$$D(N, h) \sim \int_N^{2N} \mathfrak{S}(x, h) \mathrm{d}x$$

holds for almost all $h \leq H$ if $N^{1/3+\varepsilon} \leq H \leq N^{1-\varepsilon}$.

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$$\mathrm{Res}_{s=1}\zeta^3(s)\frac{(n+H)^s - n^s}{s} + \frac{1}{2\pi i}\left(\int_{\mathcal{P}_1} + \int_{\mathcal{P}_2} + \int_{\sigma-iT}^{\sigma+iT}\right)\zeta^3(s)\left((n+H)^s - n^s\right)\frac{\mathrm{d}s}{s},$$

with $1/2 < \sigma < 1$ and $2 \leq T \leq N$.

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with $1/2<\sigma<1$ and $2\le T\le N$.

- The first term above, together with a result of Voronoi, gives the main term.

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- Combining everything, we get our first moment theorem.

Proof of the Second Moment Theorem

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- Then the range of integration is divided into the major and minor arcs.
- The major arcs are part of the interval $[0,1]$ that are close to rational numbers with small denominators and the minor arcs form the rest of the interval.

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- The major arcs, as usual, give the main term.
- Collecting everything, we have our second moment result.

Thank you for your attention!