Continued fractions for translation surfaces

T. A. Schmidt

Oregon State University

13 March 2012

Continued fractions for translation surfaces

T. A. Schmidt

Oregon State University

13 March 2012

Outline

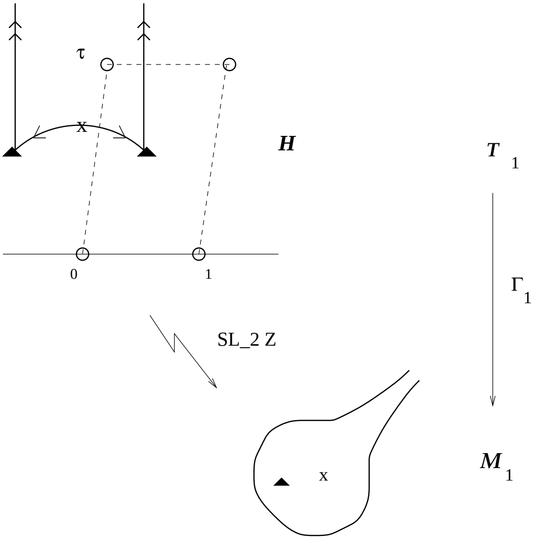
Thanks

- Thanks to organizers!

Thanks

- Thanks to organizers!

Genus one



Veech '89 examples

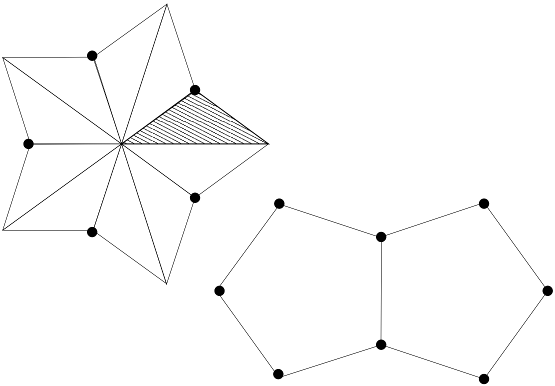


Figure: Glue parallel sides by translation, get projective curve and abelian differential

Translation Surfaces

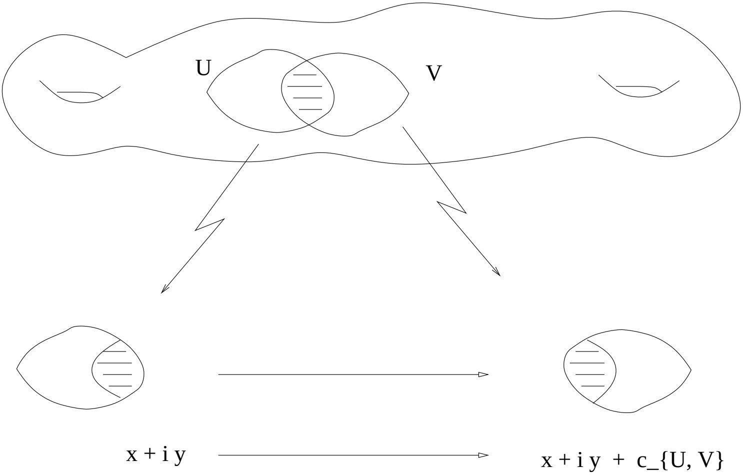
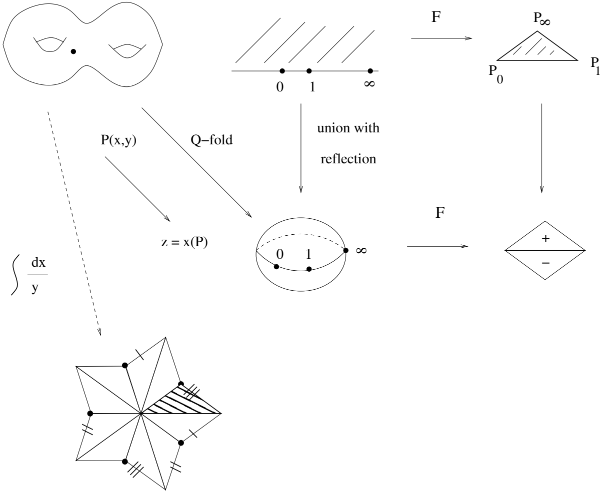
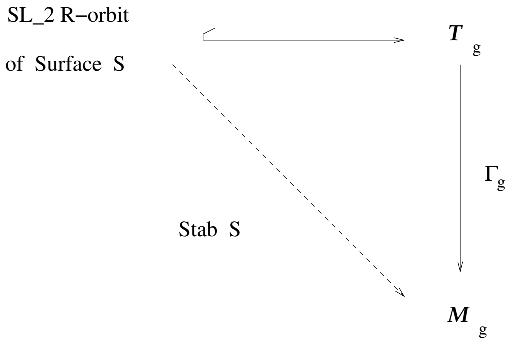


Figure: Idea of translation surface

One form gives translation structure



Teichmüller Curves

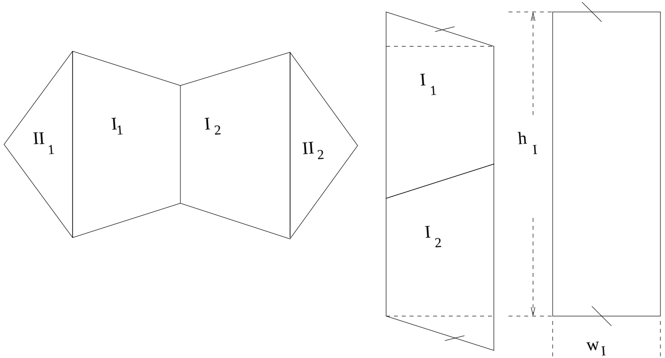


Stabilizers — Dehn twist on square cylinder

$$(x, y) \longrightarrow (x,\ x + y \bmod 1)$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Stabilizers — cylinder decomposition



Hecke groups

- The Hecke (triangle Fuchsian) group, G_q, with $q \in \{3, 4, 5, \dots\}$ is the group generated by

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\lambda = \lambda_q = 2\cos\pi/q\,.$$

Hecke groups

- The Hecke (triangle Fuchsian) group, G_q, with $q \in \{3, 4, 5, \dots\}$ is the group generated by

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\lambda = \lambda_q = 2\cos \pi/q\,.$$

- Example: $G_3 = \mathrm{PSL}(2, \mathbb{Z})$.

Hecke groups

- The Hecke (triangle Fuchsian) group, G_q, with $q \in \{3, 4, 5, \dots\}$ is the group generated by

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\lambda = \lambda_q = 2 \cos \pi / q \, .$$

- Example: $G_3 = \mathrm{PSL}(2, \mathbb{Z})$.

- The stabilizer of Veech's double pentagon example is G_5.
Non-arithmetic subgroup of Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$.

Rosen Continued Fractions

- 1952 Ph.D. dissertation, David Rosen proposed a new type of continued fraction to related to the Hecke groups.

Rosen Continued Fractions

- 1952 Ph.D. dissertation, David Rosen proposed a new type of continued fraction to related to the Hecke groups.
- Determine a_i with nearest integer multiple of λ_q

  Need $\epsilon_i = \pm 1$

$$\alpha = a_0\lambda + \cfrac{\epsilon_1}{a_1\lambda + \cfrac{\epsilon_2}{a_2\lambda + \cfrac{\epsilon_3}{\ddots}}}$$

Application to translation surfaces

- An appropriately normalized Fuchsian subgroup of a Hilbert modular group has a *special pseudo-Anosov* if the subgroup has an element of the field as a hyperbolic fixed point.

Application to translation surfaces

- An appropriately normalized Fuchsian subgroup of a Hilbert modular group has a *special pseudo-Anosov* if the subgroup has an element of the field as a hyperbolic fixed point.
- Arnoux-S. (2009) special pseudo-Anosovs exist for double 14-, 18-, 20-, 24-gon

Application to translation surfaces

- An appropriately normalized Fuchsian subgroup of a Hilbert modular group has a *special pseudo-Anosov* if the subgroup has an element of the field as a hyperbolic fixed point.
- Arnoux-S. (2009) special pseudo-Anosovs exist for double 14-, 18-, 20-, 24-gon

Application to translation surfaces

- An appropriately normalized Fuchsian subgroup of a Hilbert modular group has a *special pseudo-Anosov* if the subgroup has an element of the field as a hyperbolic fixed point.

- Arnoux-S. (2009) special pseudo-Anosovs exist for double 14-, 18-, 20-, 24-gon

- Combining above with results of Leutbecher, Wolfart and others ...

Application to translation surfaces

- An appropriately normalized Fuchsian subgroup of a Hilbert modular group has a *special pseudo-Anosov* if the subgroup has an element of the field as a hyperbolic fixed point.
- Arnoux-S. (2009) special pseudo-Anosovs exist for double 14-, 18-, 20-, 24-gon
- Combining above with results of Leutbecher, Wolfart and others ...

Theorem (Arnoux-S.)

Every Veech example of $g > 2$ has non-parabolic elements in periodic field.

Rapid growth of Rosen fraction implies Transcendence

Theorem (Bugeaud-Hubert-S.)

Fix $\lambda = 2\cos\pi/m$ for an integer $m > 3$, and denote the field extension degree $[\mathbb{Q}(\lambda) : \mathbb{Q}]$ by D. If a real number $\xi \notin \mathbb{Q}(\lambda)$ has an infinite expansion in Rosen continued fraction over $\mathbb{Q}(\lambda)$ of convergents p_n/q_n satisfying

$$\limsup_{n\to\infty} \frac{\log\log q_n}{n} > \log(2D-1)\,,$$

then ξ is transcendental.

tools for transcendence: Roth-LeVeque

Theorem

(Roth-LeVeque) Let K be a number field and ζ a real algebraic number not in K. Then for any $\epsilon > 0$, there exists a positive constant $c(\zeta, K, \epsilon)$ such that

$$|\zeta - \alpha| > \frac{c(\zeta, K, \epsilon)}{H(\alpha)^{2+\epsilon}}$$

holds for every $\alpha \in K$.

Implying

$$|\zeta - p_n/q_n| \gg H(p_n/q_n)^{-2-\epsilon}, \quad \text{for } n \geq 1. \tag{1}$$

tools for transcendence: Galois domination

Lemma

There is a $c_1 = c_1(\lambda)$ such that for all $n \geq n_0$, and any such σ field embedding of $\mathbb{Q}(\lambda)$, we have both

$$q_n \geq c_1 \, |\, \sigma(q_n) \,| \quad \text{and} \quad |p_n| \geq c_1 \, |\, \sigma(p_n) \,| \, .$$

Matrix manipulation, from

tools for transcendence: Galois domination

Lemma

There is a $c_1 = c_1(\lambda)$ such that for all $n \geq n_0$, and any such σ field embedding of $\mathbb{Q}(\lambda)$, we have both

$$q_n \geq c_1 \,|\, \sigma(q_n) \,| \;\; \text{and} \;\; |p_n| \geq c_1 \,|\, \sigma(p_n) \,|\,.$$

Matrix manipulation, from

Traces of hyperbolics dominate, Bogomoly-Schmit '04;

tools for transcendence: Galois domination

Lemma

There is a $c_1 = c_1(\lambda)$ such that for all $n \geq n_0$, and any such σ field embedding of $\mathbb{Q}(\lambda)$, we have both

$$q_n \geq c_1 \,|\,\sigma(q_n)\,| \;\; \textit{and} \;\; |p_n| \geq c_1 \,|\,\sigma(p_n)\,|\,.$$

Matrix manipulation, from

Traces of hyperbolics dominate, Bogomoly-Schmit '04;

or, Cohen-Wolfart '90 and Schmutz Schaller and Wolfart '00.

tools for transcendence: Galois domination

Lemma

There is a $c_1 = c_1(\lambda)$ such that for all $n \geq n_0$, and any such σ field embedding of $\mathbb{Q}(\lambda)$, we have both

$$q_n \geq c_1 \, |\sigma(q_n)| \;\; \text{and} \;\; |p_n| \geq c_1 \, |\sigma(p_n)| \, .$$

Matrix manipulation, from

Traces of hyperbolics dominate, Bogomoly-Schmit '04;

or, Cohen-Wolfart '90 and Schmutz Schaller and Wolfart '00.

Another proof: By a Perron-Frobenius argument for pseudo-Anosovs, and

tools for transcendence: Galois domination

Lemma

There is a $c_1 = c_1(\lambda)$ such that for all $n \geq n_0$, and any such σ field embedding of $\mathbb{Q}(\lambda)$, we have both

$$q_n \geq c_1 \, |\, \sigma(q_n) \,| \quad \textit{and} \quad |p_n| \geq c_1 \, |\, \sigma(p_n) \,| \,.$$

Matrix manipulation, from

Traces of hyperbolics dominate, Bogomoly-Schmit '04;

or, Cohen-Wolfart '90 and Schmutz Schaller and Wolfart '00.

Another proof: By a Perron-Frobenius argument for pseudo-Anosovs, and the trace of any hyperbolic generates the trace field, Kenyon-Smillie '00.

proof of transcendence

Roth-LeVeque and Galois domination give that for n sufficiently large

$$q_n^{-2D-D\epsilon} \ll \left|\zeta - p_n/q_n\right|,$$

proof of transcendence

Roth-LeVeque and Galois domination give that for n sufficiently large

$$q_n^{-2D-D\epsilon} \ll |\zeta - p_n/q_n|\,,$$

But, for any G_q-irrational ζ, there exists $c_2 = c_2(\lambda_q)$ such that

$$\left|\zeta - \frac{p_n}{q_n}\right| < \frac{c_2}{q_n q_{n+1}}\,. \tag{2}$$

proof of transcendence

Roth-LeVeque and Galois domination give that for n sufficiently large

$$q_n^{-2D-D\epsilon} \ll |\zeta - p_n/q_n| \,,$$

But, for any G_q-irrational ζ, there exists $c_2 = c_2(\lambda_q)$ such that

$$\left| \zeta - \frac{p_n}{q_n} \right| < \frac{c_2}{q_n q_{n+1}} \,. \qquad (2)$$

Therefore, there exists a constant c_3 such that

$$q_{n+1} < c_3 q_n^{2D-1+D\epsilon} \,.$$

proof of transcendence

Roth-LeVeque and Galois domination give that for n sufficiently large

$$q_n^{-2D-D\epsilon} \ll |\zeta - p_n/q_n| \, ,$$

But, for any G_q-irrational ζ, there exists $c_2 = c_2(\lambda_q)$ such that

$$\left| \zeta - \frac{p_n}{q_n} \right| < \frac{c_2}{q_n q_{n+1}} \, . \tag{2}$$

Therefore, there exists a constant c_3 such that

$$q_{n+1} < c_3 q_n^{2D-1+D\epsilon} \, .$$

Now can find that any algebraic ζ has limit on rate of growth of q_n.

Continued fractions for Ward

This section is all work with K. Calta.

Consider $(3, m, \infty)$ family of Fuchsian groups, shown by Ward '98 to stabilize translation surfaces. Fix m and set $\tau = 1 + 2\cos\pi/m$. Let $\mathbb{I} = \mathbb{I}_m = [-\tau, 0)$ and define

$$\begin{aligned} g : \mathbb{I} &\to \mathbb{I} \\ x &\mapsto -k\tau + 1 - 1/x\,, \end{aligned}$$

where $k = k(x)$ is the unique positive integer such that $g(x) \in \mathbb{I}$.

Natural extension

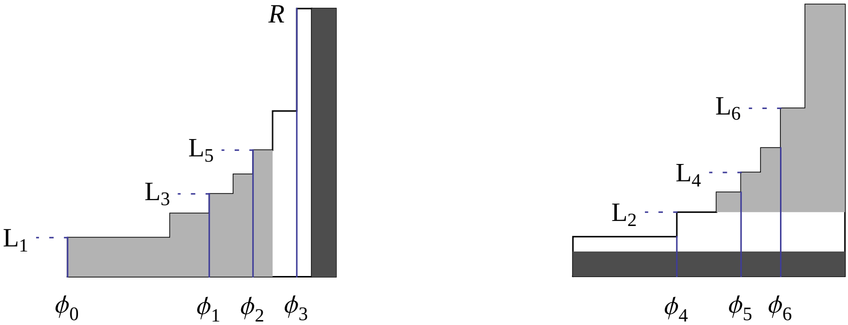


Figure: Natural extension map $\mathcal{S}(x, y) = (M_k \cdot x, N_k \cdot y)$ where $g(x) = M_k \cdot x$ for $x \in \Delta_k$ and $N_k \cdot y = -1/(M_k \cdot (-1/y))$. Here region for $m = 5$.

Natural extension, cont'd

- Locally leaves invariant

$$d\mu = (1 + x\,y)^{-2}\, dx\, dy$$

Natural extension, cont'd

- Locally leaves invariant

$$d\mu = (1 + x\,y)^{-2}\,dx\,dy$$

- $$\mathcal{S}^n(x, 0) = (\,g^n(x),\; q_{n-1}/q_n\,)\,.$$

Natural extension, cont'd

- Locally leaves invariant

$$d\mu = (1 + x\,y)^{-2}\,dx\,dy$$

- $$\mathcal{S}^n(x, 0) = (\,g^n(x),\; q_{n-1}/q_n\,)\,.$$

No finite c such that

$$\left|\,\zeta - \frac{p_n}{q_n}\,\right| < \frac{c}{q_n q_{n+1}}\,. \qquad (3)$$

Accelerated map

1. Naive map had infinite invariant measure, poor approximation properties and failed for proof of transcendence.

Accelerated map

1. Naive map had infinite invariant measure, poor approximation properties and failed for proof of transcendence.

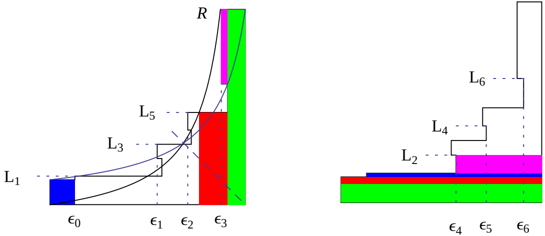
2. 

Figure: Accelerate

Accelerated is nice

Theorem

For each $m \geq 4$, the following hold:

(i.) *Every f-irrational x is the limit of its f-approximants:*

$$\lim_{n\to\infty} |\, x - p_n/q_n \,| = 0\,.$$

(ii.) *For every f-irrational x and every $n \geq 1$,*

$$\min\{\Theta_{n-1}, \ldots, \Theta_{m+n-1}\} \leq \tau\,,$$

and the constant τ is best possible.

(iii.) *f is ergodic with respect to the finite invariant measure ν on $\mathbb{I}$.*

Detects transcendence

Theorem

Let $\lambda = 2\cos(\pi/m)$ for any integer $m \geq 4$ and let $d = [\mathbb{Q}(\lambda) : \mathbb{Q}]$. If a real number $\xi \notin Q(\lambda)$ is f-irrational with convergents p_n/q_n such that

$$\limsup_{n\to\infty} \frac{\log\log q_n}{n} > \log(2d-1)$$

then ξ is transcendental.

Detects transcendence

Theorem

Let $\lambda = 2\cos(\pi/m)$ for any integer $m \geq 4$ and let $d = [\mathbb{Q}(\lambda) : \mathbb{Q}]$. If a real number $\xi \notin Q(\lambda)$ is f-irrational with convergents p_n/q_n such that

$$\limsup_{n\to\infty} \frac{\log \log q_n}{n} > \log(2d-1)$$

then ξ is transcendental.

That is, acceleration gives also finite c such that

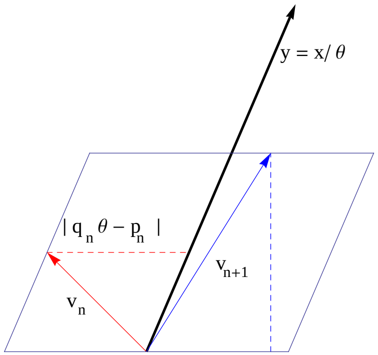
$$\left| \zeta - \frac{p_n}{q_n} \right| < \frac{c}{q_n q_{n+1}} \,. \tag{4}$$

Cheung: Approximate by connection vectors

- Yitwah Cheung introduced in 2011 his Z-fractions — connecting singularities on translation surface defines a set Z of *connection vectors*.

Cheung: Approximate by connection vectors

- Yitwah Cheung introduced in 2011 his Z-fractions — connecting singularities on translation surface defines a set Z of *connection vectors*.
- Given a direction, approximate by elements of Z.



Vector approximation catches transcendence

Theorem (Hubert-S.)

Suppose that S is a Veech surface normalized so that: $\Gamma(S) \subset SL_2(\mathbb{K})$; the horizontal direction is periodic; and, both components of every saddle connection vector of S lie in $\mathbb{K}$, where $\mathbb{K}$ is the trace field of S. Let $D = [\mathbb{K} : \mathbb{Q}]$ be the field extension degree of $\mathbb{K}$ over the field of rational numbers. If a real number $\xi \in [0,1] \setminus \mathbb{K}$ has an infinite $V_{sc}(S)$-expansion, whose convergents p_n/q_n satisfy

$$\limsup_{n\to\infty} \frac{\log\log q_n}{n} > \log(\,2D-1\,)\,,$$

then ξ is transcendental.

Minkowski constant is finite

Definition

The *Minkowski constant* of Z is

$$\mu(Z) = \frac{1}{4} \sup \text{area}(\mathcal{C})$$

where $\mathcal{C}$ varies through bounded, convex, $(0,0)$-symmetric sets that are disjoint from Z.

Cheung-Hubert-Masur: If $\mu(Z) < \infty$, then every direction other than a direction of a vector of Z has infinite expansion.

Minkowski constant is finite

Definition

The *Minkowski constant* of Z is

$$\mu(Z) = \frac{1}{4} \sup \operatorname{area}(\mathcal{C})$$

where $\mathcal{C}$ varies through bounded, convex, $(0,0)$-symmetric sets that are disjoint from Z.

Cheung-Hubert-Masur: If $\mu(Z) < \infty$, then every direction other than a direction of a vector of Z has infinite expansion.

Theorem

Let S be a compact translation surface, and $Z = V_{sc}(S)$ the set of saddle connection vectors of S. Then

$$\mu(Z) \leq \pi \, vol(S) \, .$$

End ...

Thank you!