Generalising epidemic dynamics: Deriving a fractional SIR model

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25th November 2015

Epidemic Modelling

In simulating the dynamics of a disease throughout a population we break up the whole population into 3 classes.

(S) for Susceptibles, (I) for Infecteds and (R) for Removed

The dynamics of the interactions of the populations can be modelled most simply via the law of mass action.

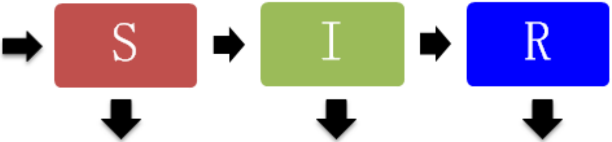
Law of Mass Action

The principle that the rate of change of a chemical reaction is proportional to the concentration of the reactants.

SIR Equations

Standard Equations with vital dynamics

$$\frac{dS(t)}{dt} = \lambda - \sigma S(t)I(t) - \gamma S(t)$$
$$\frac{dI(t)}{dt} = \sigma S(t)I(t) - \beta I(t) - \gamma I(t)$$
$$\frac{dR(t)}{dt} = \beta I(t) - \gamma R(t)$$



Extensions to SIR Model

- Addition of new compartments
- Disease vector interaction of different species
- The consideration of spatial diffusion
- Introduction of fractional derivatives

Fractional Derivatives

Riemann-Liouville Integral and Corresponding Derivative

$$
{}_0\mathcal{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi
$$

$$
{}_0\mathcal{D}_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi
$$

A fractional derivative is the ordinary derivative of a fractional integral.

Fractional Derivatives

Using Laplace transforms of a fractional derivative we can write:

$$ {}_0\mathcal{D}_t^{1-\alpha} f(t) = \mathcal{L}_s^{-1}\{s^{1-\alpha}\mathcal{L}_t\{f(t)\}|t\}, $$

when $t > 0$ this can be simplified using the convolution theorem,

Fractional Derivative in terms of Laplace Transforms

$$ {}_0\mathcal{D}_t^{1-\alpha} f(t) = \int_0^t \mathcal{L}_s^{-1}\left[s^{1-\alpha}\middle|t'\right] f(t-t')dt' $$

Existing Fractional SIR Models

Previously the inclusion of fractional derivatives, replaces the time derivatives in an ad hoc fashion,

Simple Fractional Model

$$
\begin{aligned}
{}_0\mathcal{D}_t^{1-\alpha_1} S(t) &= \lambda - \sigma S(t) I(t) - \gamma S(t) \\
{}_0\mathcal{D}_t^{1-\alpha_2} I(t) &= \sigma S(t) I(t) - \beta I(t) - \gamma I(t) \\
{}_0\mathcal{D}_t^{1-\alpha_3} R(t) &= \beta I(t) - \gamma R(t) \\
\alpha_1, \alpha_2, \alpha_3 &\in [0, 1)
\end{aligned}
$$

Such models have been used for Dengue fever[1]

[1] Diethelm, 2012, Pooseh et. al., 2011

Problems with the ad hoc approach

- Dimensional affect on the rates, and their physical interpretation

$$ {}_0\mathcal{D}_t^{\alpha} S(t) = \frac{d^{\alpha} S(t)}{dt^{\alpha}} $$

- Use of different fractional exponents causes the flux-balance to be lost
- This system has not been derived from a physical stochastic process and hence interpretation of the fractional derivative is difficult

Derivation of General Equations

We can solve these problems by considering a physical stochastic process. We derive master equations that allow for the inclusion of a physically motivated fractional derivative.

We begin by considering the epidemic as a continuous time random walk.[1]

Flux into the Infected Compartment

$$q^{+}(I,t) = \int_0^t \overbrace{\sigma(t,t')}^{\text{Infectivity}} \underbrace{S(t)}_{\text{Susceptibles}} \overbrace{\Phi(t,t')}^{\text{Survival Rate}} \underbrace{q^{+}(I,t')}_{\text{Flux}} \, dt'$$

Where the infectivity can be represented as,

$$\sigma(t,t') = \omega(t)\rho(t-t')$$

[1]For more depth see Angstmann, Henry, McGann 2015.

Derivation of General Equations

Total Population of the Infected Compartment

$$I(t) = \int_0^t \Phi(t, t')q^+(I, t')dt'$$

Where $\Phi(t, t')$ represents the composite effect of being able to leave the infected compartment, either by death or by recovery. Assuming these two effects are independent, and that the death and recovery rate are $\gamma(t)$ and $\mu(t)$ respectively, we can write:

$$\Phi(t, t') = e^{-\int_{t'}^{t} \gamma(s)+\mu(s)ds}$$
$$\Phi(t, s) = \Phi(t, t')\Phi(t', s) \ \ \forall \ s < t' < t,$$

Infectious Compartment

Making use of the chain rule, Leibniz rule, and algebra:

$$\frac{dI(t)}{dt} = q^{+}(I,t) - (\mu(t) + \gamma(t))I(t),$$

which can be expressed as,

$$\frac{dI(t)}{dt} = \omega(t)S(t)\int_0^t \rho(t-t')\Phi(t,t')q^{+}(I,t')dt' - (\mu(t)+\gamma(t))I(t).$$

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We would like the first term to be in terms of $S(t)$,$I(t)$ or $R(t)$.

Laplace Transform

We can make use of Laplace transforms to express $q^+(I, t')$ in terms of population size. We begin by rewriting $I(t)$ as:

$$\frac{I(t)}{\Phi(t,0)} = \int_0^t \frac{q^+(I,t')}{\Phi(t',0)} dt'.$$

Taking the Laplace transform gives:

$$\mathcal{L}\left\{\frac{q^+(I,t)}{\Phi(t,0)}\right\} = s\mathcal{L}\left\{\frac{I(t)}{\Phi(t,0)}\right\}.$$

Laplace Transform

The flux can then be expressed in terms of $I(t)$ by noting,

$$\omega(t)S(t)\int_0^t \rho(t-t')\frac{q^+(I,t')}{\Phi(t',0)}dt' = \omega(t)S(t)\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\rho(t)\right\}\mathcal{L}\left\{\frac{q^+(I,t)}{\Phi(t,0)}\right\}\right\}$$
$$= \omega(t)S(t)\int_0^t \kappa(t-t')\frac{I(t')}{\Phi(t',0)}dt'.$$

Where we introduce a *memory kernel*:

$$\kappa(t) = \mathcal{L}^{-1}\{s\mathcal{L}\{\rho(t)\}\}$$

Derived Master Equations

$$
\begin{aligned}
\frac{dS(t)}{dt} =& \lambda(t) - \omega(t)S(t)\Phi(t,0)\int_0^t \kappa(t-t')\frac{I(t')}{\Phi(t',0)}dt' - \gamma(t)S(t) \\
\frac{dI(t)}{dt} =& \omega(t)S(t)\Phi(t,0)\int_0^t \kappa(t-t')\frac{I(t')}{\Phi(t',0)}dt' - (\mu(t)+\gamma(t))I(t) \\
\frac{dR(t)}{dt} =& \mu(t)I(t) - \gamma(t)R(t)
\end{aligned}
$$

Consistency with original SIR model

If we take $\rho(t - t') = \rho$ to be a constant, we will get the resulting memory kernel,

$$\kappa(t) = \rho\delta(t).$$

When substituted into the master equations this recovers the original SIR equations.

Derivation of the Fractional Model

If we consider the time since infection dependent infectivity, $\rho(t)$ to be,

$$\rho(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

The Laplace transform of the resulting memory kernel is,

$$\mathcal{L}\left\{\kappa(t)\right\} = s^{1-\alpha}.$$

Hence the integral becomes,

$$\int_0^t \kappa(t-t')\frac{I(t')}{\Phi(t',0)}dt' = \mathcal{L}^{-1}\left\{s^{1-\alpha}\mathcal{L}\left\{\frac{I(t)}{\Phi(t,0)}\right\}\right\}.$$

A fractional infectivity SIR model

Set of fiSIR Equations

$$\frac{dS(t)}{dt} = \lambda(t) - \omega(t)S(t)\Phi(t,0)\,{}_0\mathcal{D}_t^{1-\alpha}\left(\frac{I(t)}{\Phi(t,0)}\right) - \gamma(t)S(t)$$

$$\frac{dI(t)}{dt} = \omega(t)S(t)\Phi(t,0)\,{}_0\mathcal{D}_t^{1-\alpha}\left(\frac{I(t)}{\Phi(t,0)}\right) - \mu(t)I(t) - \gamma(t)I(t)$$

$$\frac{dR(t)}{dt} = \mu(t)I(t) - \gamma(t)R(t)$$

Observations on the fiSIR Model

This fractional model naturally arises when the disease has a power-law time since infection infectivity.

A dimensional analysis yields the time dependent infectivity, $\omega(t)$, to have $[\text{population}]^{-1}[\text{time}]^{-\alpha}$, considering the dimensions of the total infectivity,

$$\sigma(t, t') = \frac{[\text{population}]}{[\text{time}]^{1-\alpha}} \frac{1}{[\text{population}][\text{time}]^{\alpha}} = [\text{time}]^{-1}$$

Equilibrium solutions exist, stability analysis is difficult.

Fractional Recovery Master Equations

Other fractional SIR models are derivable from the same method.
A model which captures chronic infection is the fractional recovery model.

Set of frSIR Equations

$$\frac{dS(t)}{dt} = \lambda - \omega(t)S(t)I(t) - \gamma(t)S(t)$$

$$\frac{dI(t)}{dt} = \omega(t)S(t)I(t) - \beta\theta(t,0)\mathcal{D}_t^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right) - \gamma(t)I(t)$$

$$\frac{dR(t)}{dt} = \beta\theta(t,0)\mathcal{D}_t^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right) - \gamma(t)R(t)$$

Example

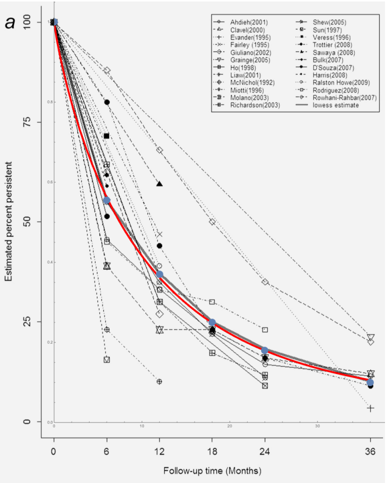

Figure: A fit of the Mittag-Leffler survival function to HPV recovery data, where $\alpha = 0.874$, $\tau = 8.351$ months.

Summary

- The classic SIR model has been extended in many ways
- Recent fractional SIR models often do not consider dimensions and flux
- The fiSIR model provides a random walk stochastic process method to include a fractional derivative
- A fractional recovery SIR model can be constructed in a similar way.
- A fractional recovery SIR model allows for the consideration of chronic diseases