On the distribution of arg $\zeta(\sigma + it)$ in the half-plane $\sigma > 1/2$

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Joint work with

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In memory of Alf van der Poorten

Abstract

We consider the distribution of $\arg\zeta(\sigma+it)$ on fixed lines $\sigma > 1/2$, and in particular the density

$$d(\sigma) = \lim_{T\to+\infty} \frac{1}{2T}\left|\{t\in[-T,+T] : |\arg\zeta(\sigma+it)| > \pi/2\}\right|,$$

where $|\{\cdots\}|$ denotes Lebesgue measure. We also consider the closely related density

$$d_-(\sigma) = \lim_{T\to+\infty} \frac{1}{2T}\left|\{t\in[-T,+T] : \Re\zeta(\sigma+it) < 0\}\right|.$$

Using classical results of Bohr and Jessen, we obtain an explicit expression for the characteristic function $\psi_\sigma(x)$ of $\arg\zeta(\sigma+it)$. We give explicit expressions for $d(\sigma)$ and $d_-(\sigma)$ in terms of $\psi_\sigma(x)$. Finally, we consider the (difficult) problem of evaluating these expressions to obtain accurate numerical values of $d(\sigma)$ and $d_-(\sigma)$. For example,

$$d(1) \approx 3.7886623606688718671 \times 10^{-7}.$$

Alf van der Poorten

Like almost all number theorists, Alf van der Poorten was interested in the Riemann zeta function. For example, Alf gave a very clear exposition of Apéry's proof of the irrationality of $\zeta(3)$, in his 1979 paper *A proof that Euler missed . . . Apéry's proof of the irrationality of* $\zeta(3)$.[1]

Thus, although my only joint work with Alf was on continued fractions of algebraic numbers, I decided to talk today on a topic related to the Riemann zeta function.

[1] Already mentioned by Jeffrey Shallit in his talk on Tuesday.

Some notation and definitions

P is the set of primes and $p \in P$ is a prime.

We always have $s = \sigma + it \in \mathbb{C}$.

$\log\zeta(s)$ denotes the main branch defined in the usual way on the open set G equal to the complex plane $\mathbb{C}$ with cuts along the half-lines $(-\infty + i\gamma, \beta + i\gamma]$ for each zero or pole $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq 1/2$. Thus $\log\zeta(s)$ is real and positive in $(1, +\infty)$.

For $s \in G$, we can define $\arg\zeta(s)$ by

$$\log\zeta(s) = \log|\zeta(s)| + i \cdot \arg\zeta(s).$$

$|B|$ denotes the Lebesgue measure of a set B.

Usually $\sigma > 1/2$ is fixed, $t \in \mathbb{R}$ is regarded as the independent variable, and $b = p^\sigma$.

Motivation

Several authors (Gram, Titchmarsh, Edwards, ...) have observed that $\Re\zeta(s)$ is "usually" positive, at least for $\sigma \geq 1/2$. This is plausible because the Dirichlet series

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

starts with a positive term, and the other terms n^{-s} may have positive or negative real part, depending on the sign of $\cos(t \log n)$.

Our aim is to quantify this observation.

Remarks

Let $\sigma_0 = 1.1923473371\cdots$ be the real root in $(1, +\infty)$ of

$$\sum_p \arcsin(p^{-\sigma}) = \frac{\pi}{2}.$$

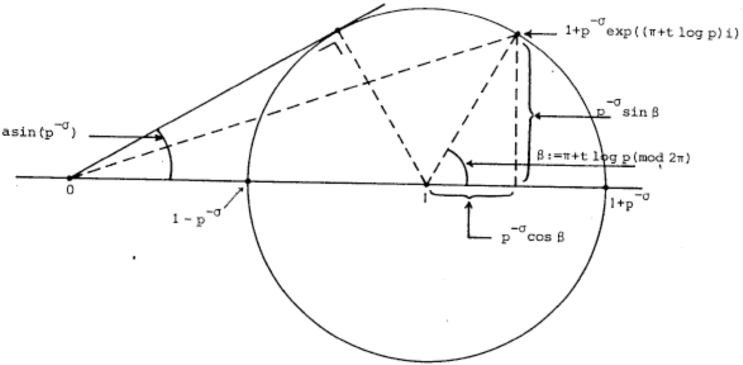
It was shown by Jan van de Lune (1983) that

$$(\forall \sigma \geq \sigma_0) \;\; \Re\zeta(\sigma + it) > 0\,.$$

Also, for any $\sigma \in (1, \sigma_0)$, there exist arbitrarily large t such that $\Re\zeta(\sigma + it) < 0$. The proof uses Kronecker's theorem.[2]
The result is also true for $\sigma \in [1/2, 1]$.

[2] Titchmarsh, §8.3.

Figure from Van de Lune (1983)



From the Figure, we see that the prime p contributes at most $\arcsin(p^{-\sigma})$ to $\arg\zeta(\sigma + it)$. If $\zeta(\sigma + it) < 0$ then we must have $|\arg\zeta(\sigma + it)| > \pi/2$.

Some numerical results

If we compute $\zeta(s)$ for "randomly" chosen s with $\sigma = \Re(s) \geq 0.6$ (say), we are unlikely to find any negative values of $\Re\zeta(s)$.

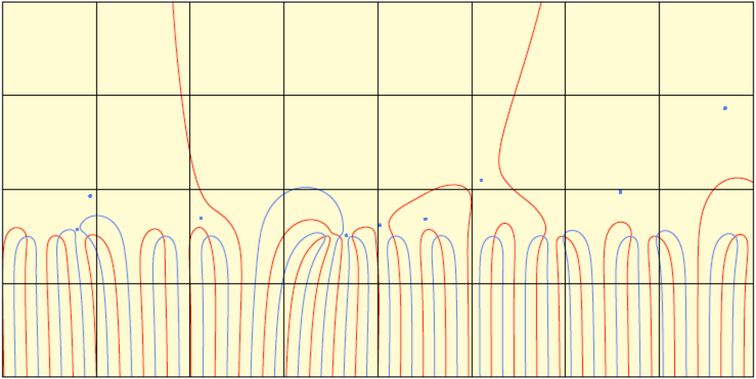
For example, taking $\sigma = 1$, it can be shown that $\Re\zeta(1 + it) > 0$ for all $t \in (0, 682112]$. Near $t = 682112.9169$ there is an interval of length 0.0529 on which $\Re\zeta(1 + it) < 0$.

For $t \in (0, 16656259]$ there are 50 intervals on which $\Re\zeta(1 + it) < 0$; the total length of these intervals is < 6.484.

Note that $16656259/6.484 \approx 2.57 \times 10^6$. Thus, the chance of finding a value of t such that $\Re\zeta(1 + it) < 0$ by random sampling is very small.

The seventh interval where $\Re\zeta(1+it) < 0$

Here is an "x-ray" of $\zeta(\sigma + it)$ for $\sigma \in [-1, 3]$, $t \in [2195052, 2195060]$, enclosing the seventh interval where $\Re\zeta(1+it)$ is negative. $\Re\zeta$ vanishes on the blue lines, and $\Im\zeta$ vanishes on the red lines. The blue dots are zeros of ζ'. The picture is rotated so that the critical strip is horizontal.



Asymptotic densities

Fix $\sigma > 1/2$, and define

$$d_+(\sigma) := \lim_{T\to+\infty} \frac{1}{2T}\left|\{t \in [-T, +T] : \Re\zeta(\sigma + it) > 0\}\right|,$$

$$d_-(\sigma) := \lim_{T\to+\infty} \frac{1}{2T}\left|\{t \in [-T, +T] : \Re\zeta(\sigma + it) < 0\}\right|.$$

It can be shown that the limits exist.

$d_+(\sigma)$ can be regarded as the probability that a randomly chosen point $\sigma + it$ gives a positive value of $\Re\zeta(\sigma + it)$; similarly for $d_-(\sigma)$ and negative values.

Since $\Re\zeta(s)$ vanishes on a set of Lebesgue measure zero, we have $d_+(\sigma) + d_-(\sigma) = 1$.

Approximation of $d_\pm(\sigma)$ via $\arg\zeta(s)$

It is easier to work with

$$d(\sigma) = \lim_{T\to+\infty} \frac{1}{2T} |\{t \in [-T, +T] : |\arg\zeta(\sigma + it)| > \pi/2\}| .$$

Observe that $\Re\zeta(s) < 0$ iff

$$|\arg\zeta(s)| \in (\pi/2, 3\pi/2) \cup (5\pi/2, 7\pi/2) \cup \cdots$$

Thus

$$d_-(\sigma) \leq d(\sigma),$$

and $d_-(\sigma) \approx d(\sigma)$ if $\arg\zeta(s)$ is "usually" small, i.e. unless σ is close to $1/2$ (more on this later).

A mean-value result

Using Chebyshev's inequality and a mean-value result[3] for $|\Re\zeta(1+it)|^2$, we can show that

$$d_-(1) \leq \frac{\zeta(2)-1}{\zeta(2)+1} = 0.243837\ldots < 1/4\,.$$

However, this result is far from the truth. We shall see later that

$$d_-(1) \approx 3.8 \times 10^{-7}\,.$$

[3] The proof is similar to that of Theorem 7.2 of Titchmarsh, which gives a mean-value result for $|\zeta(1+it)|^2$.

Intuition

Informally, the idea is that, for a prime p and large t,

$$p^{it} = \exp(it \log(p))$$

behaves like a random variable distributed uniformly on the unit circle.

Moreover, for different primes, the random variables are independent (because the $\log p$ are independent over $\mathbb{Q}$).

A theorem of Bohr and Jessen (to be stated soon) justifies this intuition.

The probability space Ω

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle with the usual probability measure μ (that is $\frac{d\theta}{2\pi}$ if we identify $\mathbb{T}$ with the interval $[0, 2\pi)$ via $z = \exp(i\theta)$).

Define $\Omega := \mathbb{T}^P$ with the product measure $\mathbb{P} = \mu^P$.

Each point of Ω is a sequence $\omega = (x_p)_{p \in P}$, with each $x_p \in \mathbb{T}$.

This formalises the idea that the x_p may be considered as a random variables, uid[4] on the unit circle.

[4] Uniformly and independently distributed.

The measure $\mathbb{P}_\sigma$ of Bohr and Jessen

Before stating the theorem of Bohr and Jessen we need to define a measure $\mathbb{P}_\sigma$.

Fix $\sigma > 1/2$. The sum of random variables

$$S = -\sum_{p\in P} \log(1 - p^{-\sigma} x_p) := \sum_{p\in P}\sum_{k=1}^{\infty} \frac{1}{k} p^{-k\sigma} x_p^k$$

converges almost everywhere, so S is a well-defined random variable.

The measure $\mathbb{P}_\sigma$ of Bohr and Jessen is defined to be the distribution of S, i.e. for each Borel set $\mathcal{B} \subseteq \mathbb{C}$ we have

$$\mathbb{P}_\sigma(\mathcal{B}) := \mathbb{P}\{\omega = (x_p) \in \Omega : S(\omega) \in \mathcal{B}\}.$$

A theorem of Bohr and Jessen

In modern language, Bohr and Jessen (1930/31) showed that, for each rectangle $\mathcal{B}$ with sides parallel to the axes,

$$\mathbb{P}_\sigma(\mathcal{B}) = \lim_{T\to\infty} \frac{1}{2T} |\{t \in [-T, +T], \log\zeta(\sigma + it) \in \mathcal{B}\}|$$

(and the limit exists).

The measure μ_σ on $\mathbb{R}$

We can specialise[5] to sets $\mathcal{B}$ of the form $\mathbb{R} \times B$. For Jordan subsets $B \subseteq \mathbb{R}$, define

$$\mu_\sigma(B) := \mathbb{P}_\sigma(\mathbb{R} \times B).$$

Then

$$d(\sigma) = \mu_\sigma(\mathbb{R} \backslash [-\pi/2, \pi/2]) \,.$$

The measure μ_σ is the distribution function of the random variable $\Im S$:

$$\begin{aligned} \mu_\sigma(B) = \mathbb{P}_\sigma(\mathbb{R} \times B) &= \mathbb{P}\{\omega \in \Omega : S(\omega) \in \mathbb{R} \times B\} \\ &= \mathbb{P}\{\omega \in \Omega : \Im S(\omega) \in B\}. \end{aligned}$$

[5]Since $\mathbb{R}$ is not bounded, we need a limiting argument to justify this.

The characteristic function ψ_σ

Recall that the *characteristic function* $\psi(y)$ of a random variable X is defined by

$$\psi(y) := \mathrm{E}[\exp(iXy)]\,.$$

This is just a Fourier transform; we omit a factor 2π in the exponent to agree with the statistical literature.

The characteristic function associated with μ_σ is the characteristic function ψ_σ of the random variable $\Im S$:

$$\psi_\sigma(x) := \int_\Omega e^{ix\Im S(\omega)}\,d\omega\,.$$

ψ_σ as an infinite product over the primes

We have

$$\psi_\sigma(x) = \prod_p I(p^\sigma, x)\,,$$

where (as usual) the product is over all primes p, and

$$I(b, x) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\arg(1-b^{-1}e^{it})}\, dt\,.$$

Sketch of the proof

By definition

$$\psi_\sigma(x) = \int_\Omega e^{ix\Im S(\omega)}\,d\omega = \int_\Omega \prod_p e^{-ix\arg(1-p^{-\sigma}x_p)}\,d\omega.$$

By independence the integral of the product is the product of the integrals:

$$\psi_\sigma(x) = \prod_p \int_\Omega e^{-ix\arg(1-p^{-\sigma}x_p)}\,d\omega.$$

Each random variable x_p is distributed as $e^{i\theta}$ on the unit circle. $\square$

$I(b,x)$ as an integral

Assume $b > 1$. Recall the definition

$$I(b,x) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\arg(1-b^{-1}e^{it})}\,dt\,.$$

Then we have two equivalent expressions for $I(b,x)$ as a definite integral:

$$\begin{aligned} I(b,x) &= \frac{1}{\pi}\int_0^{\pi}\cos\left(x\arctan\left(\frac{\sin\theta}{b-\cos\theta}\right)\right)\,d\theta \\ &= \frac{2}{\pi}\int_0^{1}\cos\left(x\arcsin\frac{u}{b}\right)\frac{du}{\sqrt{1-u^2}}\,. \end{aligned}$$

What does ψ_σ look like?

$\psi_\sigma(x)$ is a product:

$$\psi_\sigma(x) = \prod_p I(p^\sigma, x)\,.$$

Each factor in the product has infinitely many real zeros, and the same is true for $\psi_\sigma(x)$.

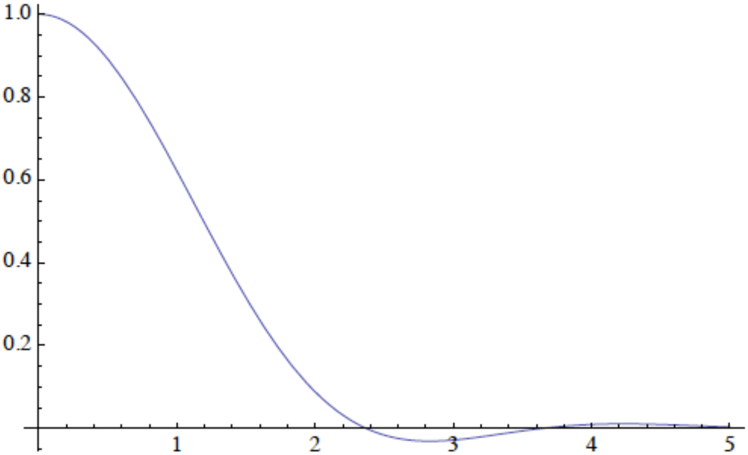
We have the bound

$$|\psi_\sigma(x)| \le C \exp\left(-c\frac{x^{1/\sigma}}{\log x}\right)$$

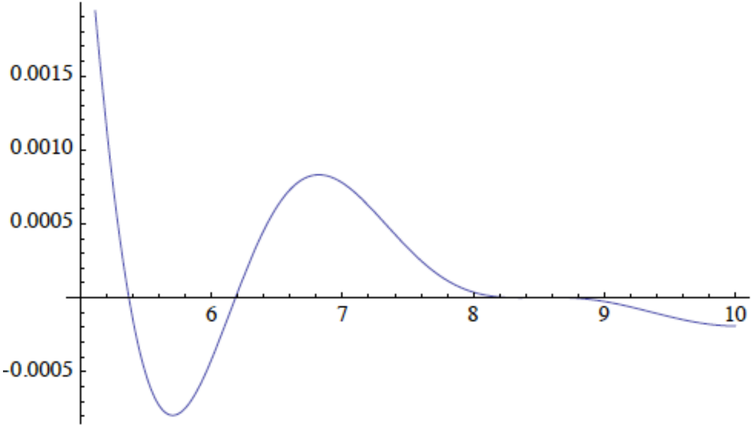
for some positive constants $C = C(\sigma)$, $c = c(\sigma)$ and $x \ge 2$.

Pictures of $\psi_{1.02}(x)$

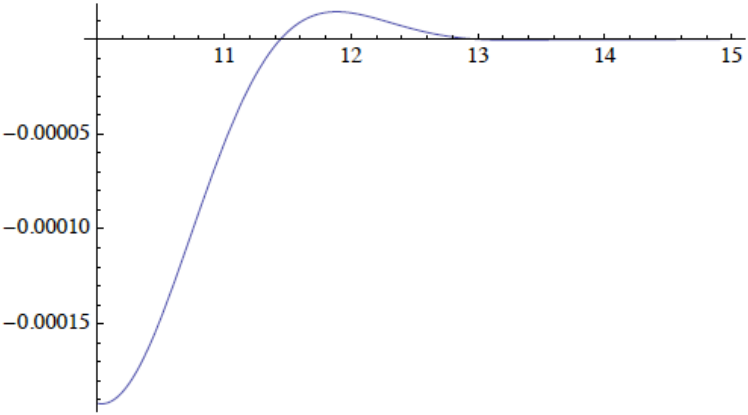
Due to the exponential decay of $|\psi_\sigma(x)|$ it is difficult to give a single plot that shows its behaviour. Following are plots for $\sigma = 1.02$ and x in the intervals $[0, 5]$, $[5, 10]$, $[10, 15]$, $[15, 20]$.



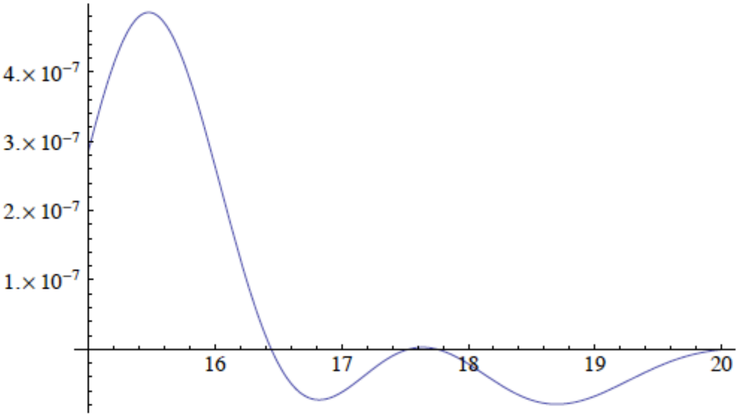
$x \in [5, 10]$



$x \in [10, 15]$



$x \in [15, 20]$



$\cos(2x \arcsin u)$ as a hypergeometric function

For $|u| < 1$ and all $x \in \mathbb{C}$,

$$\cos(2x \arcsin u) = 1 + \sum_{n=1}^{\infty} \frac{(2u)^{2n}}{(2n)!} \prod_{j=0}^{n-1} (j^2 - x^2).$$

To prove this, let $f(u) :=$ LHS, $g(u) :=$ RHS above. Then

$$(1 - u^2) f''(u) - u f'(u) + 4x^2 f(u) = 0.$$

Also, $g(u)$ satisfies the same differential equation. Since $g(0) = f(0) = 1$ and $g'(0) = f'(0) = 0$, the two solutions coincide. □

Corollary

For $|b| > 1$ we have

$$I(b, 2x) = 1 + \sum_{n=1}^{\infty} \frac{1}{b^{2n} n!^2} \prod_{j=0}^{n-1} (j^2 - x^2).$$

Combining this result with the product formula for ψ_σ, we obtain an explicit expression for ψ_σ, valid for $\sigma > 1/2$ (next slide).

An explicit expresson for ψ_σ

The characteristic function ψ_σ is given by the following product

$$\psi_\sigma(2x) = \prod_p \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!^2} \prod_{j=0}^{n-1} (j^2 - x^2) \cdot \frac{1}{p^{2n\sigma}}\right).$$

Marc Kac[6] gave the probability of $\log(\varphi(n)/n)$ being in a given interval (ω_1, ω_2) as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega_2\xi} - e^{i\omega_1\xi}}{i\xi} \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \exp\left[i\xi \log\left(1 - \frac{1}{p}\right)\right]\right) d\xi\,,$$

and referred to this as "an explicit but nearly useless formula".
Is our formula for $\psi_\sigma(2x)$ in the same category ?

[6]Mark Kac, *Statistical Independence in Probability, Analysis and Number Theory*, 1959, page 64.

Remarks

- Formally, the problem is solved. We can use a Fourier transform to compute the distribution function μ_σ from ψ_σ, and hence compute $d(\sigma)$, $d_\pm(\sigma)$ etc.
- In practice there are still severe difficulties (recall the quote from Kac). The product over prime p converges slowly, and we need to compute ψ_σ accurately to compensate for cancellation in computing the Fourier transform.
  Also, some of the results of interest, such as $d_-(\sigma)$ for $\sigma \in [1, \sigma_0)$, are tiny, so we need to compute the Fourier transform accurately.
- In the following slides we show how these difficulties can be overcome.

Computing sums/products over primes

There is a well-known technique, going back at least to Wrench (1961), for accurately computing certain sums/products over primes.

The idea is to express what we want to compute in terms of the *prime zeta function*

$$P(s) := \sum_p p^{-s} \quad (\Re(s) > 1).$$

The prime zeta function can be computed from $\log \zeta(s)$ using Möbius inversion:

$$P(s) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \log \zeta(rs)\,.$$

This formula was essentially known to Euler (1748).

Application to computation of ψ_σ

Recall that

$$\psi_\sigma(x) = \prod_p I(p^\sigma, x),$$

so the obvious approach is to take logarithms of each side:

$$\log \psi_\sigma(x) = \sum_p \log I(p^\sigma, x)\,,$$

and try to express the RHS as a sum $\sum_{k\geq 1} a_k(x,\sigma)P(k\sigma)$.

Problem – $I(p^\sigma, x)$, considered as a function of x, has zeros. Thus, the series for $\log \psi_\sigma(x)$ fails to converge.

Solution – If we consider x fixed, then $I(p^\sigma, x) > 0$ for $p > p_0(x, \sigma)$. Thus, we can apply a variant of the "prime zeta function" technique after all, provided we sum over $p > p_0$ rather than over all primes. The finite number of cases $p \leq p_0$ can be handled in a different manner.

The function $\log I(b, x)$

Suppose $x > 0$. There exist even polynomials $Q_n(x)$ of degree $2n$ such that

$$\log I(b, 2x) = -\sum_{n=1}^{\infty} \frac{Q_n(x)}{n!^2} \frac{1}{b^{2n}} \,,$$

and the series converges for $b > \max(1, |x|)$.

The polynomials $Q_n(x)$ are determined by the following recurrence:

$$Q_{n+1}(x) = (n!)^2 x^2 + \sum_{j=0}^{n-1} \binom{n}{j} \binom{n}{j+1} Q_{j+1}(x) Q_{n-j}(x), \;\; n \geq 0.$$

An algorithm for the computation of ψ_σ

We want to compute

$$\psi_\sigma(2x) = \prod_p I(p^\sigma, 2x).$$

Choose $p_0 > |x|$ (a good choice is $p_0 \approx 8|x|$).
Split the product at p_0, so $\psi_\sigma(2x) = AB$ say.
Then $A = \prod_{p \le p_0} \cdots$ is computed using

$$A = \prod_{p \le p_0} \left(1 + \sum_{n=1}^{\infty} \frac{1}{p^{2n\sigma} n!^2} \prod_{j=0}^{n-1} (j^2 - x^2) \right),$$

and $B = \prod_{p > p_0} \cdots$ is computed using

$$B = \exp\left(- \sum_{n=1}^{\infty} \frac{Q_n(x^2)}{n!^2} \left[P(2n\sigma) - \sum_{p \le p_0} p^{-2n\sigma} \right] \right).$$

Remarks

1. The summations required for A involve some cancellation, so the working precision has to be increased to compensate.
2. There is inevitably cancellation in computing

$$\left[P(2n\sigma) - \sum_{p \le p_0} p^{-2n\sigma} \right] ,$$

so here too the working precision has to be increased to compensate.

The density ρ_σ

Suppose for the moment that $\sigma > 1$. Then the support of the measure μ_σ is the interval $[-L(\sigma), L(\sigma)]$, where

$$L(\sigma) := \sum_p \arcsin(p^{-\sigma}).$$

Recall that

$$\psi_\sigma(x) = \int_{\mathbb{R}} e^{ixt}\, d\mu_\sigma(t).$$

μ_σ is the Fourier transform of a function in $L^2(\mathbb{R})$, so it is a measure with density

$$\rho_\sigma(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\sigma(x) e^{-itx}\, dx.$$

Computation of the density $d(\sigma)$

Theorem

Let $\sigma > 1$ *and* $\ell > \max(\pi/2, L(\sigma))$*. Then we have*

$$d(\sigma) = 1 - \frac{\pi}{2\ell} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\psi\Big(\frac{\pi n}{\ell}\Big)\sin\Big(\frac{n\pi^2}{2\ell}\Big).$$

Remarks

1. The formula for $d(\sigma)$ is exact, on the assumption that $\sigma > 1$ and $\ell > \max(\pi/2, L(\sigma))$. If $\sigma \in (1/2, 1]$ the formula is approximate, but converges rapidly to $d(\sigma)$ as $\ell \to \infty$, because ψ_σ is exponentially small outside a small compact interval $[-L, L]$.
2. If we take $m := 4\ell/\pi$ in the theorem, we get the slightly simpler form

$$d(\sigma) = 1 - \frac{2}{m} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\psi_\sigma\left(\frac{4n}{m}\right)\sin\left(\frac{2\pi n}{m}\right)$$

for $m > \max(2, M(\sigma))$, where $M(\sigma) = 4L(\sigma)/\pi$.

Computation of $d_-(\sigma)$

Recall that $d_-(\sigma)$ is the probability that $\Re(\zeta(\sigma+it)<0$.
Let a_k be the probability that $|\arg\zeta(\sigma+it)|>(2k+1)\pi/2$.
Then

$$d_-(\sigma)=\sum_{k=0}^{\infty}(a_{2k}-a_{2k+1}).$$

We have seen that

$$a_0=d(\sigma)=1-\frac{2}{m}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\psi_\sigma\left(\frac{4n}{m}\right)\sin\left(\frac{2\pi n}{m}\right).$$

Similarly, we have

$$a_k=1-\frac{4k+2}{m}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\psi_\sigma\left(\frac{4n}{m}\right)\sin\left(\frac{(4k+2)\pi n}{m}\right).$$

Some numerical results

We have three varieties of numerical results:

1. *Monte Carlo results.* Here we replace p^{it} in the (truncated) Euler product by a pseudo-random variable uniformly distributed on the unit circle. In this way we can estimate $d(\sigma)$ or $d_-(\sigma)$ from the outcome of a number of trials. To give one example, we estimated

$$d(1) = (3.806 \pm 0.020) \times 10^{-7}$$

from 10^{11} trials taking about 22 hours of computer time.

The method is time-consuming and inaccurate when $d(\sigma)$ is small. It is also inaccurate when σ is close to $1/2$.

On the positive side, the Monte Carlo method was easy to program and provided a "sanity check". It was very helpful for debugging the "exact" method.

Numerical results continued

2. *Exhaustive search in an interval* $(0, T]$. For example, we already mentioned that for $t \in (0, 16656259]$ there are 50 intervals on which $\Re\zeta(1 + it) < 0$, and the total length of these intervals is < 6.484.

   The problems with this approach are:

   - It is slow, requiring computation of $\zeta(s)$ at many points.
   - The results may not be "typical", because T is limited by our computational power.

Numerical results continued

3. *"Exact" computation.* Using our algorithm for the computation of $d(\sigma)$ via ψ_σ, as described above, we have computed the following results (next slide), believed to be correct to the number of decimals given.

   We used two independent programs, one written in Mathematica and the other in Magma.

   The results are consistent with those obtained by the Monte Carlo method, at least in the range $0.7 \leq \sigma \leq 1.05$ where Monte Carlo is feasible.

Values of $d(\sigma)$

Table: $d(\sigma)$ for various $\sigma \in [1/2, \sigma_0]$, $\sigma_0 = 1.192347\cdots$

σ	$d(\sigma)$
$0.5+$	$1-$
$0.5 + 10^{-11}$	0.6533592249148917497
$0.5 + 10^{-5}$	0.4962734204446697434
0.6	$7.9202919267432753125 \times 10^{-2}$
0.7	$2.5228782796068962969 \times 10^{-2}$
0.8	$5.1401888600187247641 \times 10^{-3}$
0.9	$3.1401743610642112427 \times 10^{-4}$
1.0	$3.7886623606688718671 \times 10^{-7}$
1.1	$6.3088749952505014038 \times 10^{-22}$
1.15	$1.3815328080907034247 \times 10^{-103}$
1.16	$1.1172074815779368125 \times 10^{-194}$
1.165	$1.2798207752318534603 \times 10^{-283}$
1.19234	positive
σ_0	zero

$d(\sigma)$ and $d_{-}(\sigma)$

For $\sigma > 0.8$, there is no appreciable difference between $d(\sigma)$ and $d_{-}(\sigma)$. This is because the probability that $|\arg\zeta(\sigma+it)| > 3\pi/2$ is very small in this region.

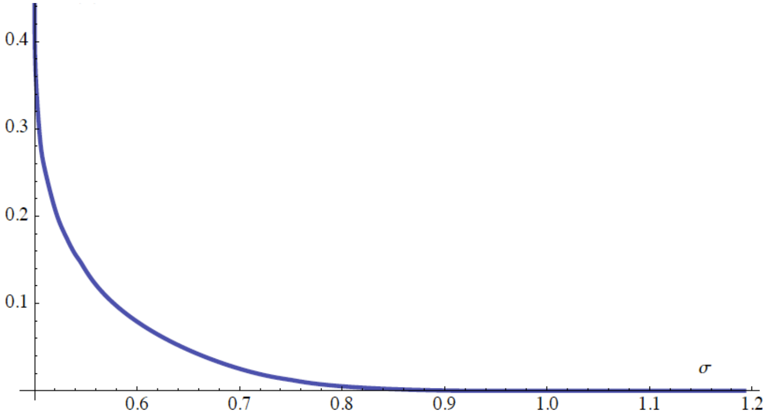
There is an appreciable difference very close to the critical line. For example,

$$d_{-}(0.5+10^{-11}) = 0.4986058426,$$
$$d(0.5+10^{-11}) = 0.6533592249.$$

Table: The difference $d(\sigma) - d_{-}(\sigma)$

σ	$d(\sigma) - d_{-}(\sigma)$
$0.5+$	$0.5-$
$0.5+10^{-11}$	0.1547533823
0.6	$8.073328981 \times 10^{-11}$
0.7	$2.676004881 \times 10^{-32}$
0.8	$7.655052120 \times 10^{-210}$
$\sigma_1 \approx 1.006823$	0

Plot of $d_-(\sigma)$



This is a plot of $d_-(\sigma)$ for $0.5 < \sigma \leq \sigma_0$.

A plot of $d(\sigma)$ is indistinguishable to the naked eye, but

$$\lim_{\sigma \downarrow 0.5} d(\sigma) = 1 \neq \lim_{\sigma \downarrow 0.5} d_-(\sigma) = 0.5\,.$$

Asymptotics of $d(\sigma)$ near the critical line

Using the asymptotic behaviour of $\psi_\sigma(x)$ for σ close to $1/2$, we expect (though have not proved) that

$$1 - d(\sigma) \sim c/\sqrt{-\log(2\sigma - 1)} \ \text{as} \ \sigma \downarrow 1/2.$$

A good fit to the numerical data is

$$d(\sigma) \approx 1 - \frac{A}{\sqrt{B - \log(2\sigma - 1)}}$$

with $A = 1.7786$, $B = 1.6479$.

This explains why the convergence of $d(\sigma)$ to 1 as $\sigma \downarrow 1/2$ is so slow.

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Due to the time constraint, I had to omit many details and almost all the proofs from this talk. If you want to hear more, please attend my one-hour talk next Monday at the Macquarie Workshop[7] next week!

[7] Workshop on Number Theory and its Applications in Memory of Alf van der Poorten, Macquarie University, 19–20 March, 2012.
http://comp.mq.edu.au/~igor/NT-AvdP_Workshop.html

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