A "difficult and deep" identity of Ramanujan

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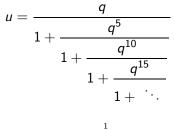
A "difficult and deep" identity of Ramanujan

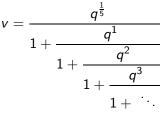
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The identity

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then

$$v^{5} = u \frac{1 - 2u + 4u^{2} - 3u^{3} + u^{4}}{1 + 3u + 4u^{2} + 2u^{3} + u^{4}}.$$

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The identity

The Rogers–Ramanujan identities

Notation: |q| < 1

$$(a;q)_{\infty}=\prod_{n\geq 0}(1-aq^n),$$

$$(a_1,a_2,\cdots,a_k;q)_{\infty}=(a_1;q)_{\infty}(a_2;q)_{\infty}\cdots(a_k;q)_{\infty}.$$

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Preliminaries

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The Rogers–Ramanujan identities

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Notation: |q| < 1

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 $(a_1,a_2,\cdots,a_k;q)_{\infty}=(a_1;q)_{\infty}(a_2;q)_{\infty}\cdots(a_k;q)_{\infty}.$

We start with Jacobi's triple product identity

$$(-a^{-1}q,-aq,q;q^2)_{\infty}=\sum_{-\infty}^{\infty}a^nq^{n^2}.$$

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Preliminaries

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We start with Jacobi's triple product identity

$$(-a^{-1}q,-aq,q;q^2)_{\infty}=\sum_{-\infty}^{\infty}a^nq^{n^2}.$$

Replace q by $q^{\frac{1}{2}}$, replace a by $-aq^{\frac{1}{2}}$, obtain

$$(1\!-\!a^{-1})(a^{-1}q,aq,q;q)_\infty = \sum_{k\geq 0} (-1)^k (a^k\!-\!a^{-k-1})q^{(k^2+k)/2}.$$

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Preliminaries continued

Notation: |q| < 1

$$(a;q)_{\infty}=\prod_{n\geq 0}(1-aq^n),$$

 $(a_1, a_2, \cdots, a_k; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_k; q)_{\infty}.$

We start with Jacobi's triple product identity

$$(-a^{-1}q,-aq,q;q^2)_{\infty}=\sum_{-\infty}^{\infty}a^nq^{n^2}.$$

Replace q by $q^{\frac{1}{2}}$, replace a by $-aq^{\frac{1}{2}}$, obtain $(1-a^{-1})(a^{-1}q, aq, q; q)_{\infty} = \sum_{k\geq 0} (-1)^k (a^k - a^{-k-1})q^{(k^2+k)/2}.$

Divide by $(1 - a^{-1})$, obtain $(a^{-1}q, aq, q; q)_{\infty} = 1 + \sum_{k \ge 1} (-1)^k (a^k + a^{k-1} + \dots + a^{-k}) q^{(k^2 + k)/2}$

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Preliminaries

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Put a = 1, obtain Jacobi's identity

$$(q;q)_{\infty}^{3} = \sum_{k\geq 0} (-1)^{k} (2k+1)q^{(k^{2}+k)/2} = \sum_{-\infty}^{\infty} (4k+1)q^{2k^{2}+k}.$$

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Put a = 1, obtain Jacobi's identity

$$(q;q)^3_{\infty} = \sum_{k\geq 0} (-1)^k (2k+1)q^{(k^2+k)/2} = \sum_{-\infty}^{\infty} (4k+1)q^{2k^2+k}.$$

$$(a^{-1}q,aq,q;q)_{\infty} = 1 + \sum_{k \geq 1} (-1)^k (a^k + a^{k-1} + \dots + a^{-k}) q^{(k^2+k)/2}$$

Put $a = \eta$, a fifth root of unity other than 1.

$$\eta^{k} + \eta^{k-1} + \cdots + \eta^{-k} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{5}, \\ -(\eta^{2} + \eta^{-2} & \text{if } k \equiv 1 \pmod{5}, \\ 0 & \text{if } k \equiv 2 \pmod{5}, \\ \eta^{2} + \eta^{-2} & \text{if } k \equiv 3 \pmod{5}, \\ -1 & \text{if } k \equiv 4 \pmod{5}. \end{cases}$$

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$\begin{aligned} &(\eta^{-1}q,\eta q,q;q)_{\infty} \\ &= (q^{10},q^{15},q^{25};q^{25})_{\infty} + (\eta^2+\eta^{-2})q(q^5,q^{20},q^{25};q^{25})_{\infty}. \end{aligned}$

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The identity

Preliminaries

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The Rogers–Ramanujan identities

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Important identity of Ramanujan

Write
$$A = (q^{10}, q^{15}, q^{25}; q^{25})_{\infty}, \quad B = (q^5, q^{20}, q^{25}; q^{25})_{\infty}.$$

 $(\eta^{-1}q, \eta q, q; q)_{\infty} = A + (\eta^2 + \eta^{-2})qB.$

A "difficult and deep" identity of Ramanujan

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The identity

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Important identity

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Important identity of Ramanujan

Write
$$A = (q^{10}, q^{15}, q^{25}; q^{25})_{\infty}, \ B = (q^5, q^{20}, q^{25}; q^{25})_{\infty}.$$

 $(\eta^{-1}q, \eta q, q; q)_{\infty} = A + (\eta^2 + \eta^{-2})qB.$

Put η^2 for η , obtain

$$(\eta^{-2}q, \eta^2 q, q; q)_{\infty} = A + (\eta + \eta^{-1})qB.$$

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The identity

Preliminaries

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Important identity

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Important identity of Ramanujan Write $A = (q^{10}, q^{15}, q^{25}; q^{25})_{\infty}, B = (q^5, q^{20}, q^{25}; q^{25})_{\infty}.$ $(\eta^{-1}q, \eta q, q; q)_{\infty} = A + (\eta^2 + \eta^{-2})qB.$

Put η^2 for η , obtain

$$(\eta^{-2}q,\eta^2q,q;q)_{\infty}=A+(\eta+\eta^{-1})qB.$$

In one or the other order, these are

$$\begin{split} &\prod_{k\geq 1}(1+\alpha q^k+q^{2k})(1-q^k)=A-\beta qB,\\ &\prod_{k\geq 1}(1+\beta q^k+q^{2k})(1-q^k)=A-\alpha qB. \end{split}$$
 where $\alpha=(1+\sqrt{5})/2, \ \ \beta=(1-\sqrt{5})/2.$

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Important identity continued

Multiply these, then divide by $(q^5; q^5)_{\infty}$, obtain

$$\begin{aligned} &(q;q)_{\infty} \\ &= (q^{25};q^{25})_{\infty} \\ &\times \left(\begin{pmatrix} q^{10},q^{15} \\ q^5,q^{20};q^{25} \end{pmatrix}_{\infty} - q - q^2 \begin{pmatrix} q^5,q^{20} \\ q^{10},q^{15};q^{25} \end{pmatrix}_{\infty} \right) \end{aligned}$$

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Down to business

Let
$$\eta = e^{\frac{2\pi i}{5}}$$
. Then $\eta + \eta^{-1} = -\beta$, $\eta^2 + \eta^{-2} = -\alpha$. We saw

$$A - \beta qB = \prod_{k \ge 1} (1 + \alpha q^k + q^{2k})(1 - q^k),$$

$$A - \alpha qB = \prod_{k \ge 1} (1 + \beta q^k + q^{2k})(1 - q^k).$$

Set $\eta^2 q$ and $\eta^{-2} q$ for q in the first, multiply, obtain

$$\begin{split} &A^2 - qAB + \beta^2 q^2 B^2 \\ &= \prod_{k \ge 1} (1 + \alpha \eta^{2k} q^k + \eta^{4k} q^{2k}) (1 + \alpha \eta^{-2k} q^k + \eta^{-4k} q^{2k}) \\ &\times (1 - \eta^{2k} q^k) (1 - \eta^{-2k} q^k). \end{split}$$

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Set ηq and $\eta^{-1}q$ for q in the second, multiply, obtain

$$\begin{aligned} &A^2 - qAB + \alpha^2 q^2 B^2 \\ &= \prod_{k \ge 1} (1 + \beta \eta^k q^k + \eta^{2k} q^{2k}) (1 + \beta \eta^{-k} q^k + \eta^{-2k} q^{2k}) \\ &\times (1 - \eta^k q^k) (1 - \eta^{-k} q^k). \end{aligned}$$

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$$\begin{split} & A^2 - qAB + \alpha^2 q^2 B^2 \\ &= \prod_{k \geq 1} (1 + \beta \eta^k q^k + \eta^{2k} q^{2k}) (1 + \beta \eta^{-k} q^k + \eta^{-2k} q^{2k}) \\ &\times (1 - \eta^k q^k) (1 - \eta^{-k} q^k). \end{split}$$

Multiply these last two, obtain

$$\begin{aligned} A^{4} - 2qA^{3}B + 4q^{2}A^{2}B^{2} - 3q^{3}AB^{3} + q^{4}B^{4} \\ &= \prod_{k \ge 1} (1 + \alpha \eta^{2k}q^{k} + \eta^{4k}q^{2k})(1 + \alpha \eta^{-2k}q^{k} + \eta^{-4k}q^{2k}) \\ &\times (1 + \beta \eta^{k}q^{k} + \eta^{2k}q^{2k})(1 + \beta \eta^{-k}q^{k} + \eta^{-2k}q^{2k}) \\ &\times (1 - \eta^{2k}q^{k})(1 - \eta^{-2k}q^{k})(1 - \eta^{k}q^{k})(1 - \eta^{-k}q^{k}) \end{aligned}$$

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continuing...

Rogers–Ramanujan identities

Set ηq and $\eta^{-1}q$ for q in the second, multiply, obtain

$$\begin{split} & A^2 - qAB + \alpha^2 q^2 B^2 \\ &= \prod_{k \geq 1} (1 + \beta \eta^k q^k + \eta^{2k} q^{2k}) (1 + \beta \eta^{-k} q^k + \eta^{-2k} q^{2k}) \\ &\times (1 - \eta^k q^k) (1 - \eta^{-k} q^k). \end{split}$$

Multiply these last two, obtain

$$\begin{aligned} A^{4} - 2qA^{3}B + 4q^{2}A^{2}B^{2} - 3q^{3}AB^{3} + q^{4}B^{4} \\ &= \prod_{k\geq 1} (1 + \alpha\eta^{2k}q^{k} + \eta^{4k}q^{2k})(1 + \alpha\eta^{-2k}q^{k} + \eta^{-4k}q^{2k}) \\ &\times (1 + \beta\eta^{k}q^{k} + \eta^{2k}q^{2k})(1 + \beta\eta^{-k}q^{k} + \eta^{-2k}q^{2k}) \\ &\times (1 - \eta^{2k}q^{k})(1 - \eta^{-2k}q^{k})(1 - \eta^{k}q^{k})(1 - \eta^{-k}q^{k}) \end{aligned}$$

Consider cases $k \equiv 0$, $k \equiv \pm 1$, $k \equiv \pm 2 \mod 5$ separately.

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continuing ...

$$k \equiv 0 \pmod{5}$$

$$\text{prod} = \prod_{k \equiv 0 \pmod{5}} (1 + \alpha q^k + q^{2k})^2 (1 + \beta q^k + q^{2k})^2 (1 - q^k)^4$$

$$= \prod_{k \equiv 0 \pmod{5}} (1 + q^k + q^{2k} + q^{3k} + q^{4k})^2 (1 - q^k)^4$$

$$= (q^5; q^5)_{\infty}^2 (q^{25}; q^{25})_{\infty}^2.$$

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$$k \equiv 0 \pmod{5}$$

$$prod = \prod_{k \equiv 0 \pmod{5}} (1 + \alpha q^k + q^{2k})^2 (1 + \beta q^k + q^{2k})^2 (1 - q^k)^4$$

$$= \prod_{k \equiv 0 \pmod{5}} (1 + q^k + q^{2k} + q^{3k} + q^{4k})^2 (1 - q^k)^4$$

$$= (q^5; q^5)^2_{\infty} (q^{25}; q^{25})^2_{\infty}.$$

$$k \equiv \pm 1 \pmod{5} \qquad \text{prod} = (q, q^4; q^5)^2_{\infty} (q^5, q^{20}; q^{25})^2_{\infty}$$

$$k \equiv \pm 2 \pmod{5} \qquad \text{prod} = \frac{(q^{10}, q^{15}; q^{25})^3_{\infty}}{(q^2, q^3; q^5)_{\infty}}$$

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$$= (q^5; q^5)^2_{\infty} (q^{25}; q^{25})^2_{\infty}.$$

$$= (q^{10}, q^{15}; q^{25})^3_{\infty}$$

$$(not follow)$$

$$k \equiv \pm 2 \pmod{5} \qquad \text{prod} = \frac{(q^{10}, q^{15}; q^{25})^3_{\infty}}{(q^2, q^3; q^5)_{\infty}}$$

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Ramanujan Michael D. Hirschhorn

$$k \equiv 0 \pmod{5}$$

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$$= \prod_{k \equiv 0 \pmod{5}} (1 + q^k + q^{2k} + q^{3k} + q^{4k})^2 (1 - q^k)^4$$

$$= (q^5; q^5)^2_{\infty} (q^{25}; q^{25})^2_{\infty}$$

$$k \equiv \pm 1 \pmod{5} \quad \text{prod} = (q, q^4; q^5)^2_{\infty} (q^5, q^{20}; q^{25})^2_{\infty}$$

$$k \equiv \pm 2 \pmod{5} \quad \text{prod} = \frac{(q^{10}, q^{15}; q^{25})^3_{\infty}}{(q^2, q^3; q^5)_{\infty}}$$

$$A^4 - 2qA^3B + 4q^2 - 3q^3AB^3 + q^4B^4$$

$$= (q^5; q^5)^4_{\infty} \frac{(q, q^4; q^5)^2_{\infty} (q^{10}, q^{15}; q^{25})_{\infty}}{(q^2, q^3; q^5)_{\infty}}$$
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We saw

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$$A - \alpha qB = \prod_{k \ge 1} (1 + \beta q^k + q^{2k})(1 - q^k).$$

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The Rogers–Ramanujan identities

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We saw

$$egin{aligned} &A-eta qB = \prod_{k\geq 1}(1+lpha q^k+q^{2k})(1-q^k),\ &A-lpha qB = \prod_{k\geq 1}(1+eta q^k+q^{2k})(1-q^k). \end{aligned}$$

Set ηq and $\eta^{-1}q$ for q in the first, multiply, obtain $A^2 + \beta^2 q A B + \beta^2 q^2 B^2 = \text{product}$ set $\eta^2 q$ and $\eta^{-2}q$ for q in the second, multiply, obtain

$$A^2 + \alpha^2 qAB + \alpha^2 q^2B^2 =$$
product.

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We saw

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$$egin{aligned} &A-eta qB = \prod_{k\geq 1}(1+lpha q^k+q^{2k})(1-q^k), \ &A-lpha qB = \prod_{k\geq 1}(1+eta q^k+q^{2k})(1-q^k). \end{aligned}$$

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set $\eta^2 q$ and $\eta^{-2} q$ for q in the second, multiply, obtain

$$A^2 + \alpha^2 qAB + \alpha^2 q^2B^2 =$$
product.

Multiply these, consider cases $k \equiv 0$, $k \equiv \pm 1$, $k \equiv \pm 2 \pmod{5}$, obtain

$$A^{4} + 3qA^{3}B + 4q^{2}A^{2}B^{2} + 2q^{3}AB^{3} + B^{4}$$

= $(q^{5}; q^{5})_{\infty}^{4} \frac{(q^{2}, q^{3}; q^{5})_{\infty}^{2}(q^{5}, q^{20}; q^{25})_{\infty}}{(q, q^{4}; q^{5})_{\infty}^{3}}.i$

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continuing...

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$$\begin{aligned} &\frac{A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + B^4}{A^4 + 3qA^3B + 4q^2A^2B^2 + 2q^3AB^3 + B^4} \\ &= \frac{(q, q^4; q^5)_\infty^5(q^{10}, q^{15}; q^{25})_\infty}{(q^2, q^3; q^5)_\infty^5(q^5, q^{20}; q^{25})_\infty} \\ &= \left(\begin{pmatrix} q, q^4 \\ q^2, q^3 \end{pmatrix} \right)^5 \frac{A}{B}. \end{aligned}$$

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Almost there

Almost there

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$$\begin{aligned} \frac{A^4 - 2qA^3B + 4q^2A^2B^2 - 3q^3AB^3 + B^4}{A^4 + 3qA^3B + 4q^2A^2B^2 + 2q^3AB^3 + B^4} \\ &= \frac{(q, q^4; q^5)_\infty^5(q^{10}, q^{15}; q^{25})_\infty}{(q^2, q^3; q^5)_\infty^5(q^5, q^{20}; q^{25})_\infty} \\ &= \left(\frac{q, q^4}{q^2, q^3}; q^5\right)_\infty^5 \frac{A}{B}. \end{aligned}$$
$$\begin{pmatrix} q^{\frac{1}{5}} \left(\frac{q, q^4}{q^2, q^3}; q^5\right)_\infty^5 \\ &= q\frac{B}{A} \cdot \frac{1 - 2qB/A + 4q^2B^2/A^2 - 3q^3B^3/A^3 + q^4B^4/A^4}{1 + 3qB/A + 4q^2B^2/A^2 + 2q^3B^3/A^3 + q^4B^4/A^4}. \end{aligned}$$

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Guess what?

$$v = q^{rac{1}{5}} \begin{pmatrix} q, \ q^4 \ q^2, q^3; q^5 \end{pmatrix}_{\infty}, \ \ u = q \begin{pmatrix} q^5, q^{20} \ q^{15}; q^{25} \end{pmatrix}_{\infty} = q rac{B}{A},$$

so we are done ...

A "difficult and deep" identity of Ramanujan

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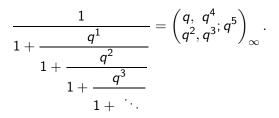
continued...

Guess what?

$$v = q^{rac{1}{5}} \begin{pmatrix} q, \ q^4 \\ q^2, \ q^3; \ q^5 \end{pmatrix}_{\infty}, \ \ u = q \begin{pmatrix} q^5, \ q^{20} \\ q^{10}, \ q^{15}; \ q^{25} \end{pmatrix}_{\infty} = q rac{B}{A},$$

so we are done ...

... provided we believe that



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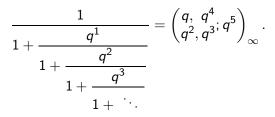
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Here's a proof: Let

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$$F(a) = \sum_{k \ge 0} a^k \frac{q^{k^2}}{(q;q)_k}$$

where $(q;q)_0 = 1$, $(q;q)_k = (1-q) \cdots (1-q^k)$ for $k \ge 1$.

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continued...

It is easy to show that

$$F(a) = F(aq) + aqF(aq^2).$$

It follows that

$$rac{F(aq)}{F(a)} = rac{1}{1+rac{aq^1}{\left(rac{F(aq)}{F(aq^2)}
ight)}}.$$

By iteration

$$\frac{F(aq)}{F(a)} = \frac{1}{1 + \frac{aq^1}{1 + \frac{aq^2}{1 + \frac{aq^3}{1 + \frac{aq^3}{1 + \frac{\cdot}{\cdot}}}}}}.$$

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continued...

The Rogers–Ramanujan identities

Put a = 1.

$$\frac{1}{1 + \frac{q^1}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^3}{1 + \frac{\cdots}{(q;q)_k}}}}} = \frac{F(q)}{F(1)}$$
$$= \frac{\sum_{k \ge 0} \frac{q^{k^2 + k}}{(q;q)_k}}{\sum_{k \ge 0} \frac{q^{k^2}}{(q;q)_k}}.$$

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continued...

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Now all we need is the Rogers-Ramanujan identities:

$$\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2,q^3;q^5)_{\infty}}, \quad \sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q,q^4;q^5)_{\infty}}.$$

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the "deep" bit?

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continued...

The Rogers-Ramanujan identities

are the special cases a = 1 and a = q of the one identity,

$$(aq;q)_{\infty} \sum_{n\geq 0} \sum_{k\geq 0} a^{k} \frac{q^{k^{2}}}{(q;q)_{k}}$$

= $1 + \sum_{k\geq 1} (-1)^{k} a^{2k} q^{(5k^{2}-k)/2} (1 - aq^{2k}) \frac{(aq;q)_{k-1}}{(q;q)_{k}}.$

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The Rogers-Ramanujan

identities

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The Rogers–Ramanujan identities

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For a proof of this identity, see Ramanujan's Collected works, or Chapter 15 in my (forthcoming) book.

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continued