Conic Theta Functions and their relations to classical theta functions

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The University of Newcastle, Australia

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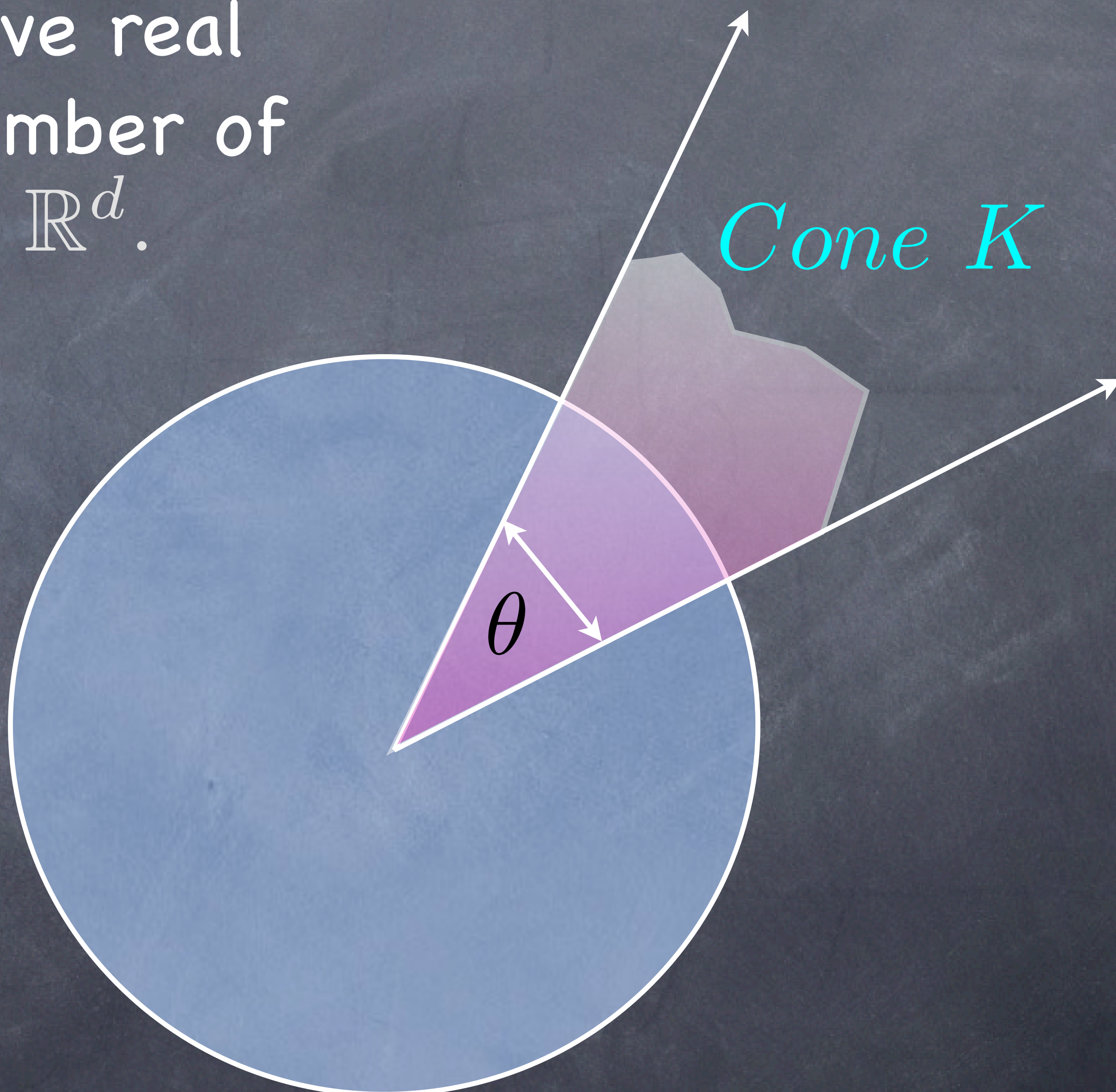
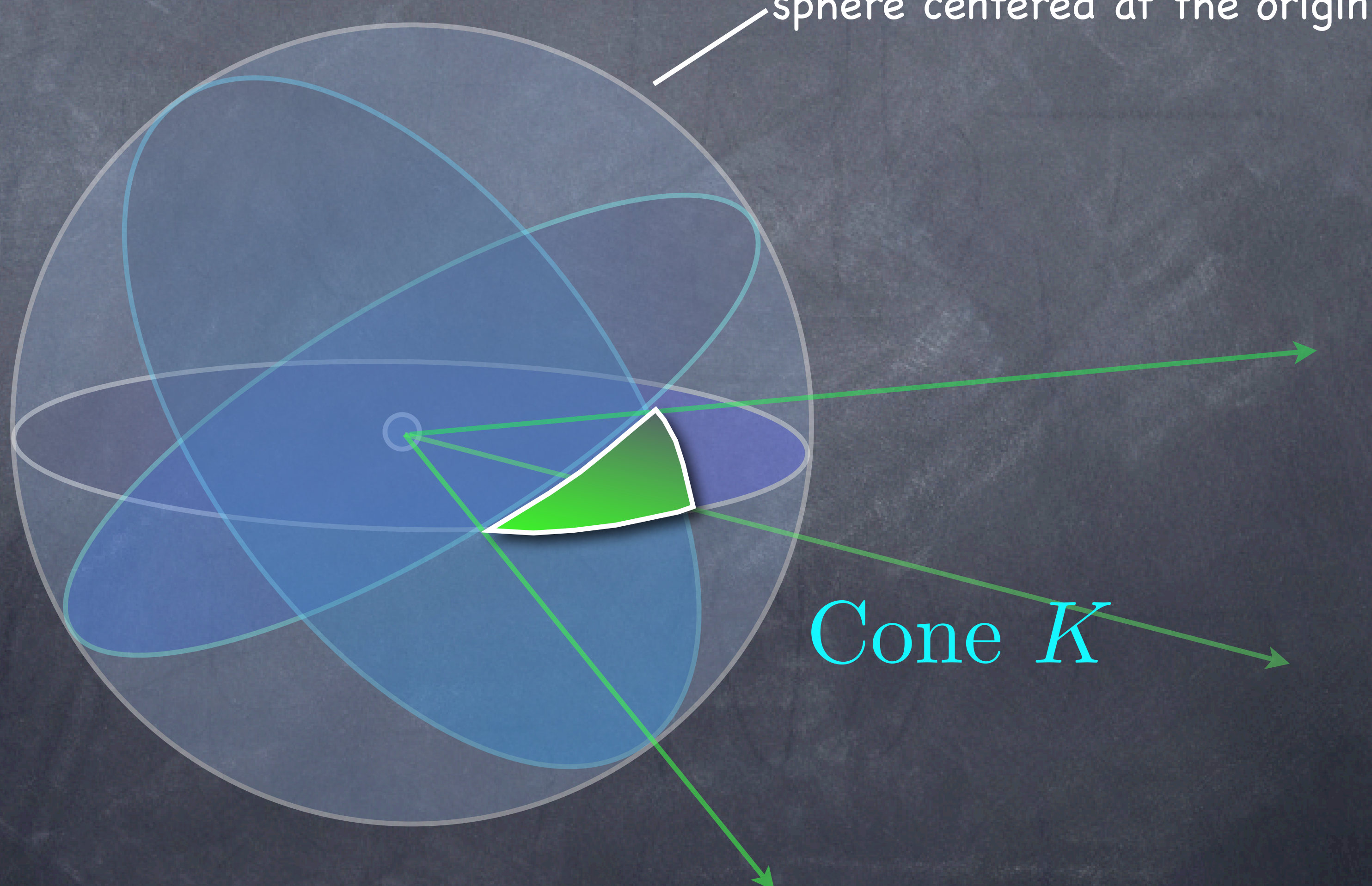
Joint work with A. Folsom and W. Kohnen

Def. A polyhedral cone K is the non-negative real span of a finite number of vectors in $\mathbb{R}^d$.



Extending angles to higher dimensions: Defining the solid angle at a vertex of a cone

This is a geodesic triangle on the sphere, representing the solid angle ω_K



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A third equivalent condition is that $\omega_K = \int_K e^{\pi ||x||^2} dx$.

Motivation

Problem 1. Which polyhedral cones K give rise to spherical polytopes whose volume is a rational number?

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Problem 2. Considering a certain conic theta function Φ_K, associated to any polyhedral cone K, how close is Φ_K to a modular form?

Let L be a rank d lattice in $\mathbb{R}^d$, let $K \subset \mathbb{R}^d$ be a polyhedral cone, and let $\tau \in H$, the upper complex half plane.

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We define the conic theta function

$$\Phi_{K,L}(\tau) = \sum_{m \in K \cap L} e^{\pi i \tau ||m||^2}.$$

Conic theta functions are discretized volumes of spherical polytopes

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Lemma. (Folsom, Kohnen, R. 2011)
Let $L \subset \mathbb{R}^d$ be a full rank lattice. Then

$$\lim_{t \to 0} t^{\frac{d}{2}} \Phi_{K,L}(it) = \frac{\omega_K}{|detL|}.$$

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Proof. Look carefully at the Riemann sum definition of $\omega_K := \int_K e^{-\pi||x||^2} dx$.

By contrast, the classical theta function associated to the lattice $L \subset \mathbb{R}^d$ is defined by:

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Fact. If L is an even, integral lattice, then $\Theta_L(\tau)$ is a modular form of weight $d/2$ on $\Gamma_0(N)$.

We define R to be the ring of all finite, rational linear combinations of theta functions Θ_L, varying over all d-dimensional even integral lattices $L \subset \mathbb{R}^d$.

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Theorem. (A.Folsom, W.Kohnen, R. 2011)

If K is the Weyl chamber of a finite reflection group, then the conic theta function $\Phi_{K,2L}(\tau)$ lies in the graded ring R.

On the other hand, if the solid angle ω_K (a.k.a. the volume of a spherical polytope) if irrational, then it turns out that $\Phi_{K,L}(\tau)$ is not a modular form.

More precisely, let $K \subset \mathbb{R}^d$ be a polyhedral cone, and let $L := A(\mathbb{Z}^d)$ be an even integral lattice of full rank.

Theorem. (A. Folsom, W. Kohnen, R. 2011)
If $\frac{\omega_K}{|det A|}$ is irrational, then $\Phi_{K,L}$ is not a modular form of weight k on any congruence subgroup, and any $k \in \frac{1}{2}\mathbb{Z}$, $k \geq \frac{1}{2}$.

Tools

1. The q-expansion principle, due to Deligne and Rapoport, tells us that if an integer weight modular form f has rational Fourier coefficients at the cusp $i\infty$, then the Fourier expansion of f at all other cusps must also have rational coefficients.

2. Combinatorial geometry of cones.

Open problems

1. If $\frac{\omega_K}{|detA|} \in \mathbb{Q}$, we do not have proofs of non-modularity for $\Phi_{K,L}$, except for some special cases.
(recalling that $L := A(\mathbb{Z}^d)$)

2. If we have an integer cone K, does its conic theta function $\Phi_{K,L}$ lie in the ring R of rational linear combinations of classical theta functions ?

Thank you