A “difficult and deep” identity of Ramanujan

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The identity

If

$$u = \cfrac{q}{1 + \cfrac{q^5}{1 + \cfrac{q^{10}}{1 + \cfrac{q^{15}}{1 + \ddots}}}}$$

and

$$v = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q^1}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}}$$

then

$$v^5 = u\,\frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}.$$

Preliminaries

Notation: $|q| < 1$

$$(a;q)_\infty = \prod_{n\geq 0}(1 - aq^n),$$

$$(a_1, a_2, \cdots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty.$$

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Preliminaries

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$$(a;q)_\infty = \prod_{n\geq 0}(1-aq^n),$$

$$(a_1,a_2,\ \cdots\ ,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty.$$

We start with Jacobi's triple product identity

$$(-a^{-1}q,-aq,q;q^2)_\infty = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

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Preliminaries

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We start with Jacobi's triple product identity

$$(-a^{-1}q,-aq,q;q^2)_\infty = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

Replace q by $q^{\frac{1}{2}}$, replace a by $-aq^{\frac{1}{2}}$, obtain

$$(1-a^{-1})(a^{-1}q,aq,q;q)_\infty = \sum_{k\geq 0}(-1)^k(a^k-a^{-k-1})q^{(k^2+k)/2}.$$

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Preliminaries

Notation: $|q| < 1$

$$(a;q)_\infty = \prod_{n\geq 0}(1-aq^n),$$

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$$(-a^{-1}q,-aq,q;q^2)_\infty = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

Replace q by $q^{\frac{1}{2}}$, replace a by $-aq^{\frac{1}{2}}$, obtain

$$(1-a^{-1})(a^{-1}q,aq,q;q)_\infty = \sum_{k\geq 0}(-1)^k(a^k-a^{-k-1})q^{(k^2+k)/2}.$$

Divide by $(1-a^{-1})$, obtain

$$(a^{-1}q,aq,q;q)_\infty = 1+\sum_{k\geq 1}(-1)^k(a^k+a^{k-1}+\cdots+a^{-k})q^{(k^2+k)/2}.$$

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Put $a = 1$, obtain Jacobi's identity

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{(k^2+k)/2} = \sum_{-\infty}^{\infty}(4k+1)q^{2k^2+k}.$$

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Put $a = 1$, obtain Jacobi's identity

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{(k^2+k)/2} = \sum_{-\infty}^{\infty}(4k+1)q^{2k^2+k}.$$

$$(a^{-1}q, aq, q; q)_\infty = 1+\sum_{k\geq 1}(-1)^k(a^k+a^{k-1}+\cdots+a^{-k})q^{(k^2+k)/2}.$$

Put $a = \eta$, a fifth root of unity other than 1.

$$\eta^k + \eta^{k-1} + \cdots + \eta^{-k} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 5, \\ -(\eta^2+\eta^{-2}) & \text{if } k \equiv 1 \pmod 5, \\ 0 & \text{if } k \equiv 2 \pmod 5, \\ \eta^2+\eta^{-2} & \text{if } k \equiv 3 \pmod 5, \\ -1 & \text{if } k \equiv 4 \pmod 5. \end{cases}$$

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$$(\eta^{-1}q, \eta q, q; q)_\infty$$
$$= (q^{10}, q^{15}, q^{25}; q^{25})_\infty + (\eta^2 + \eta^{-2})q(q^5, q^{20}, q^{25}; q^{25})_\infty.$$

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Important identity of Ramanujan

Write $A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty.$

$$(\eta^{-1}q, \eta q, q; q)_\infty = A + (\eta^2 + \eta^{-2})qB.$$

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Important identity of Ramanujan

Write $A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty$.

$$(\eta^{-1}q, \eta q, q; q)_\infty = A + (\eta^2 + \eta^{-2})qB.$$

Put η^2 for η, obtain

$$(\eta^{-2}q, \eta^2 q, q; q)_\infty = A + (\eta + \eta^{-1})qB.$$

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Important identity of Ramanujan

Write $A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty.$

$$(\eta^{-1}q, \eta q, q; q)_\infty = A + (\eta^2 + \eta^{-2})qB.$$

Put η^2 for η, obtain

$$(\eta^{-2}q, \eta^2 q, q; q)_\infty = A + (\eta + \eta^{-1})qB.$$

In one or the other order, these are

$$\prod_{k\geq 1}(1 + \alpha q^k + q^{2k})(1 - q^k) = A - \beta qB,$$

$$\prod_{k\geq 1}(1 + \beta q^k + q^{2k})(1 - q^k) = A - \alpha qB.$$

where $\alpha = (1 + \sqrt{5})/2, \quad \beta = (1 - \sqrt{5})/2.$

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Multiply these, then divide by $(q^5;q^5)_\infty$, obtain

$$
\begin{aligned}
&(q;q)_\infty \\
&= (q^{25};q^{25})_\infty \\
&\times \left(\left(\begin{matrix} q^{10}, q^{15} \\ q^5, q^{20} \end{matrix} ; q^{25} \right)_\infty - q - q^2 \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix} ; q^{25} \right)_\infty \right).
\end{aligned}
$$

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Let $\eta = e^{\frac{2\pi i}{5}}$. Then $\eta + \eta^{-1} = -\beta, \quad \eta^2 + \eta^{-2} = -\alpha$.
We saw

$$A - \beta q B = \prod_{k\geq 1}(1 + \alpha q^k + q^{2k})(1 - q^k),$$

$$A - \alpha q B = \prod_{k\geq 1}(1 + \beta q^k + q^{2k})(1 - q^k).$$

Set $\eta^2 q$ and $\eta^{-2} q$ for q in the first, multiply, obtain

$$\begin{aligned} &A^2 - qAB + \beta^2 q^2 B^2 \\ &= \prod_{k\geq 1}(1 + \alpha\eta^{2k}q^k + \eta^{4k}q^{2k})(1 + \alpha\eta^{-2k}q^k + \eta^{-4k}q^{2k}) \\ &\times(1 - \eta^{2k}q^k)(1 - \eta^{-2k}q^k). \end{aligned}$$

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Set ηq and $\eta^{-1}q$ for q in the second, multiply, obtain

$$
\begin{aligned}
&A^2 - qAB + \alpha^2 q^2 B^2 \\
&= \prod_{k\geq 1}(1+\beta\eta^k q^k + \eta^{2k}q^{2k})(1+\beta\eta^{-k}q^k + \eta^{-2k}q^{2k}) \\
&\times(1-\eta^k q^k)(1-\eta^{-k}q^k).
\end{aligned}
$$

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&\times(1-\eta^k q^k)(1-\eta^{-k}q^k).
\end{aligned}
$$

Multiply these last two, obtain

$$
\begin{aligned}
&A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + q^4B^4 \\
&= \prod_{k\geq 1}(1+\alpha\eta^{2k}q^k+\eta^{4k}q^{2k})(1+\alpha\eta^{-2k}q^k+\eta^{-4k}q^{2k}) \\
&\qquad \times(1+\beta\eta^k q^k+\eta^{2k}q^{2k})(1+\beta\eta^{-k}q^k+\eta^{-2k}q^{2k}) \\
&\qquad \times(1-\eta^{2k}q^k)(1-\eta^{-2k}q^k)(1-\eta^k q^k)(1-\eta^{-k}q^k)
\end{aligned}
$$

Set ηq and $\eta^{-1}q$ for q in the second, multiply, obtain

$$\begin{aligned}
&A^2 - qAB + \alpha^2 q^2 B^2 \\
&= \prod_{k\geq 1}(1+\beta\eta^k q^k + \eta^{2k}q^{2k})(1+\beta\eta^{-k}q^k+\eta^{-2k}q^{2k}) \\
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\end{aligned}$$

Multiply these last two, obtain

$$\begin{aligned}
&A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + q^4B^4 \\
&= \prod_{k\geq 1}(1+\alpha\eta^{2k}q^k + \eta^{4k}q^{2k})(1+\alpha\eta^{-2k}q^k+\eta^{-4k}q^{2k}) \\
&\quad\times(1+\beta\eta^k q^k + \eta^{2k}q^{2k})(1+\beta\eta^{-k}q^k+\eta^{-2k}q^{2k}) \\
&\quad\times(1-\eta^{2k}q^k)(1-\eta^{-2k}q^k)(1-\eta^k q^k)(1-\eta^{-k}q^k)
\end{aligned}$$

Consider cases $k\equiv 0,\quad k\equiv \pm 1,\quad k\equiv \pm 2 \mod 5$ separately.

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$k \equiv 0 \pmod 5$

$$\text{prod} = \prod_{k \equiv 0 \pmod 5} (1 + \alpha q^k + q^{2k})^2 (1 + \beta q^k + q^{2k})^2 (1 - q^k)^4$$

$$= \prod_{k \equiv 0 \pmod 5} (1 + q^k + q^{2k} + q^{3k} + q^{4k})^2 (1 - q^k)^4$$

$$= (q^5; q^5)_\infty^2 (q^{25}; q^{25})_\infty^2 .$$

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$$\text{prod} = \prod_{k\equiv 0 \pmod 5} (1+\alpha q^k + q^{2k})^2(1+\beta q^k + q^{2k})^2(1-q^k)^4$$
$$= \prod_{k\equiv 0 \pmod 5} (1+q^k+q^{2k}+q^{3k}+q^{4k})^2(1-q^k)^4$$
$$= (q^5;q^5)_\infty^2 (q^{25};q^{25})_\infty^2.$$

$k \equiv \pm 1 \pmod 5$ $\qquad$ $\text{prod} = (q,q^4;q^5)_\infty^2(q^5,q^{20};q^{25})_\infty^2$

$k \equiv \pm 2 \pmod 5$ $\qquad$ $\text{prod} = \dfrac{(q^{10},q^{15};q^{25})_\infty^3}{(q^2,q^3;q^5)_\infty}$

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$k \equiv 0 \pmod 5$

$$\text{prod} = \prod_{k\equiv 0 \pmod 5} (1+\alpha q^k + q^{2k})^2(1+\beta q^k + q^{2k})^2(1-q^k)^4$$
$$= \prod_{k\equiv 0 \pmod 5} (1+q^k+q^{2k}+q^{3k}+q^{4k})^2(1-q^k)^4$$
$$= (q^5;q^5)_\infty^2(q^{25};q^{25})_\infty^2.$$

$k \equiv \pm 1 \pmod 5$ $\qquad$ $\text{prod} = (q,q^4;q^5)_\infty^2(q^5,q^{20};q^{25})_\infty^2$

$k \equiv \pm 2 \pmod 5$ $\qquad$ $\text{prod} = \dfrac{(q^{10},q^{15};q^{25})_\infty^3}{(q^2,q^3;q^5)_\infty}$

$$A^4 - 2qA^3B + 4q^2 - 3q^3AB^3 + q^4B^4$$
$$= (q^5;q^5)_\infty^4 \frac{(q,q^4;q^5)_\infty^2(q^{10},q^{15};q^{25})_\infty}{(q^2,q^3;q^5)_\infty}.$$

We saw

$$A - \beta q B = \prod_{k\geq 1}(1+\alpha q^k + q^{2k})(1-q^k),$$

$$A - \alpha q B = \prod_{k\geq 1}(1+\beta q^k + q^{2k})(1-q^k).$$

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$$A - \alpha q B = \prod_{k\geq 1}(1+\beta q^k + q^{2k})(1-q^k).$$

Set ηq and $\eta^{-1} q$ for q in the first, multiply, obtain

$$A^2 + \beta^2 q AB + \beta^2 q^2 B^2 = \text{product}$$

set $\eta^2 q$ and $\eta^{-2} q$ for q in the second, multiply, obtain

$$A^2 + \alpha^2 q AB + \alpha^2 q^2 B^2 = \text{product}.$$

We saw

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set $\eta^2 q$ and $\eta^{-2}q$ for q in the second, multiply, obtain

$$A^2 + \alpha^2 qAB + \alpha^2 q^2 B^2 = \text{product}.$$

Multiply these, consider cases $k \equiv 0, \quad k \equiv \pm 1, \quad k \equiv \pm 2 \pmod 5$, obtain

$$\begin{aligned} &A^4 + 3qA^3B + 4q^2A^2B^2 + 2q^3AB^3 + B^4 \\ &\quad = (q^5;q^5)_\infty^4 \frac{(q^2,q^3;q^5)_\infty^2 (q^5,q^{20};q^{25})_\infty}{(q,q^4;q^5)_\infty^3}.i \end{aligned}$$

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$$\frac{A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + B^4}{A^4 + 3qA^3B + 4q^2A^2B^2 + 2q^3AB^3 + B^4}$$
$$= \frac{(q, q^4; q^5)_\infty^5 (q^{10}, q^{15}; q^{25})_\infty}{(q^2, q^3; q^5)_\infty^5 (q^5, q^{20}; q^{25})_\infty}$$
$$= \begin{pmatrix} q, & q^4 \\ q^2, & q^3 \end{pmatrix}; q^5 \Big)^5 \frac{A}{B}.$$

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$$\frac{A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + B^4}{A^4 + 3qA^3B + 4q^2A^2B^2 + 2q^3AB^3 + B^4}$$
$$= \frac{(q, q^4; q^5)_\infty^5 (q^{10}, q^{15}; q^{25})_\infty}{(q^2, q^3; q^5)_\infty^5 (q^5, q^{20}; q^{25})_\infty}$$
$$= \left(\begin{matrix} q, & q^4 \\ q^2, & q^3 \end{matrix}; q^5 \right)^5 \frac{A}{B}.$$

$$\left(q^{\frac{1}{5}} \left(\begin{matrix} q, & q^4 \\ q^2, & q^3 \end{matrix}; q^5 \right)_\infty \right)^5$$
$$= q\frac{B}{A} \cdot \frac{1 - 2qB/A + 4q^2B^2/A^2 - 3q^3B^3/A^3 + q^4B^4/A^4}{1 + 3qB/A + 4q^2B^2/A^2 + 2q^3B^3/A^3 + q^4B^4/A^4}.$$

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Guess what?

$$v = q^{\frac{1}{5}} \begin{pmatrix} q, & q^4 \\ q^2, & q^3 \end{pmatrix}; q^5 \Big)_\infty, \quad u = q \begin{pmatrix} q^5, & q^{20} \\ q^{10}, & q^{15} \end{pmatrix}; q^{25} \Big)_\infty = q\,\frac{B}{A},$$

so we are done ...

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Guess what?

$$v = q^{\frac{1}{5}} \begin{pmatrix} q, & q^4 \\ q^2, & q^3 \end{pmatrix}_{\infty} ; q^5 , \quad u = q \begin{pmatrix} q^5, & q^{20} \\ q^{10}, & q^{15} \end{pmatrix}_{\infty} ; q^{25} = q\,\frac{B}{A},$$

so we are done ...

...provided we believe that

$$\cfrac{1}{1+\cfrac{q^1}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} = \left(\begin{matrix} q, & q^4 \\ q^2, & q^3 \end{matrix} ; q^5\right)_{\infty} .$$

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Guess what?

$$v = q^{\frac{1}{5}} \left(\begin{matrix} q, & q^4 \\ q^2, & q^3 \end{matrix} ; q^5 \right)_\infty , \quad u = q \left(\begin{matrix} q^5, & q^{20} \\ q^{10}, & q^{15} \end{matrix} ; q^{25} \right)_\infty = q \, \frac{B}{A},$$

so we are done ...

...provided we believe that

$$\cfrac{1}{1 + \cfrac{q^1}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}} = \left(\begin{matrix} q, & q^4 \\ q^2, & q^3 \end{matrix} ; q^5 \right)_\infty .$$

Here's a proof: Let

$$F(a) = \sum_{k \geq 0} a^k \frac{q^{k^2}}{(q;q)_k}$$

where $(q;q)_0 = 1, \; (q;q)_k = (1-q) \cdots (1-q^k)$ for $k \geq 1$.

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It is easy to show that

$$F(a) = F(aq) + aqF(aq^2).$$

It follows that

$$\frac{F(aq)}{F(a)} = \cfrac{1}{1 + \cfrac{aq^1}{\left(\cfrac{F(aq)}{F(aq^2)}\right)}}.$$

By iteration

$$\frac{F(aq)}{F(a)} = \cfrac{1}{1 + \cfrac{aq^1}{1 + \cfrac{aq^2}{1 + \cfrac{aq^3}{1 + \ddots}}}}.$$

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Put $a = 1$.

$$\cfrac{1}{1+\cfrac{q^1}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} = \frac{F(q)}{F(1)}$$

$$= \frac{\sum_{k\geq 0} \dfrac{q^{k^2+k}}{(q;q)_k}}{\sum_{k\geq 0} \dfrac{q^{k^2}}{(q;q)_k}}.$$

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$$\cfrac{1}{1+\cfrac{q^1}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} = \frac{F(q)}{F(1)}$$

$$= \frac{\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k}}{\sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k}}.$$

Now all we need is the Rogers–Ramanujan identities:

$$\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2,q^3;q^5)_\infty}, \quad \sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q,q^4;q^5)_\infty}.$$

Put $a = 1$.

$$\cfrac{1}{1+\cfrac{q^1}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} = \frac{F(q)}{F(1)}$$

$$= \frac{\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k}}{\sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k}}.$$

Now all we need is the Rogers–Ramanujan identities:

$$\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2,q^3;q^5)_\infty}, \quad \sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q,q^4;q^5)_\infty}.$$

the "deep" bit?

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Almost there
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The Rogers–Ramanujan identities

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$$= 1 + \sum_{k\geq 1}(-1)^k a^{2k} q^{(5k^2-k)/2}(1-aq^{2k})\frac{(aq;q)_{k-1}}{(q;q)_k}.$$

The identity
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continued...
Important identity
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Down to business
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For a proof of this identity, see Ramanujan's Collected works, or Chapter 15 in my (forthcoming) book.

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