Integer zeroes of generalized exponential polynomials

Joint work with Vichian Laohakosol

Ouamporn Phuksuwan

Department of Mathematics and Computer Science, Chulalongkorn University, Thailand

March 16, 2012

Skolem-Mahler-Lech Theorem

If an exponential polynomial has infinitely many integer zeros, then all but finitely many of such zeros form a finite union of arithmetic progressions.

T. Skolem, Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen, Comptes Rendus du Huitième Congrès des Mathématicians Scandinaves, Stockholm, 1934, pp. 163-188.

K. Mahler, Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen, *Akad. Wetensch. Amsterdam Proc.* **38**(1935), 50–60.

C. Lech, A note on recurring series, *Ark. Mat.* **2** (1953), 417–421.

K. Mahler, On the Taylor coefficients of rational functions, *Proc. Cambridge Phil. Soc.* **52**(1956), 39–48.

Skolem-Mahler-Lech Theorem

If an exponential polynomial has infinitely many integer zeros, then all but finitely many of such zeros form a finite union of arithmetic progressions.

T. Skolem, Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen, Comptes Rendus du Huitième Congrès des Mathématicians Scandinaves, Stockholm, 1934, pp. 163-188.

K. Mahler, Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen, *Akad. Wetensch. Amsterdam Proc.* **38**(1935), 50–60.

C. Lech, A note on recurring series, *Ark. Mat.* **2** (1953), 417–421.

K. Mahler, On the Taylor coefficients of rational functions, *Proc. Cambridge Phil. Soc.* **52**(1956), 39–48.

A generalized exponential polynomial

$$F(x) = \sum_{i=1}^{m} P_i(x) A_i^{Q(x)},$$

where A_i's are distinct elements of $\mathbb{C} \setminus \{0\}$, $P_i(x) \in \mathbb{C}[x] \setminus \{0\}$ and $Q(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$.

Does the classical Skolem-Mahler-Lech assertion remains valid for generalized exponential polynomials?

A generalized exponential polynomial

$$F(x) = \sum_{i=1}^{m} P_i(x) A_i^{Q(x)},$$

where A_i's are distinct elements of $\mathbb{C} \setminus \{0\}$, $P_i(x) \in \mathbb{C}[x] \setminus \{0\}$ and $Q(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$.

Does the classical Skolem-Mahler-Lech assertion remains valid for generalized exponential polynomials?

Factorization Theorem

If F has infinitely many integer zeros, then there exist $T \in \mathbb{N}$ and a subset $E \subseteq \{0, 1, \ldots, T-1\}$ such that

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

where η is a primitive T^{th} root of unity, $m_r \in \mathbb{N}$, and G is a generalized exponential polynomial but with finitely many integer zeros.

This factorization generalizes the one for ordinary exponential polynomials, i.e., those with $Q(x) = x$, due to Shapiro in 1959.

Factorization Theorem

If F has infinitely many integer zeros, then there exist $T \in \mathbb{N}$ and a subset $E \subseteq \{0, 1, \ldots, T-1\}$ such that

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

where η is a primitive T^{th} root of unity, $m_r \in \mathbb{N}$, and G is a generalized exponential polynomial but with finitely many integer zeros.

This factorization generalizes the one for ordinary exponential polynomials, i.e., those with $Q(x) = x$, due to Shapiro in 1959.

Outline of Proof .

Lemma

If F has infinitely many integer zeros, then for each $i \in \{1, \ldots, k\}$, there exists $j \neq i$ such that $A_i A_j^{-1}$ is a root of unity, and there exists $T \in \mathbb{N}$ such that for each $r \in \{0, \ldots, T-1\}$, we have either $F(xT + r) = 0$ for all $x \in \mathbb{Z}$, or there are only finitely many $x \in \mathbb{Z}$ for which $F(xT + r) = 0$.

- Denote by $C_1, ..., C_m$, all those A_i $(i = 1, \ldots, k)$ having the property that none of the $C_k C_\ell^{-1}$ is a root of unity for $k \neq \ell$.

  Write $A_i = C_{k_i} \eta^{s_i} \quad (i = 1, \ldots, k)$, where η is a primitive T-th root of unity, and $s_i \in \mathbb{N}$.

Outline of Proof .

Lemma

If F has infinitely many integer zeros, then for each $i \in \{1, \ldots, k\}$, there exists $j \neq i$ such that $A_i A_j^{-1}$ is a root of unity, and there exists $T \in \mathbb{N}$ such that for each $r \in \{0, \ldots, T-1\}$, we have either $F(xT + r) = 0$ for all $x \in \mathbb{Z}$, or there are only finitely many $x \in \mathbb{Z}$ for which $F(xT + r) = 0$.

- Denote by $C_1, ..., C_m$, all those A_i $(i = 1, \ldots, k)$ having the property that none of the $C_k C_\ell^{-1}$ is a root of unity for $k \neq \ell$.

  Write $A_i = C_{k_i} \eta^{s_i} \quad (i = 1, \ldots, k)$, where η is a primitive T-th root of unity, and $s_i \in \mathbb{N}$.

- Thus,

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)},$$

where $n_k \in \mathbb{N}$, H_i's are polynomials, and $s_{i,k} \in \mathbb{N}$.

- If $r \in \{0, \ldots, T-1\}$ is such that $F(xT+r)$ has infinitely many integer zeros, then so does

$$\sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(xT+r)} = \sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

- Since $C_k C_l^{-1}$ are not roots of unity for $k \neq l$, all the coefficients

$$\sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

are zero

- and $\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)}$ is also zero.

- Thus,

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)},$$

where $n_k \in \mathbb{N}$, H_i's are polynomials, and $s_{i,k} \in \mathbb{N}$.

- If $r \in \{0, \ldots, T-1\}$ is such that $F(xT+r)$ has infinitely many integer zeros, then so does

$$\sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(xT+r)} = \sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r}$$

- Since $C_k C_l^{-1}$ are not roots of unity for $k \neq l$, all the coefficients

$$\sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

are zero

- and $\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)}$ is also zero.

- Thus,

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)},$$

where $n_k \in \mathbb{N}$, H_i's are polynomials, and $s_{i,k} \in \mathbb{N}$.

- If $r \in \{0, \ldots, T-1\}$ is such that $F(xT+r)$ has infinitely many integer zeros, then so does

$$\sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(xT+r)} = \sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

- Since $C_k C_l^{-1}$ are not roots of unity for $k \neq l$, all the coefficients

$$\sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

are zero

- and $\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)}$ is also zero.

- Thus,

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)},$$

  where $n_k \in \mathbb{N}$, H_i's are polynomials, and $s_{i,k} \in \mathbb{N}$.

- If $r \in \{0, \ldots, T-1\}$ is such that $F(xT+r)$ has infinitely many integer zeros, then so does

$$\sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(xT+r)} = \sum_{k=1}^{m} C_k^{Q(xT+r)} \sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r}$$

- Since $C_k C_l^{-1}$ are not roots of unity for $k \neq l$, all the coefficients

$$\sum_{i=1}^{n_k} H_i(xT+r)\eta^{s_{i,k}Q(r)}$$

  are zero
- and $\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)}$ is also zero.

- Thus, F can be written as

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \left(\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)} - \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)} \right)$$
$$= \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x) \left(\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)} \right).$$

- Since $\eta^{Q(x)} - \eta^{Q(r)}$ divides each factor $\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)}$, we get

$$F(x) = \left(\eta^{Q(x)} - \eta^{Q(r)} \right) G(x),$$

where G is a generalized exponential polynomial of the same form $\sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} \mathcal{H}_i(x)\eta^{m_{i,k}Q(x)}$.

- The desired factorization follows in a finite number of similar repeated operations.

- Thus, F can be written as

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \left(\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)} - \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)} \right)$$
$$= \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x) \left(\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)} \right).$$

- Since $\eta^{Q(x)} - \eta^{Q(r)}$ divides each factor $\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)}$, we get

$$F(x) = \left(\eta^{Q(x)} - \eta^{Q(r)} \right) G(x),$$

where G is a generalized exponential polynomial of the same form $\sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} \mathcal{H}_i(x)\eta^{m_{i,k}Q(x)}$.

- The desired factorization follows in a finite number of similar repeated operations.

- Thus, F can be written as

$$F(x) = \sum_{k=1}^{m} C_k^{Q(x)} \left(\sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(x)} - \sum_{i=1}^{n_k} H_i(x)\eta^{s_{i,k}Q(r)} \right)$$
$$= \sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} H_i(x) \left(\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)} \right).$$

- Since $\eta^{Q(x)} - \eta^{Q(r)}$ divides each factor $\eta^{s_{i,k}Q(x)} - \eta^{s_{i,k}Q(r)}$, we get

$$F(x) = \left(\eta^{Q(x)} - \eta^{Q(r)} \right) G(x),$$

where G is a generalized exponential polynomial of the same form $\sum_{k=1}^{m} C_k^{Q(x)} \sum_{i=1}^{n_k} \mathcal{H}_i(x)\eta^{m_{i,k}Q(x)}$.

- The desired factorization follows in a finite number of similar repeated operations.

Theorem

If a generalized exponential polynomial has infinitely many integer zeros, then all but finitely many of such zeros form a finite union of arithmetic progressions.

Outline of proof. From the factorization theorem,

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

To determine the integer zeros of $F(x)$, we must solve $\eta^{Q(x)-Q(r)} = 1$.

This means $Q(x) - Q(r)$ is a multiple of T.

We need to find all integral solutions (x, y) of the Diophantine equation

$$Q(x) - Q(r) = yT$$

Outline of proof. From the factorization theorem,

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

To determine the integer zeros of $F(x)$, we must solve $\eta^{Q(x)-Q(r)} = 1$.

This means $Q(x) - Q(r)$ is a multiple of T.

We need to find all integral solutions (x, y) of the Diophantine equation

$$Q(x) - Q(r) = yT$$

Outline of proof. From the factorization theorem,

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

To determine the integer zeros of $F(x)$, we must solve $\eta^{Q(x)-Q(r)} = 1$.

This means $Q(x) - Q(r)$ is a multiple of T.

We need to find all integral solutions (x, y) of the Diophantine equation

$$Q(x) - Q(r) = yT$$

Outline of proof. From the factorization theorem,

$$F(x) = \left(\prod_{r \in E} \left(\eta^{Q(x)} - \eta^{Q(r)} \right)^{m_r} \right) G(x),$$

To determine the integer zeros of $F(x)$, we must solve $\eta^{Q(x)-Q(r)} = 1$.

This means $Q(x) - Q(r)$ is a multiple of T.

We need to find all integral solutions (x, y) of the Diophantine equation

$$Q(x) - Q(r) = yT$$

Let $Q(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$. Then

$$\begin{aligned} yT &= Q(x) - Q(r) \\ &= a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) + \ldots + a_1(x - r) \\ &= (x - r)\left[a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + \cdots + a_1\right]. \end{aligned}$$

Thus, $x - r = kT_1$, where T_1 is a positive divisor of T and $k \in \mathbb{Z}$, and then

$$f(k) := a_n((kT_1 + r)^n - r^n) + \ldots + a_1((kT_1 + r) - r) = yT. \quad (1)$$

Write $T = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

To solve (1) is equivalent to solving the system of congruences

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \quad (i = 1, 2, \ldots, s). \quad (2)$$

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. Then

$$\begin{aligned} yT &= Q(x) - Q(r) \\ &= a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) + ... + a_1(x - r) \\ &= (x - r)\left[a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + \cdots + a_1\right]. \end{aligned}$$

Thus, $x - r = kT_1$, where T_1 is a positive divisor of T and $k \in \mathbb{Z}$, and then

$$f(k) := a_n((kT_1 + r)^n - r^n) + ... + a_1((kT_1 + r) - r) = yT. \quad (1)$$

Write $T = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

To solve (1) is equivalent to solving the system of congruences

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \quad (i = 1, 2, \ldots, s). \quad (2)$$

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. Then

$$\begin{aligned} yT &= Q(x) - Q(r) \\ &= a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) + ... + a_1(x - r) \\ &= (x - r)\left[a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + \cdots + a_1\right]. \end{aligned}$$

Thus, $x - r = kT_1$, where T_1 is a positive divisor of T and $k \in \mathbb{Z}$, and then

$$f(k) := a_n((kT_1 + r)^n - r^n) + ... + a_1((kT_1 + r) - r) = yT. \qquad (1)$$

Write $T = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

To solve (1) is equivalent to solving the system of congruences

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \quad (i = 1, 2, \ldots, s). \qquad (2)$$

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. Then

$$\begin{aligned} yT &= Q(x) - Q(r) \\ &= a_n(x^n - r^n) + a_{n-1}(x^{n-1} - r^{n-1}) + ... + a_1(x - r) \\ &= (x - r)\left[a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + \cdots + a_1\right]. \end{aligned}$$

Thus, $x - r = kT_1$, where T_1 is a positive divisor of T and $k \in \mathbb{Z}$, and then

$$f(k) := a_n((kT_1 + r)^n - r^n) + ... + a_1((kT_1 + r) - r) = yT. \qquad (1)$$

Write $T = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$.

To solve (1) is equivalent to solving the system of congruences

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \quad (i = 1, 2, \ldots, s). \qquad (2)$$

To solve

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \tag{2i}$$

we consider first the congruence

$$f(k) \equiv 0 \pmod{p_i}. \tag{3}$$

It has at most n integral solutions mod p_i.

For each solution $k_0 \pmod{p_i}$ of (3), the number of integral solutions of (2i) corresponding to k_0 is at most p_i.

This means the solutions of (2i) are composed from a finite number of airthmetic progressions.

From the fact that

the intersection of any pair of arithmetic progressions, if nonempty, is again an arithmetic progression,

the proof is completed.

To solve

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \tag{2i}$$

we consider first the congruence

$$f(k) \equiv 0 \pmod{p_i}. \tag{3}$$

It has at most n integral solutions mod p_i.

For each solution $k_0 \pmod{p_i}$ of (3), the number of integral solutions of (2i) corresponding to k_0 is at most p_i.

This means the solutions of (2i) are composed from a finite number of airthmetic progressions.

From the fact that

the intersection of any pair of arithmetic progressions, if nonempty, is again an arithmetic progression,

the proof is completed.

To solve

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \tag{2i}$$

we consider first the congruence

$$f(k) \equiv 0 \pmod{p_i}. \tag{3}$$

It has at most n integral solutions mod p_i.

For each solution $k_0 \pmod{p_i}$ of (3), the number of integral solutions of (2i) corresponding to k_0 is at most p_i.

This means the solutions of (2i) are composed from a finite number of airthmetic progressions.

From the fact that

the intersection of any pair of arithmetic progressions, if nonempty, is again an arithmetic progression,

the proof is completed.

To solve

$$f(k) \equiv 0 \pmod{p_i^{\alpha_i}} \tag{2i}$$

we consider first the congruence

$$f(k) \equiv 0 \pmod{p_i}. \tag{3}$$

It has at most n integral solutions mod p_i.

For each solution $k_0 \pmod{p_i}$ of (3), the number of integral solutions of (2i) corresponding to k_0 is at most p_i.

This means the solutions of (2i) are composed from a finite number of airthmetic progressions.

From the fact that

the intersection of any pair of arithmetic progressions, if nonempty, is again an arithmetic progression,

the proof is completed.

Example. The generalized exponential polynomial

$$F(x) = 1 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) e^{\frac{\pi}{3} i(x^3+x^2+x)} - \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) e^{-\frac{\pi}{3} i(x^3+x^2+x)}$$

with $Q(x) = x^3 + x^2 + x$

can be factored as

$$F(x) = \left(e^{\frac{\pi}{3} i(x^3+x^2+x)} - 1\right)\left(e^{\frac{\pi}{3} i(x^3+x^2+x)} + e^{\frac{\pi}{3} i}\right) e^{\frac{4}{3}\pi i} e^{-\frac{\pi}{3} i(x^3+x^2+x)}$$

with $\eta = e^{\frac{\pi}{3} i}$ is a primitive 6^{th} root of unity, $r = 0$, and
$G(x) = \left(e^{\frac{\pi}{3} i(x^3+x^2+x)} + e^{\frac{\pi}{3} i}\right) e^{\frac{4}{3}\pi i} e^{-\frac{\pi}{3} i(x^3+x^2+x)}$.

Example. The generalized exponential polynomial

$$F(x) = 1 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) e^{\frac{\pi}{3}i(x^3+x^2+x)} - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) e^{-\frac{\pi}{3}i(x^3+x^2+x)}$$

with $Q(x) = x^3 + x^2 + x$

can be factored as

$$F(x) = \left(e^{\frac{\pi}{3}i(x^3+x^2+x)} - 1\right)\left(e^{\frac{\pi}{3}i(x^3+x^2+x)} + e^{\frac{\pi}{3}i}\right) e^{\frac{4}{3}\pi i} e^{-\frac{\pi}{3}i(x^3+x^2+x)}$$

with $\eta = e^{\frac{\pi}{3}i}$ is a primitive 6^{th} root of unity, $r = 0$, and
$G(x) = \left(e^{\frac{\pi}{3}i(x^3+x^2+x)} + e^{\frac{\pi}{3}i}\right) e^{\frac{4}{3}\pi i} e^{-\frac{\pi}{3}i(x^3+x^2+x)}$.

- Find the integral solutions of

$$Q(x) - Q(r) = (x - r)(x^2 + xr + r^2 + x + r + 1) = 6y. \qquad (4)$$

- If $T_1 = 1$, then (4) becomes $f(k) := k^3 + k^2 + k = 6y$.
  We need to solve the system

$$f(k) \equiv 0 \pmod 2 \quad \text{and} \quad f(k) \equiv 0 \pmod 3.$$

- The set of integral solutions of the former is the progression $(2\ell_1)_{\ell_1}$.
  The set of integral solutions of the latter are the progressions $(3\ell_2)_{\ell_2}$ or $(3\ell_2 + 1)_{\ell_2}$.
- Hence, the set of integer zeros of $F(x)$ for $T_1 = 1$ is

$$\{(2\ell_1)_{\ell_1} \cap (3\ell_2)_{\ell_2}\} \cup \{(2\ell_1)_{\ell_1} \cap (3\ell_2 + 1)_{\ell_2}\} = (6\ell)_\ell \cup (6\ell + 4)_\ell.$$

- The other cases of T_1 are treated similarly.

- Find the integral solutions of

$$Q(x) - Q(r) = (x - r)(x^2 + xr + r^2 + x + r + 1) = 6y. \qquad (4)$$

- If $T_1 = 1$, then (4) becomes $f(k) := k^3 + k^2 + k = 6y$.
  We need to solve the system

$$f(k) \equiv 0 \pmod 2 \;\; \text{and} \;\; f(k) \equiv 0 \pmod 3.$$

- The set of integral solutions of the former is the progression $(2\ell_1)_{\ell_1}$.
  The set of integral solutions of the latter are the progressions $(3\ell_2)_{\ell_2}$ or $(3\ell_2 + 1)_{\ell_2}$.
- Hence, the set of integer zeros of $F(x)$ for $T_1 = 1$ is

$$\{(2\ell_1)_{\ell_1} \cap (3\ell_2)_{\ell_2}\} \cup \{(2\ell_1)_{\ell_1} \cap (3\ell_2 + 1)_{\ell_2}\} = (6\ell)_\ell \cup (6\ell + 4)_\ell.$$

- The other cases of T_1 are treated similarly.

- Find the integral solutions of

$$Q(x) - Q(r) = (x - r)(x^2 + xr + r^2 + x + r + 1) = 6y. \qquad (4)$$

- If $T_1 = 1$, then (4) becomes $f(k) := k^3 + k^2 + k = 6y$.
  We need to solve the system

$$f(k) \equiv 0 \pmod 2 \quad \text{and} \quad f(k) \equiv 0 \pmod 3.$$

- The set of integral solutions of the former is the progression $(2\ell_1)_{\ell_1}$.
  The set of integral solutions of the latter are the progressions $(3\ell_2)_{\ell_2}$ or $(3\ell_2 + 1)_{\ell_2}$.
- Hence, the set of integer zeros of $F(x)$ for $T_1 = 1$ is

$$\{(2\ell_1)_{\ell_1} \cap (3\ell_2)_{\ell_2}\} \cup \{(2\ell_1)_{\ell_1} \cap (3\ell_2 + 1)_{\ell_2}\} = (6\ell)_\ell \cup (6\ell + 4)_\ell.$$

- The other cases of T_1 are treated similarly.

- Find the integral solutions of

$$Q(x) - Q(r) = (x - r)(x^2 + xr + r^2 + x + r + 1) = 6y. \qquad (4)$$

- If $T_1 = 1$, then (4) becomes $f(k) := k^3 + k^2 + k = 6y$.
  We need to solve the system

$$f(k) \equiv 0 \pmod 2 \text{ and } f(k) \equiv 0 \pmod 3.$$

- The set of integral solutions of the former is the progression $(2\ell_1)_{\ell_1}$.
  The set of integral solutions of the latter are the progressions $(3\ell_2)_{\ell_2}$ or $(3\ell_2 + 1)_{\ell_2}$.
- Hence, the set of integer zeros of $F(x)$ for $T_1 = 1$ is

$$\{(2\ell_1)_{\ell_1} \cap (3\ell_2)_{\ell_2}\} \cup \{(2\ell_1)_{\ell_1} \cap (3\ell_2 + 1)_{\ell_2}\} = (6\ell)_\ell \cup (6\ell + 4)_\ell.$$

- The other cases of T_1 are treated similarly.

- Find the integral solutions of

$$Q(x) - Q(r) = (x - r)(x^2 + xr + r^2 + x + r + 1) = 6y. \qquad (4)$$

- If $T_1 = 1$, then (4) becomes $f(k) := k^3 + k^2 + k = 6y$.
  We need to solve the system

$$f(k) \equiv 0 \pmod 2 \;\text{ and }\; f(k) \equiv 0 \pmod 3.$$

- The set of integral solutions of the former is the progression $(2\ell_1)_{\ell_1}$.
  The set of integral solutions of the latter are the progressions $(3\ell_2)_{\ell_2}$ or $(3\ell_2 + 1)_{\ell_2}$.
- Hence, the set of integer zeros of $F(x)$ for $T_1 = 1$ is

$$\{(2\ell_1)_{\ell_1} \cap (3\ell_2)_{\ell_2}\} \cup \{(2\ell_1)_{\ell_1} \cap (3\ell_2 + 1)_{\ell_2}\} = (6\ell)_\ell \cup (6\ell + 4)_\ell.$$

- The other cases of T_1 are treated similarly.

Finally we get

T_1	The integer zeros of $F(x)$
1	$(6\ell)_\ell \cup (6\ell+4)_\ell$
2	$(6\ell)_\ell \cup (6\ell+4)_\ell$
3	$(6\ell)_\ell$
6	$(6\ell)_\ell$

Example. The generalized exponential polynomial

$$F(x) = 1 - e^{\frac{\pi}{2} ix^2} - e^{\frac{\pi}{3} ix^2} + e^{\frac{5\pi}{6} ix^2},$$

can be factorized as

$$F(x) = \left(e^{\frac{\pi}{6} ix^2} - 1\right)\left(e^{\frac{\pi}{6} ix^2} - e^{\frac{2\pi}{3} i}\right)\left(e^{\frac{\pi}{2} ix^2} + e^{\frac{\pi}{3} i(x^2+1)} - e^{\frac{\pi}{6} ix^2} + e^{\frac{4\pi}{3} i}\right)$$

where $\eta = e^{\frac{\pi}{6} i}$ is a primitive 12^{th} root of unity, $r \in \{0, 2\}$, and $G(x) = e^{\frac{\pi}{2} ix^2} + e^{\frac{\pi}{3} i(x^2+1)} - e^{\frac{\pi}{6} ix^2} + e^{\frac{4\pi}{3} i}$.

Find the integral solutions of

$$Q(x) - Q(r) = x^2 - r^2 = (x - r)(x + r) = 12y, \tag{5}$$

so

$$x - r = k_1 T_1$$
$$x + r = k_2 \frac{12}{T_1}$$

where T_1 is a positive divisor of 12.

The set of x from the integral solutions of (5) is

$$(T_1 k_1 + r)_{k_1} \cap (12k_2/T_1 - r)_{k_2} .$$

Find the integral solutions of

$$Q(x) - Q(r) = x^2 - r^2 = (x - r)(x + r) = 12y, \tag{5}$$

so

$$x - r = k_1 T_1$$
$$x + r = k_2 \frac{12}{T_1}$$

where T_1 is a positive divisor of 12.

The set of x from the integral solutions of (5) is

$$(T_1 k_1 + r)_{k_1} \cap (12k_2/T_1 - r)_{k_2}.$$

	T_1	x
$r = 0$	1	$(12 - 12n)_n$
	2	$(6 - 6n)_n$
	3	$(12 - 12n)_n$
	4	$(12 - 12n)_n$
	6	$(6 - 6n)_n$
	12	$(12 - 12n)_n$

	T_1	x
$r = 2$	1	$(10 - 12n)_n$
	2	$(4 - 6n)_n$
	3	$(2 - 12n)_n$
	4	$(10 - 12n)_n$
	6	$(2 - 6n)_n$
	12	$(2 - 12n)_n$

Thank you.