Continued fraction expansions of transcendental numbers

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<u>Question</u>: Let θ be a real algebraic number of degree ≥ 3. Is the sequence of partial quotients of θ bounded?

Liouville (1844): Explicit examples of transcendental continued fractions (with unbounded partial quotients).

Maillet (1906): Explicit examples of transcendental continued fractions with bounded partial quotients.

Theorem: $\xi := [0; a_1, a_2, \dots]$ not quadratic. If $a_\ell \leq M$ for $\ell \geqslant 1$ and if there is an increasing sequence $(\ell_k)_{k \geq 1}$ s.t.

$$a_{\ell_k} = a_{\ell_k+1} = \dots = a_{k\ell_k} = 1 \quad (k \geq 1),$$

then ξ is transcendental.

Liouville: If θ is a real algebraic number of degree $d \geq 2$, then there exists $c_1(\theta) > 0$ s.t.

$$|\theta - p/q| \geq c_1(\theta)\, q^{-d}, \quad \text{for every } p/q \text{ with } q \geq 1.$$

Maillet: If θ is a real algebraic number of degree $d \geq 3$, then there exists $c_2(\theta) > 0$ s.t.

$$|\theta - \alpha| \geq c_2(\theta)\, H(\alpha)^{-d},$$

for every real quadratic nb α.

Proof: $\xi_k := [0; a_1, \dots, a_{l_k - 1}, 1, 1, 1, \dots]$

$$|\xi - \xi_k| \leqslant q_{k l_k}^{-2} \leqslant 2^{-k l_k}$$

$$H(\xi_k) \leqslant (M+2)^{2 l_k}$$

[$H(\alpha)$: max. of the absolute values of the coeff. of the minimal pol. of α over $\mathbb{Z}$]

$$|\xi - \xi_k| \leqslant H(\xi_k)^{-k(\log 2)/(2\log(M+2))}$$

for $k \geq 1$.

Thus, ξ cannot be algebraic of degree $d \geq 3$.

In the preceeding proof, all the ξ_k belong to the same quadratic field, namely to $\mathbb{Q}(\sqrt{5})$. This observation was used by A. Baker (1962) to improve Maillet's theorem, by appealing to Roth-LeVeque's theorem instead of Maillet's transcendence criterion. A further step was the use of results of W. M. Schmidt.

Roth-LeVeque (1955): Let K be a given real quadratic field. Let $\varepsilon > 0$. Let θ be an alg. nb. outside K. Then, there is $c_3(\theta, \varepsilon) > 0$ s.t.

$$|\theta - \alpha| \geq c_3(\theta, \varepsilon)\, H(\alpha)^{-2-\varepsilon}, \text{ for every quadratic nb } \alpha \text{ in } K.$$

Schmidt (1967): Let $\varepsilon > 0$. If θ is a real algebraic number of degree $d \geq 3$, then there exists $c_4(\theta, \varepsilon) > 0$ s.t.

$$|\theta - \alpha| \geq c_4(\theta, \varepsilon)\, H(\alpha)^{-3-\varepsilon}, \text{ for every real quadratic nb } \alpha.$$

Schmidt's result was used by Davison (1989), Queffélec (1998), Allouche - Davison - Queffélec - Zamboni (2000) to prove that a real nb whose sequence of partial quotients is the Thue - Morse sequence or any Sturmian sequence is transcendental.

This result of Schmidt is a consequence of its powerful « Subspace Theorem ».

So, try to apply the Subspace Theorem to get new examples of transcendental continued fractions!!! That's what we did with Adamczewski (2005, 2007).

<u>Theorem</u> (B., 201?): Let $(a_\ell)_{\ell \geq 1}$ be a bounded sequence of positive integers. Assume that, for some $\varepsilon > 0$ and arbitrarily large integers N, there exists a word W_N of length $\lfloor \varepsilon N \rfloor$ having <u>two non-overlapping occurrences in $a_1 a_2 \cdots a_N$</u>. Then, the real number $[0; a_1, a_2, \ldots]$ is either quadratic or transcendental.

__Theorem__ (B., 201?): Let $(a_\ell)_{\ell \geq 1}$ be a bounded sequence of positive integers. Assume that, for some $\varepsilon > 0$ and arbitrarily large integers N, there exists a word W_N of length $\lfloor \varepsilon N \rfloor$ such that W_N and its mirror image $\overleftarrow{W_N}$ occur without overlapping in $a_1 a_2 \cdots a_N$. Then, the real number $[0; a_1, a_2, \ldots]$ is either quadratic or transcendental.

$\left[\; \overleftarrow{w_1 w_2 \cdots w_k} := w_k w_{k-1} \cdots w_2 w_1 \; \right]$

Let $\underline{a} := a_1 a_2 \cdots$ be an infinite word over $\mathbb{Z}_{\geq 1}$.
For $n \geq 1$, set

$$p(n, \underline{a}) := \mathrm{Card}\,\{ a_{\ell+1} \cdots a_{\ell+n} : \ell \geq 0 \}.$$

Obviously, $1 \leq p(n, \underline{a}) \leq +\infty$.

Furthermore:

* If $\underline{a}$ is ultimately periodic, then there exists C s.t. $p(n, \underline{a}) \leq C$ for $n \geq 1$.
* If $\underline{a}$ is not ultimately periodic, then $p(n, \underline{a}) \geq n+1$ for $n \geq 1$.
* There exist uncountably many $\underline{a}$ s.t. $p(n, \underline{a}) = n+1$ for $n \geq 1$ (Sturmian words).

Theorem (B., 201?):

Let $\underline{a} = a_1 a_2 \ldots$ be a sequence of positive integers which is not ultimately periodic. If the real number

$$[0; a_1, a_2, \ldots]$$

is algebraic, then

$$\lim_{n \to +\infty} \frac{p(n, \underline{a})}{n} = +\infty .$$

In other words: the sequence of partial quotients of a real alg. nb. of degree ≥ 3 cannot be « too simple ».

<u>Corollary</u> (Cobham - Loxton - van der Poorten conjecture for continued fractions):
The continued fraction expansion of an algebraic number of degree ≥ 3 cannot be generated by a finite automaton.

<u>Proof</u>: Sequences $\underline{a}$ generated by finite automata satisfy $p(n, \underline{a}) = O(n)$, a result of Cobham (1972).

Schmidt Subspace Theorem

Let $m \geq 2$ be an integer. Let $L_1, \ldots, L_m$ be linearly independent linear forms in $\underline{x} = (x_1, \ldots, x_m)$ with <u>algebraic</u> coefficients. Let $\varepsilon > 0$. Then, the set of solutions $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{Z}^m$ to

$$|L_1(\underline{x}) \times \ldots \times L_m(\underline{x})| \leq \left(\max\{|x_1|, \ldots, |x_m|\}\right)^{-\varepsilon}$$

lies in <u>finitely many proper subspaces of $\mathbb{Q}^m$</u>.

ex.: $\theta \neq 0$ algebraic, $\varepsilon > 0$

$$L_1(X_1, X_2) := \theta X_1 - X_2$$
$$L_2(X_1, X_2) := X_1$$

The set of solutions $(p, q) \in \mathbb{Z}^2$ to

$$|q\theta - p| \cdot |q| < |q|^{-\varepsilon}$$

lies in finitely many lines.

$$\rightsquigarrow \quad \left|\theta - \frac{p}{q}\right| < |q|^{-2-\varepsilon} \quad \text{if } q \neq 0.$$

Assume that ξ is algebraic, where

$$\xi := [0; a_1, a_2, \dots, a_s, a_1, a_2, \dots, a_{\lfloor \varepsilon s \rfloor}, a_{s+\lfloor \varepsilon s \rfloor + 1}, \dots]$$

for some s (large) and ε (small).

Then ξ is close to the quadratic nb

$$\alpha := [0; \overline{a_1, a_2, \dots, a_s}]$$

whose minimal polynomial is

$$P_\alpha(X) := q_{s-1} X^2 - (p_{s-1} - q_s) X - p_s .$$

Namely, $\quad |\xi - \alpha| \lesssim C^{-2(1+\varepsilon)s} \asymp q_s^{-2(1+\varepsilon)}$

[Simplification : $\quad q_\ell \asymp C^\ell \quad$ for $\ell \geq 1$]

One deduces that

$$|P_\alpha(\xi)| \lesssim |\xi - \alpha| \cdot H(P_\alpha)$$
$$\lesssim q_s^{-1-2\varepsilon}$$

Three linear forms :

$$\xi^2 X_1 - \xi X_2 - X_3, \quad X_2, \quad X_3.$$

To be evaluated in

$$(x_1, x_2, x_3) = (q_{s-1}, p_{s-1} - q_s, p_s)$$

Product $\lesssim q_s^{-1-2\varepsilon} \, q_s \, q_s$

Needed : $\varepsilon > 1/2$.

Main idea: to separate the variables.

Four linear forms:

$$\xi^2 Y_1 - \xi Y_2 + \xi Y_3 - Y_4 \ , \ \xi Y_3 - Y_4 \ , \ Y_1 \ , \ Y_4 .$$

To be evaluated in

$$(Y_1, Y_2, Y_3, Y_4) = (q_{s-1}, p_{s-1}, q_s, p_s) .$$

Note that $|\xi q_s - p_s| < q_s^{-1}$.

Product $\lesssim q_s^{-1-2\varepsilon} \cdot q_s^{-1} q_s q_s \lesssim q_s^{-2\varepsilon}$

Needed: $\varepsilon > 0$.

<u>Lemma</u>: Let θ be a quadratic real nb with ultimately periodic continued fraction expansion

$$\theta := [0; a_1, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+s}}],$$

with $r \geq 3$ and $s \geq 1$. Let θ' be its Galois conjugate. If $a_r \neq a_{r+s}$, then we have

$$|\theta - \theta'| \ll a_r^2 \max\{a_{r-1}, a_{r-2}\} \cdot q_r^{-2},$$

where $(p_\ell / q_\ell)_{\ell \geq 1}$ denotes the sequence of convergents to θ.

Another consequence of the Subspace Theorem:

Let $\varepsilon > 0$. Let ξ be a real nb which is not algebraic of degree ≤ 2. If there exist arbitrarily large positive integers q s.t.

$$\max\{\|q\xi\|, \|q\xi^2\|\} < q^{-\frac{1}{2}-\varepsilon},$$

then ξ is transcendental.

$\|\cdot\|$: distance to the nearest integer

Let $\xi := [0; a_1, a_2, \cdots]$ and assume that $a_1 a_2 \cdots a_N = a_N a_{N-1} \cdots a_1$ (palindrome).

Then, recalling that

$$\frac{q_{N-1}}{q_N} = [0; a_N, a_{N-1}, \cdots, a_1] \quad \text{(mirror formula)},$$

we get

$$\frac{q_{N-1}}{q_N} = [0; a_1, a_2, \cdots, a_N] = \frac{p_N}{q_N},$$

thus $p_N = q_{N-1}$ and ξ^2 is close to $\frac{p_{N-1}}{q_{N-1}} \cdot \frac{p_N}{q_N} = \frac{p_{N-1}}{q_N}$. We get that

$$\| q_N \xi \|, \| q_N \xi^2 \| \ll q_N^{-1}.$$

It follows that, if there are arbitrarily large integers N s.t.

$$a_1 a_2 \dots a_{N-1} a_N \text{ is a palindrome,}$$

then the real number $[0; a_1, a_2, \dots]$ is either quadratic or transcendental.

<u>ex</u>: Thue-Morse continued fraction

$\underline{t} = 1221211221121221 2112 \dots$

$\underline{t} = \tau^\infty(1)$ where $\tau(1) = 12$, $\tau(2) = 21$.

Note that $\tau^2(1) = 1221$ and $\tau^2(2) = 2112$ are palindromes. Thus, $\tau^{2n}(1)$ is a palindrome for $n \geq 1$.