On conditions under which receding horizon control delivers approximately optimal solutions

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theory based on joint work with
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Anastasia Panin (Bayreuth), Karl Worthmann (Ilmenau)

application based on joint work with
Arthur Fleig (Bayreuth), Roberto Guglielmi (Linz)

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Setup

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x$$

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with optimal control in feedback form $\mathbf{u}(n) = \mu(x_{\mathbf{u}}(n))$

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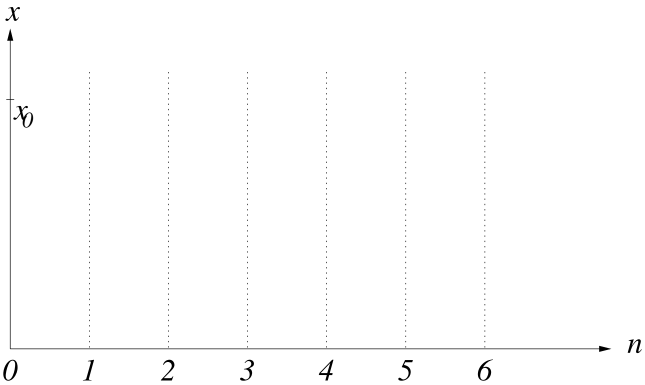
We obtain a feedback law μ_N by a receding horizon technique

MPC from the trajectory point of view



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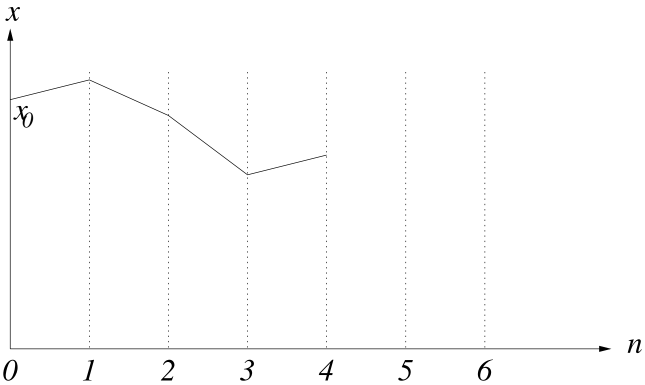

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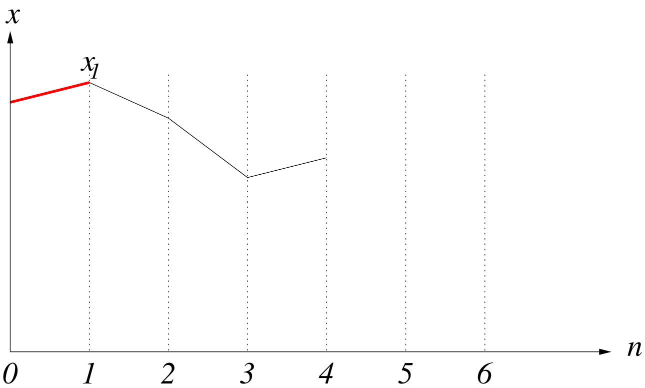

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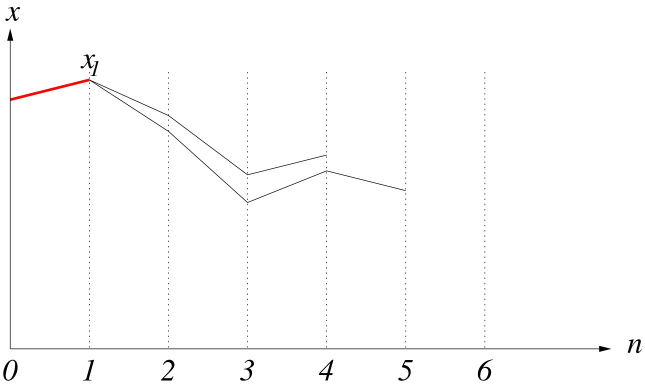

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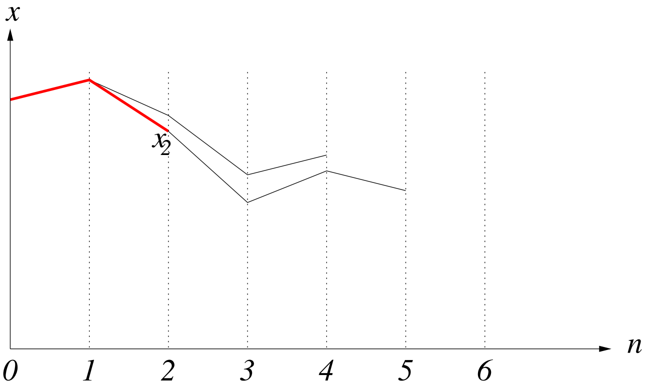

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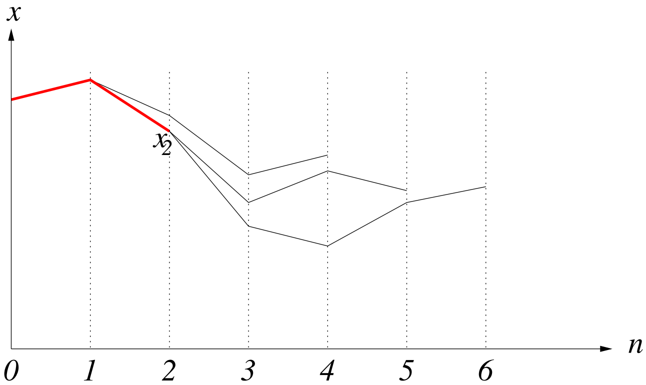

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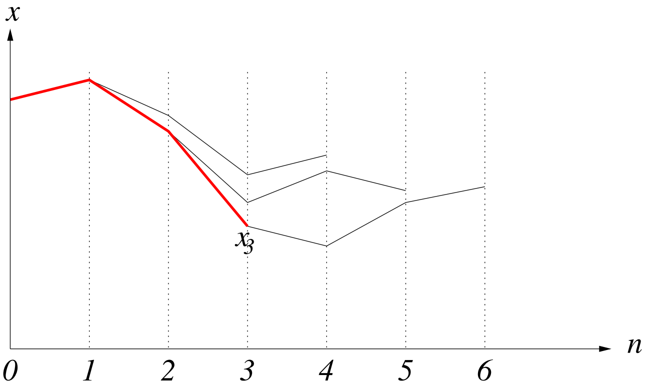

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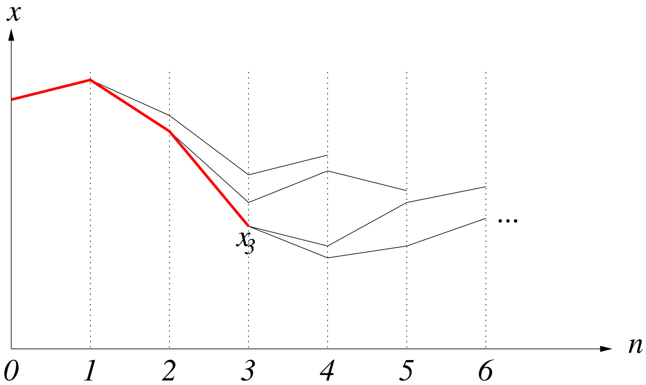

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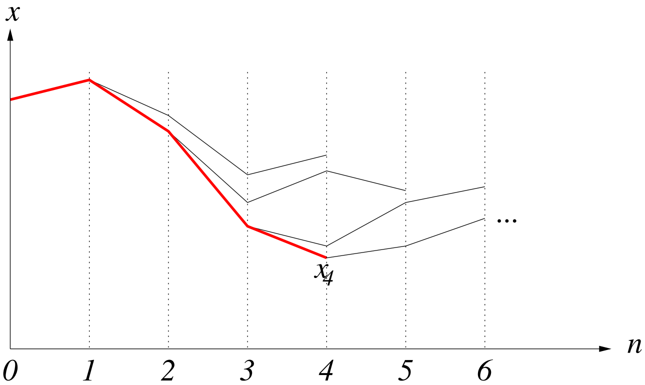

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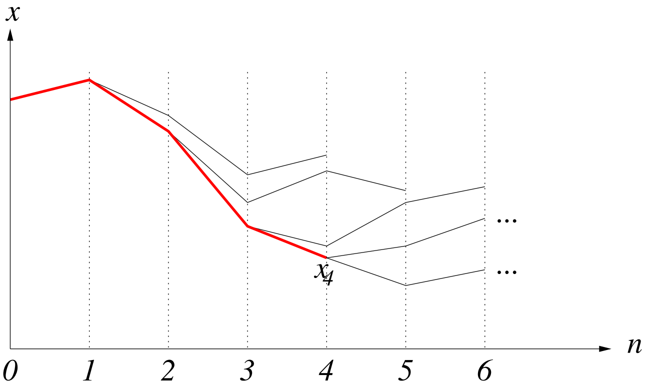

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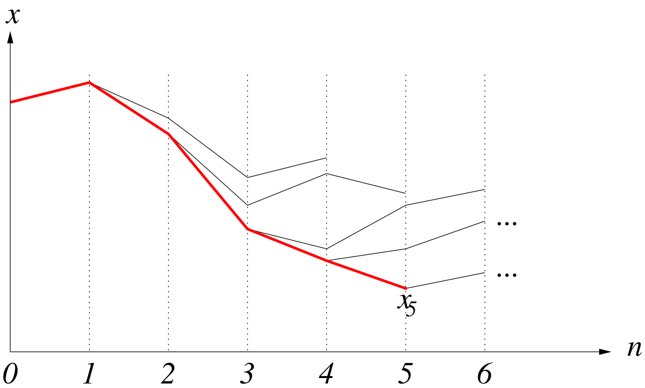

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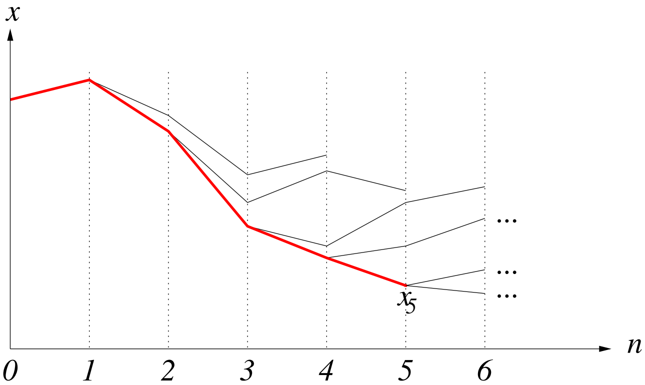

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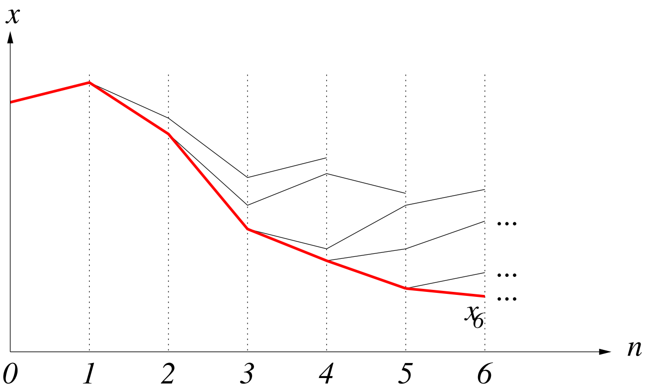

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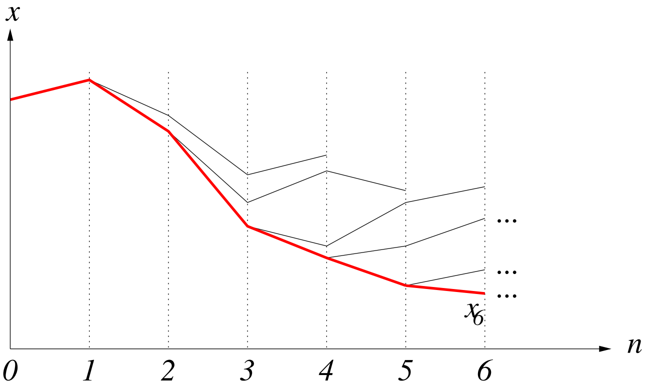

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First question: How to define performance?

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Only in special cases $K \to \infty$ makes sense

Example: minimum energy control

Example: Keep the state of the system inside the admissible set $\mathbb{X}$ minimizing the quadratic control effort

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with dynamics

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(recall: (x^e, u^e) equilibrium $\Leftrightarrow$ $f(x^e, u^e) = x^e$)

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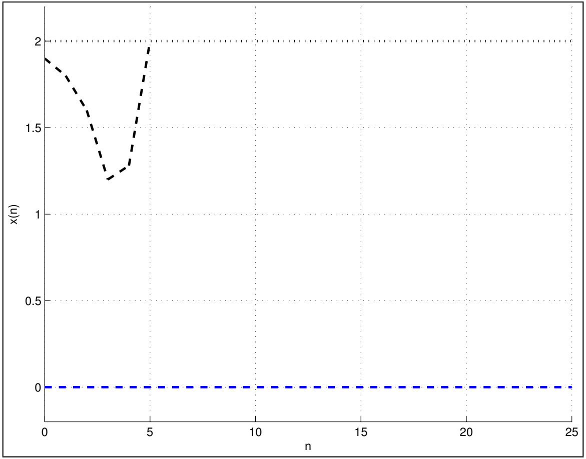

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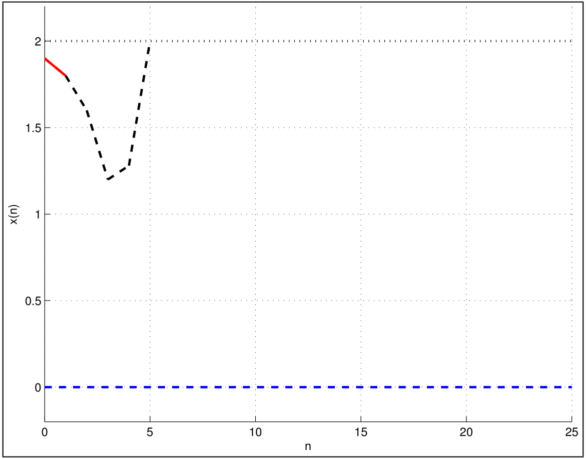

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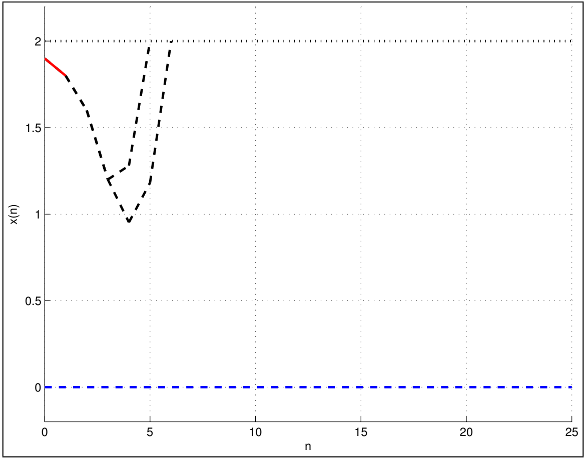

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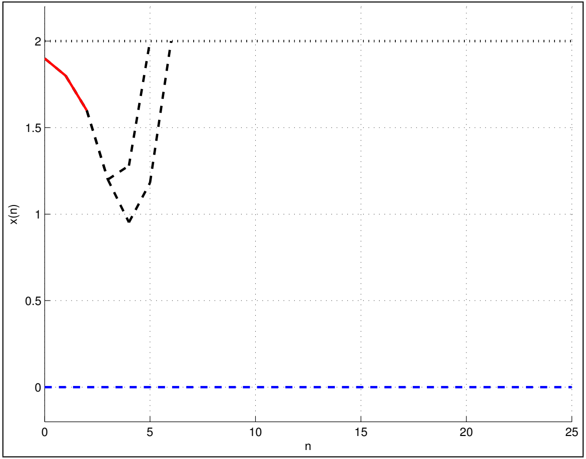

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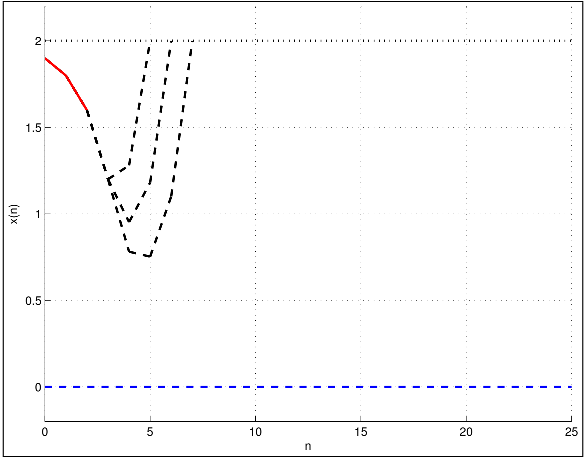

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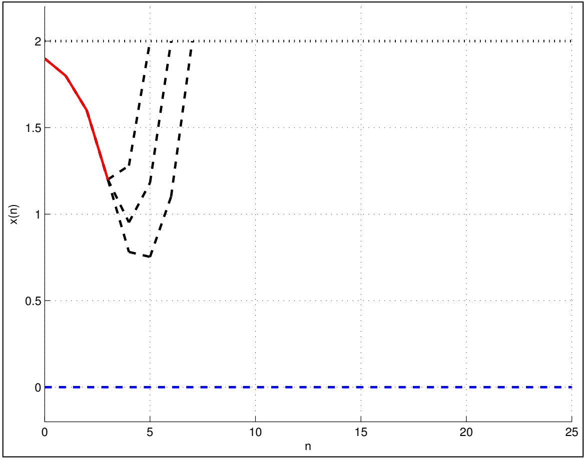

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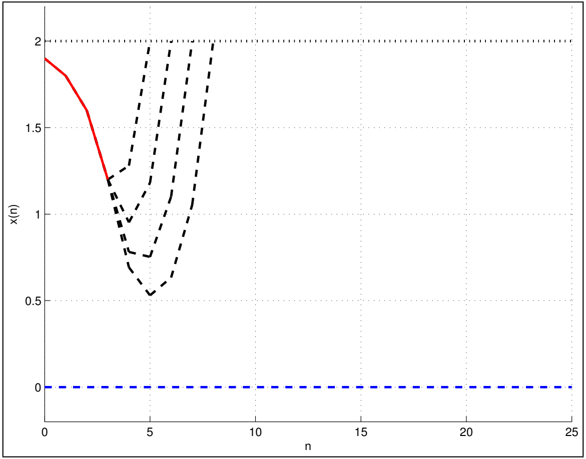

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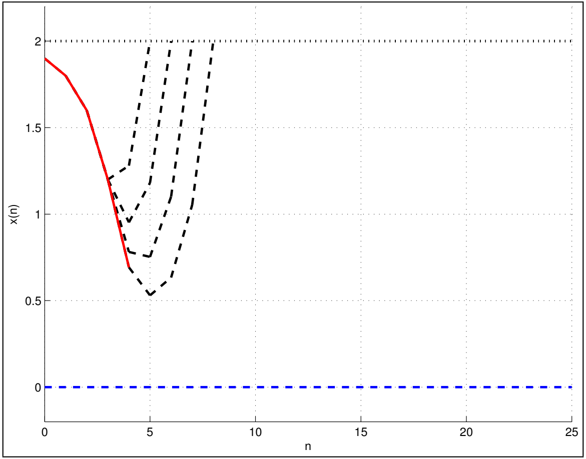

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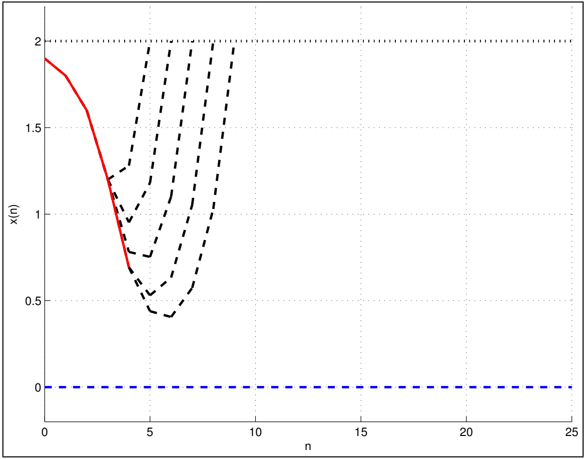

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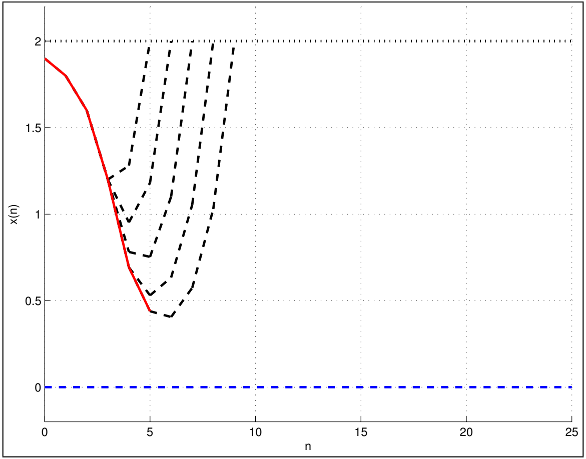

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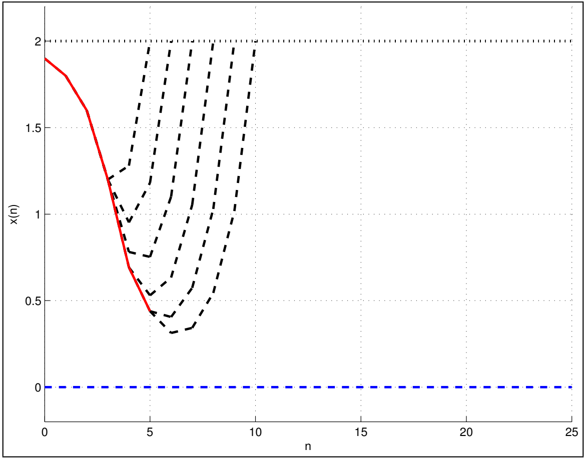
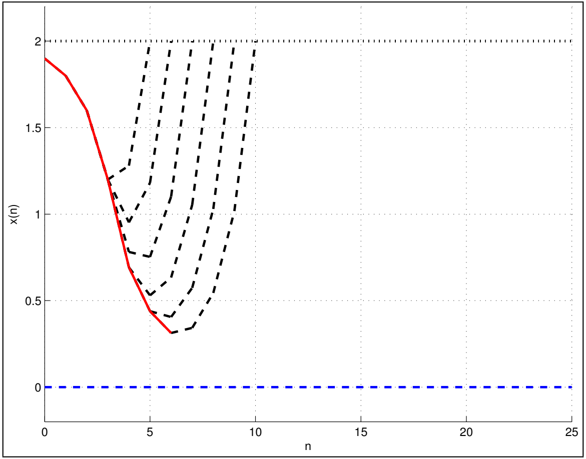


$N = 5$

Example: trajectories



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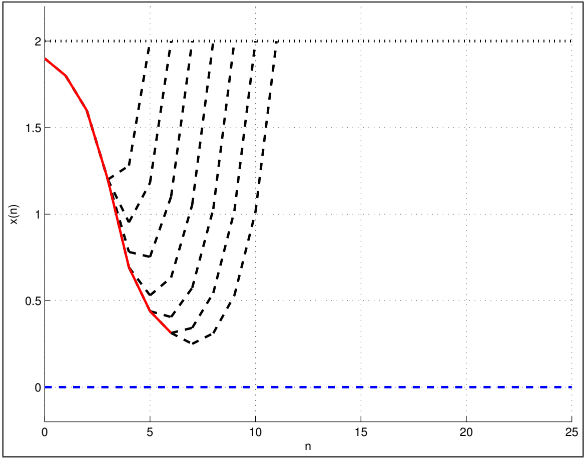

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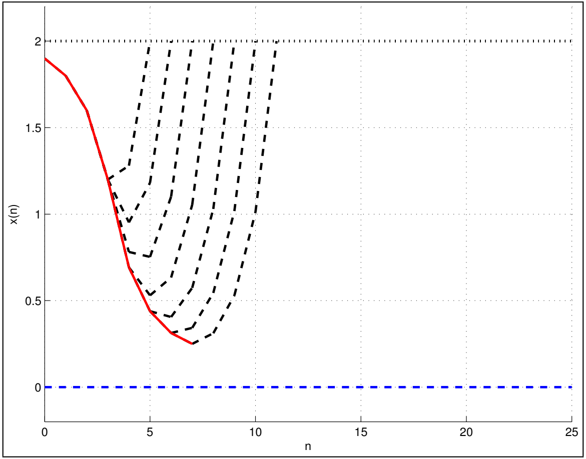

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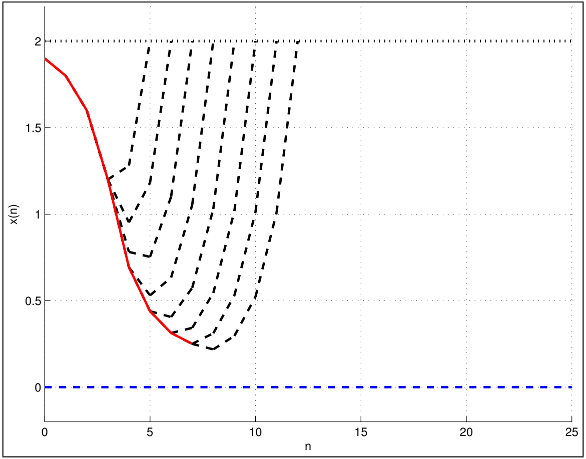

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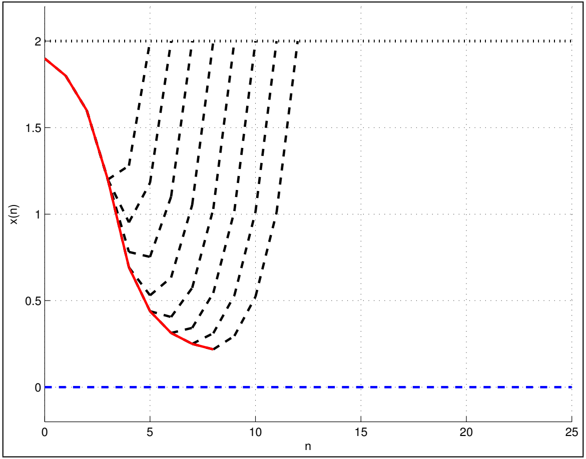

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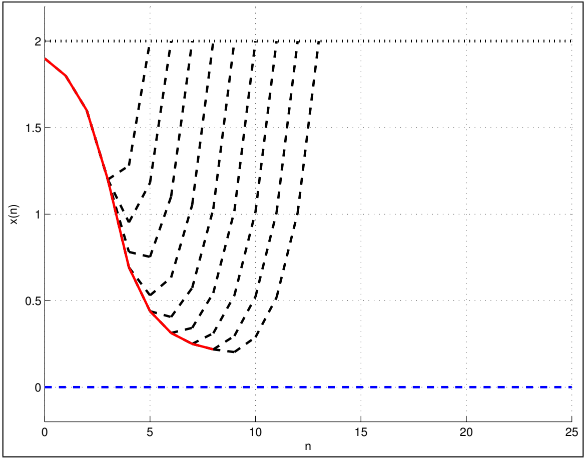

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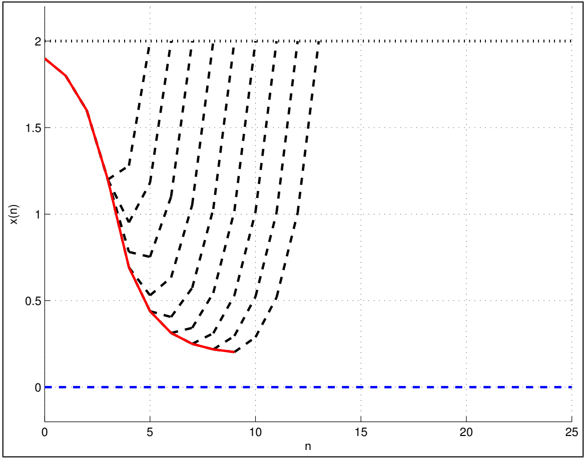

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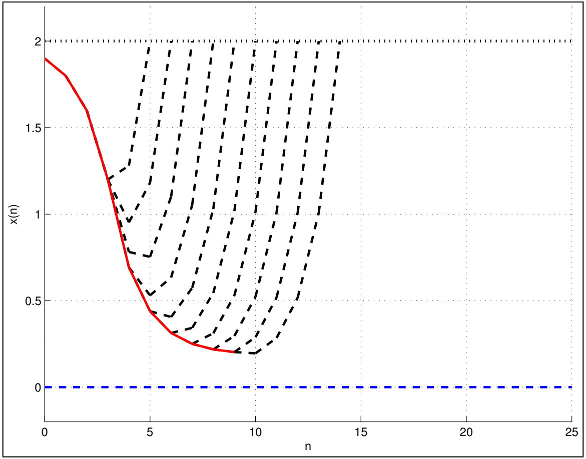

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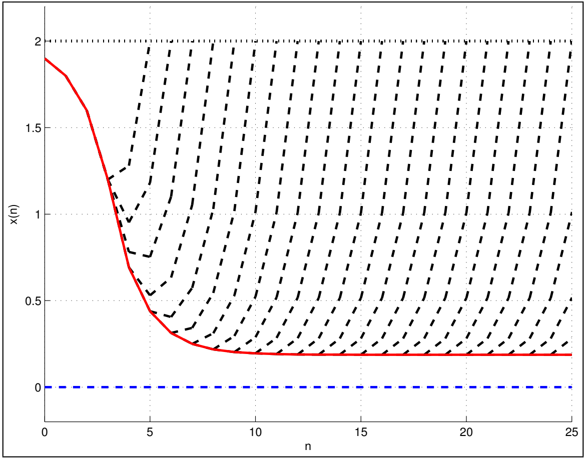

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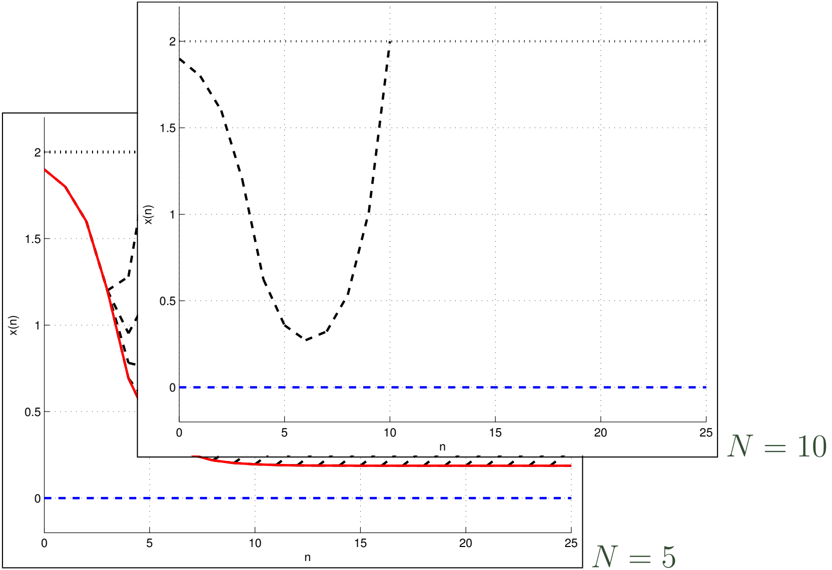

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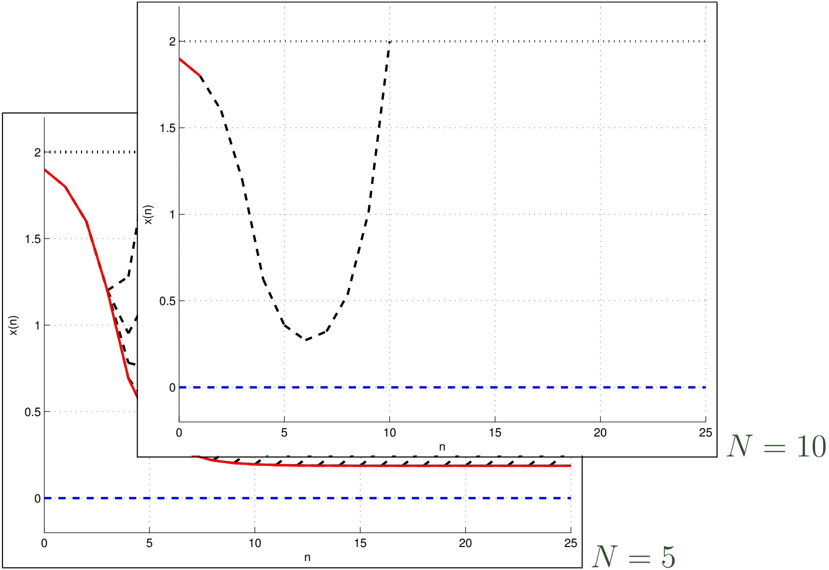

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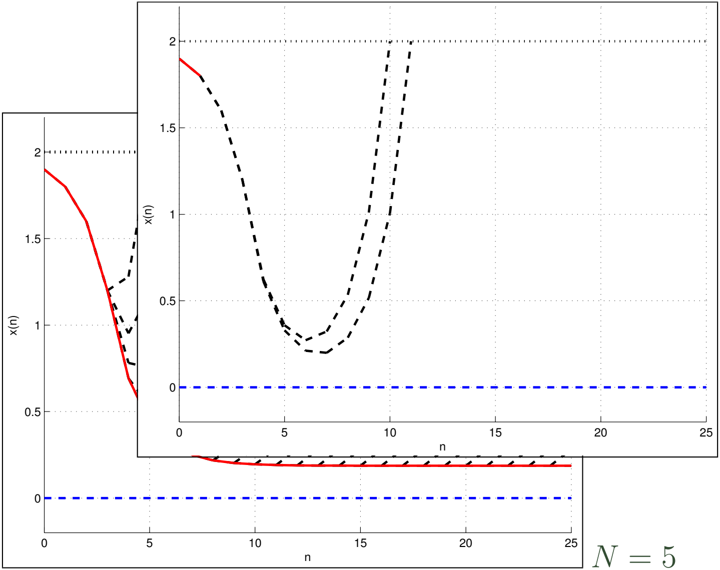

$N = 10$

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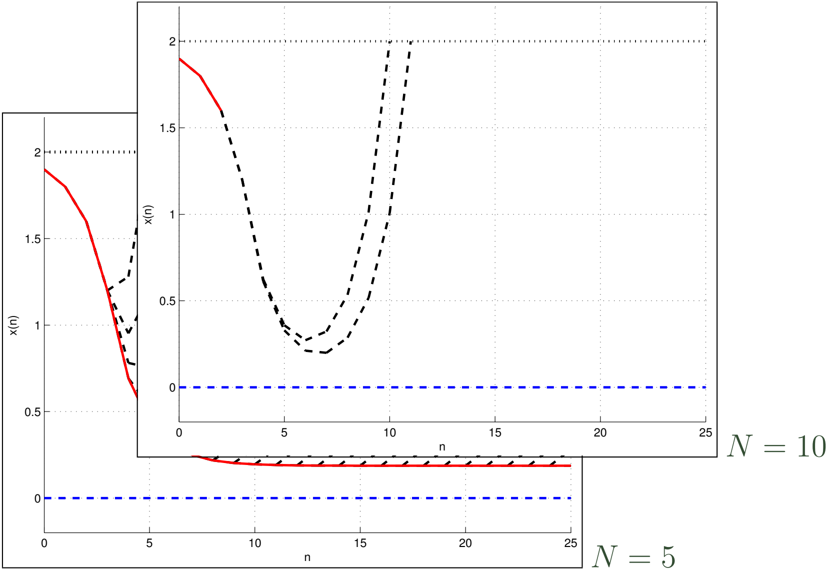

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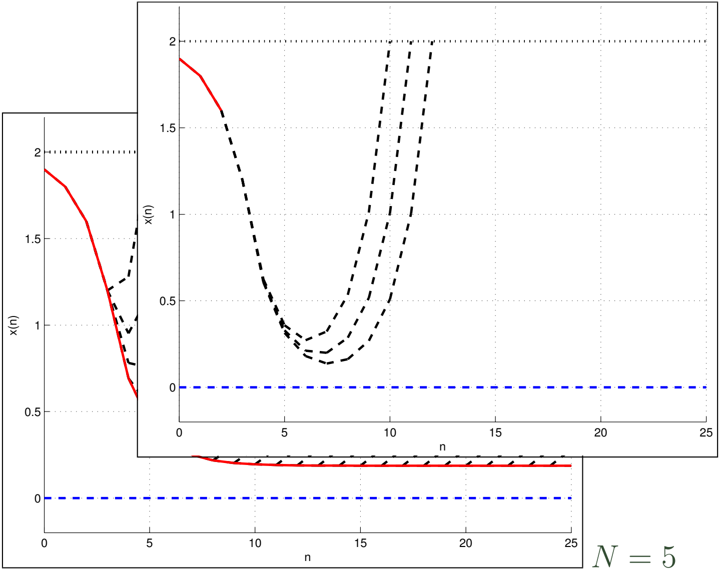

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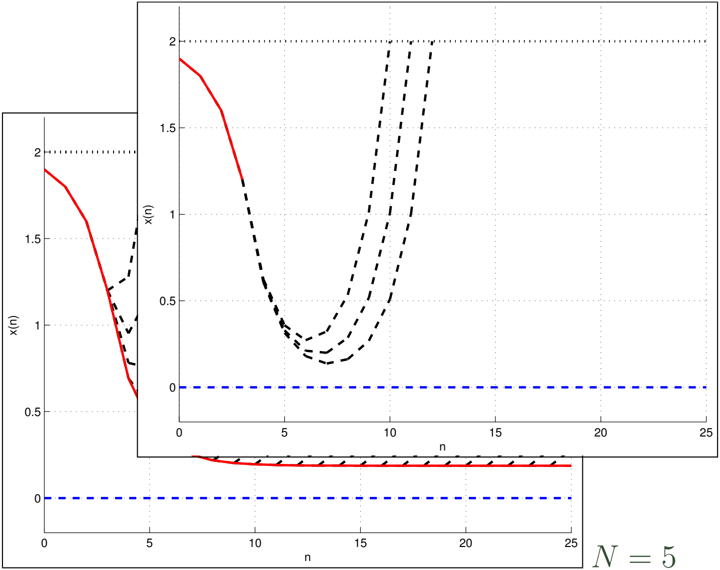

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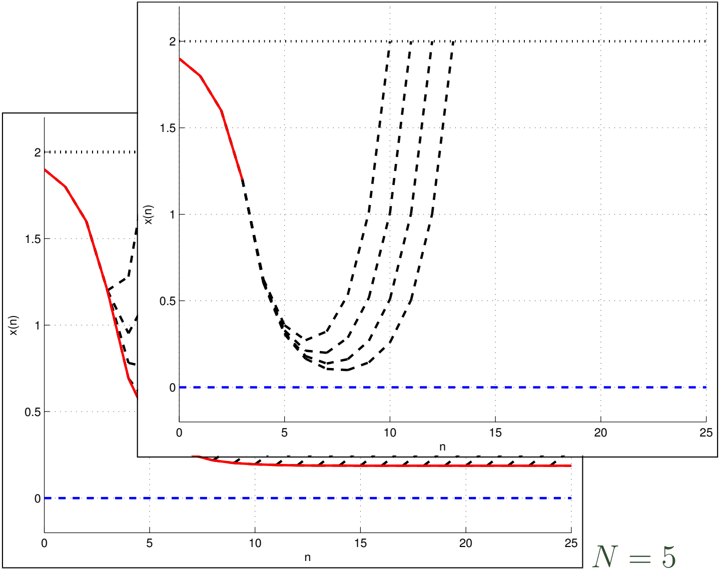

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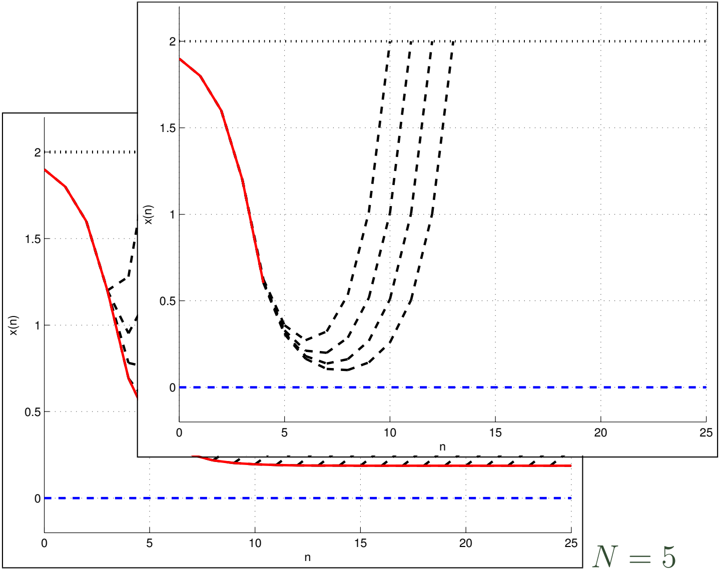

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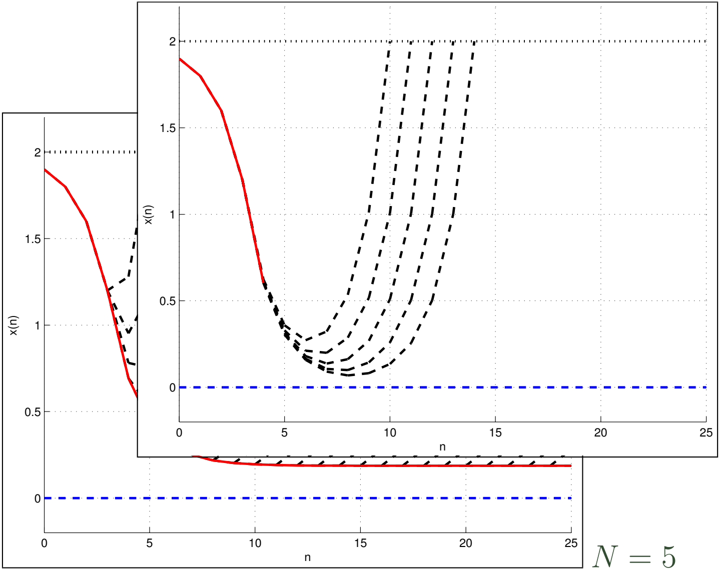

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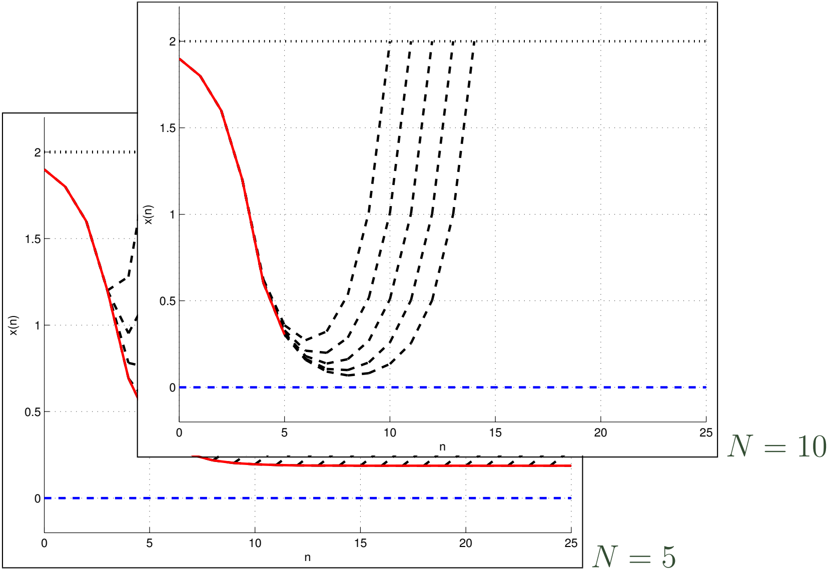

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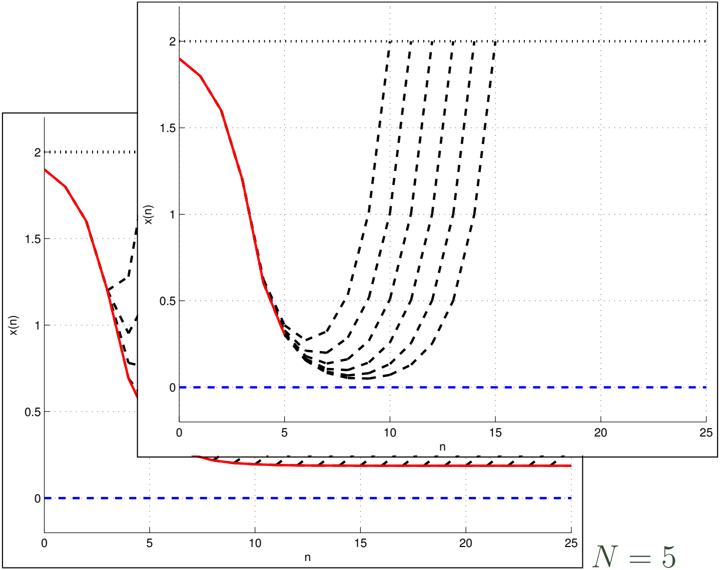

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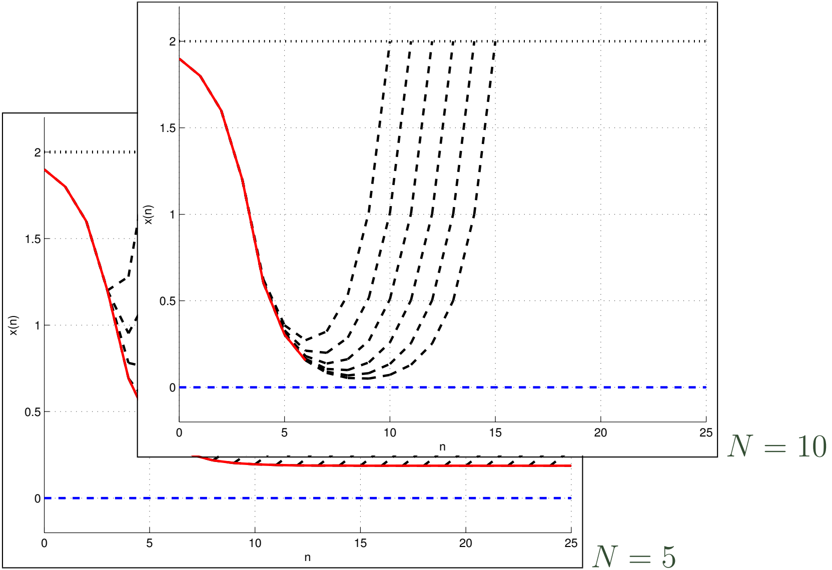

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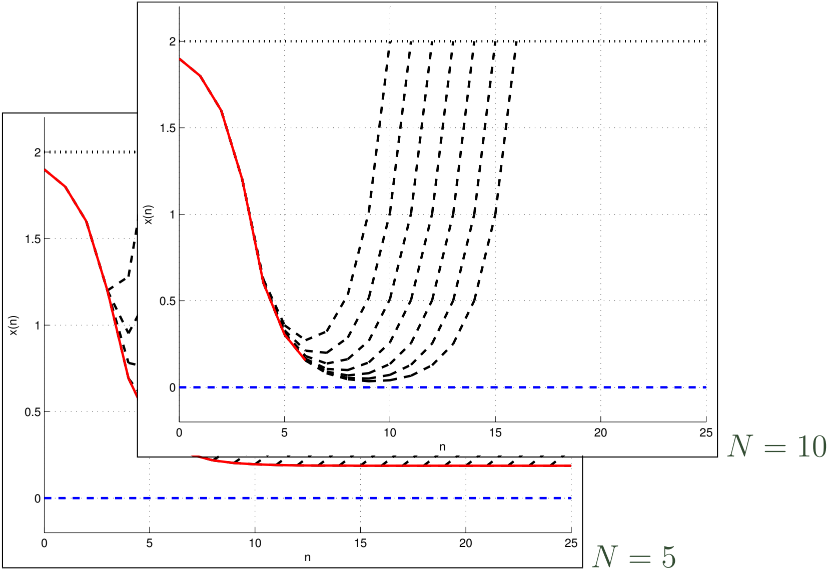

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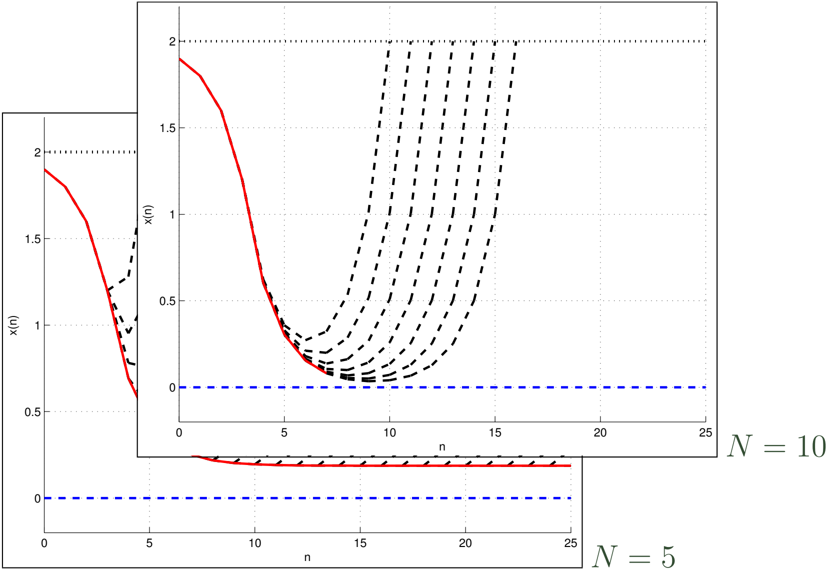

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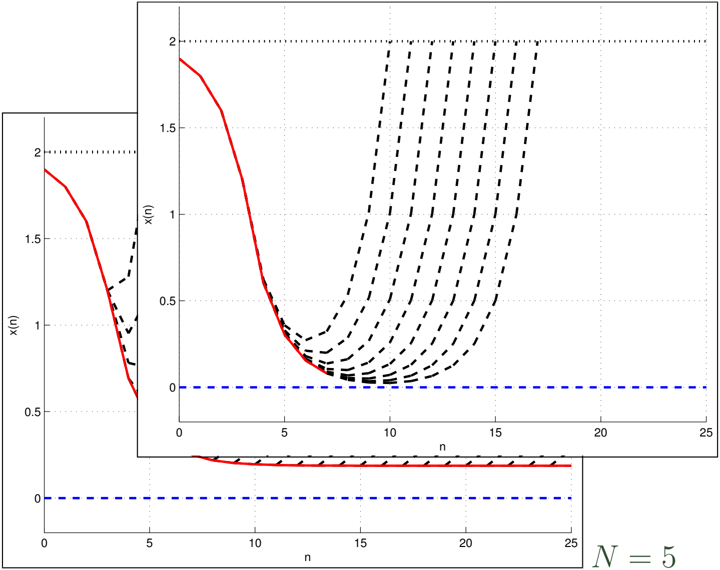

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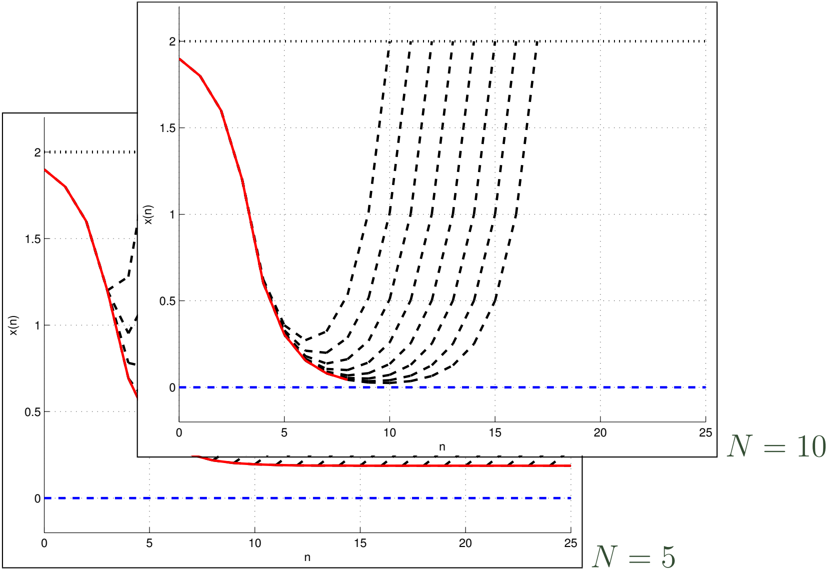

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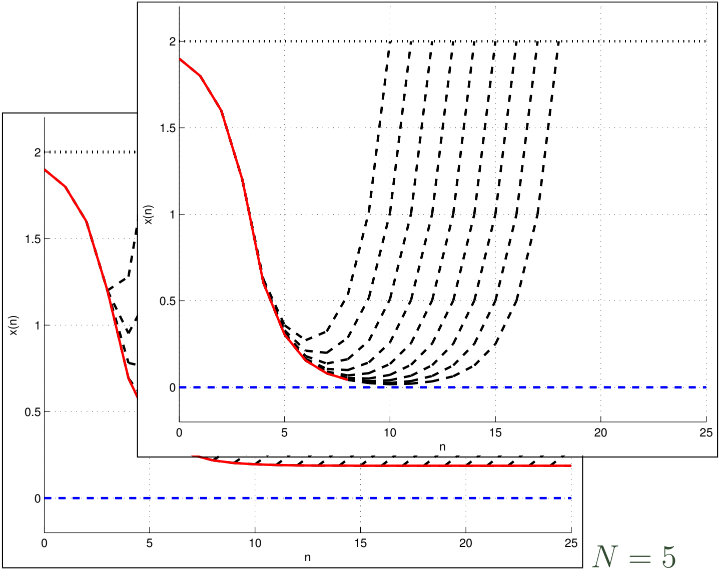

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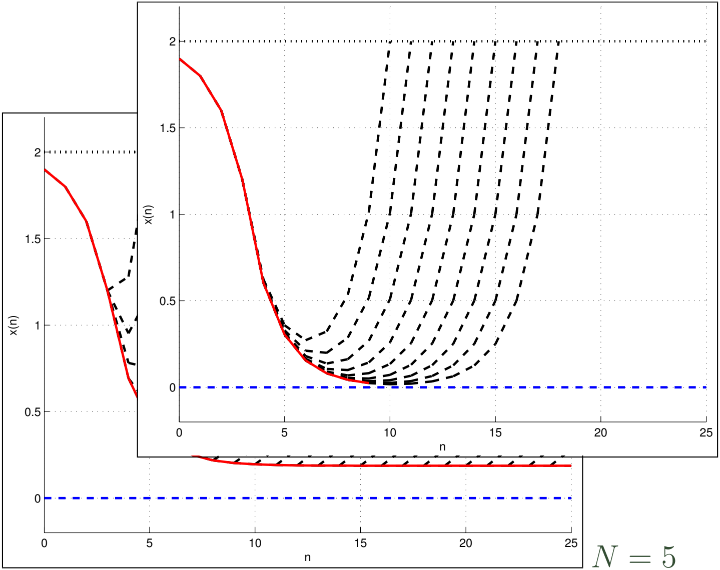

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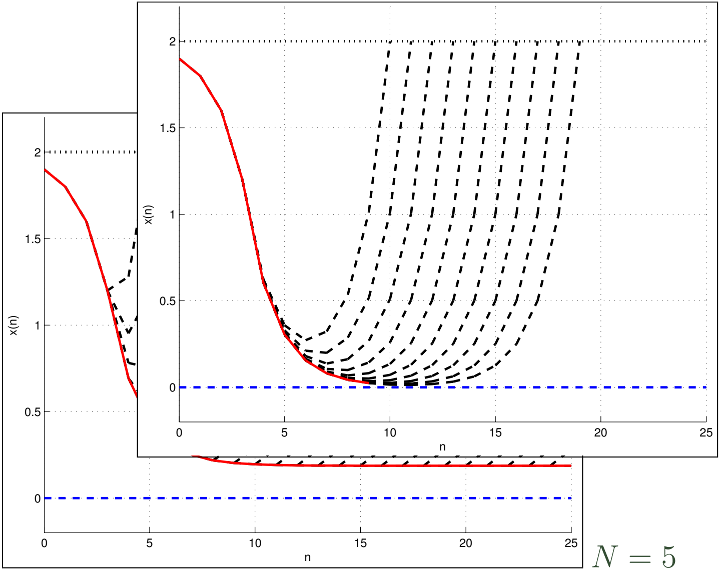

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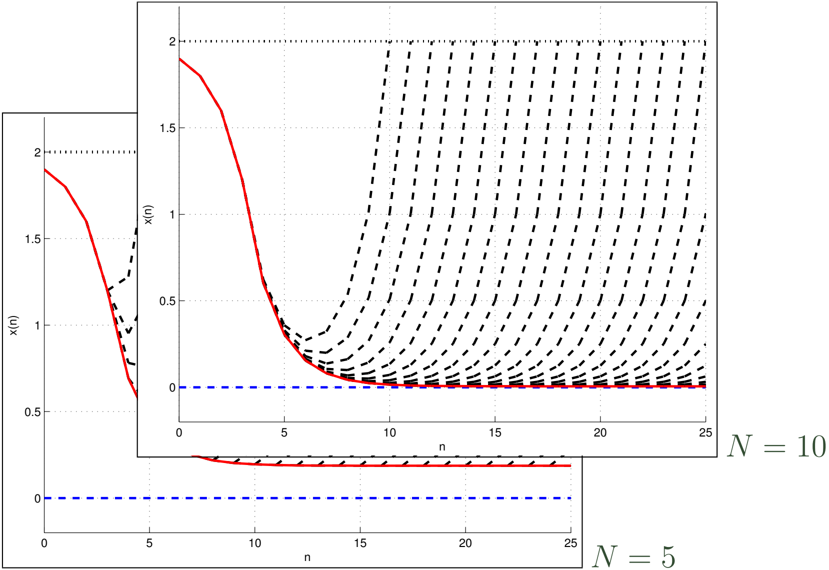
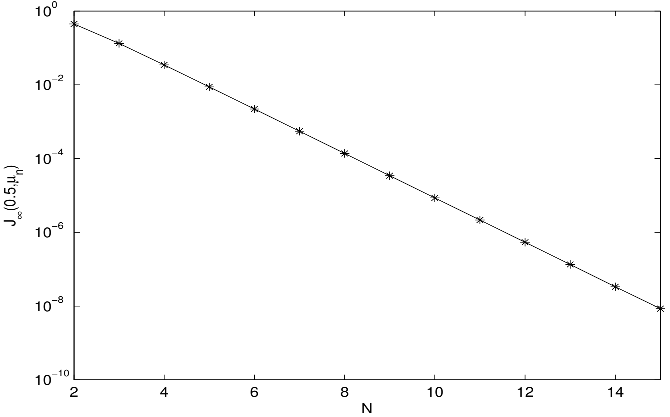
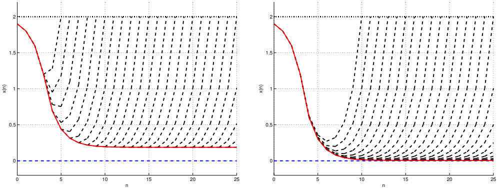


Example: averaged closed loop performance



$\overline{J}^{cl}_\infty(0.5, \mu_N) - \ell(x^e, u^e)$ depending on N, logarithmic scale

Observations



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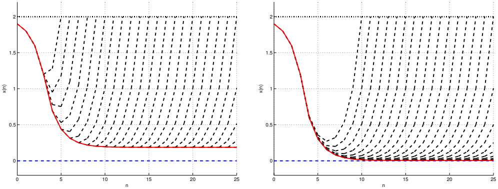

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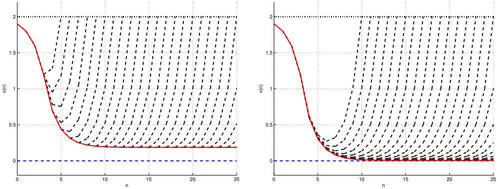

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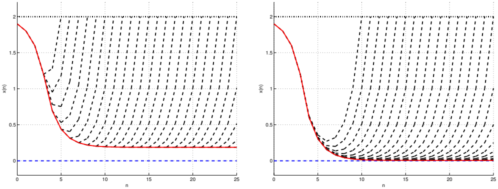

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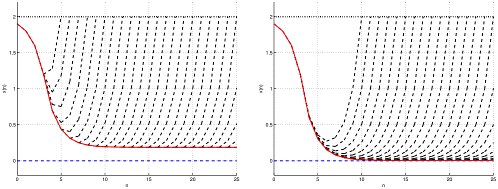

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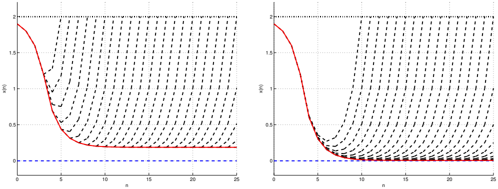

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Can we prove this behavior?

The first property will turn out to be the crucial one

Towards a performance estimate

Defining the optimal value function $V_N(x) := \inf_{\mathbf{u}} J_N(x, \mathbf{u})$,
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$$V_N(x) \leq V_{N-1}(x) + \ell(x^e, u^e) + \varepsilon$$

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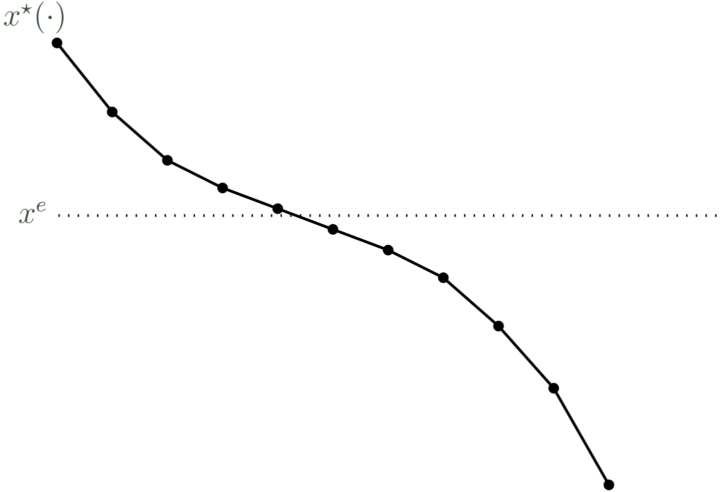
This can be achieved by prolonging the trajectory close to x^e

Prolonging near x^e

Sketch of the idea:



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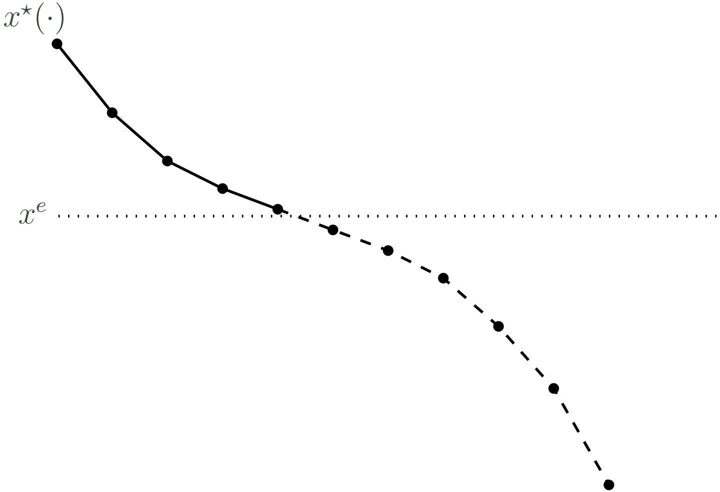

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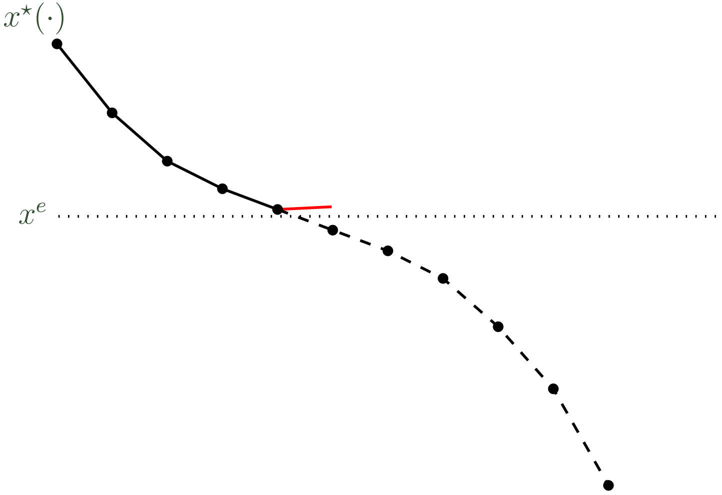

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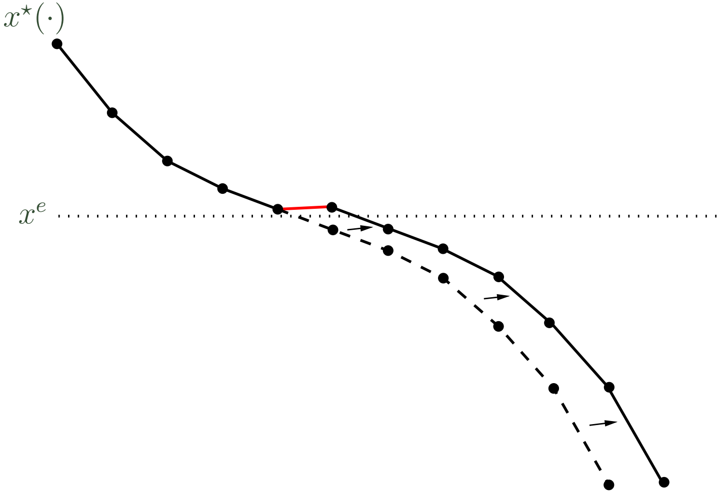

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Instead of the turnpike property, in the MPC literature another property is usually imposed: strict dissipativity

Strict dissipativity [Willems '72]

The optimal control problem is called strictly dissipative if there exists $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and $\alpha \in \mathcal{K}_\infty$ with

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Theorem: [Gr./Stieler '14]

Let f and ℓ be Lipschitz, $\mathbb{X}$ and $\mathbb{U}$ be compact and assume

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(for alternative conditions see also [Porretta/Zuazua '13]
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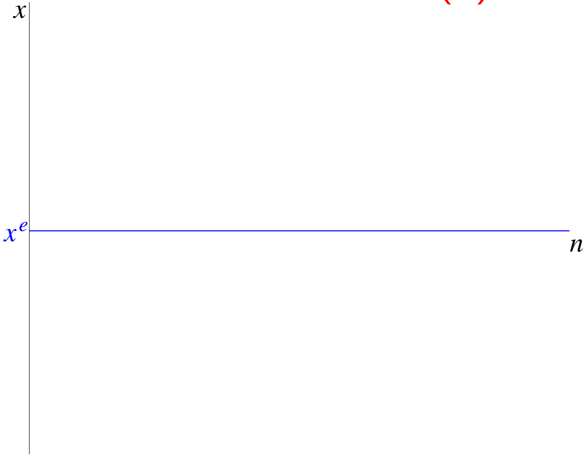
for all admissible $\mathbf{u}$ with $\|x_{\mathbf{u}}(K,x) - x^e\| \leq \beta(\|x - x^e\|, K) + \varepsilon_1(N)$

Illustration of (2) and (3)



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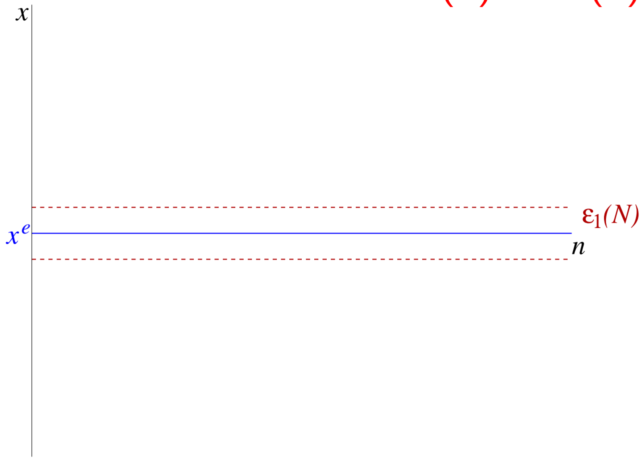

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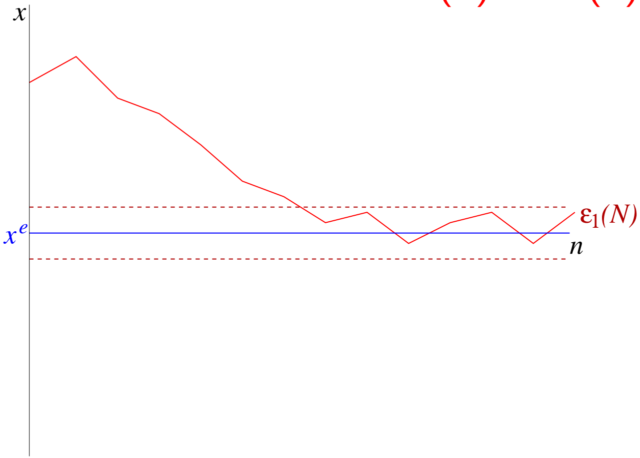

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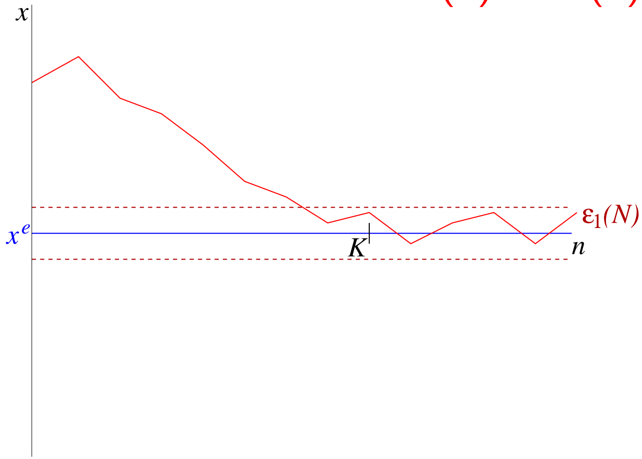

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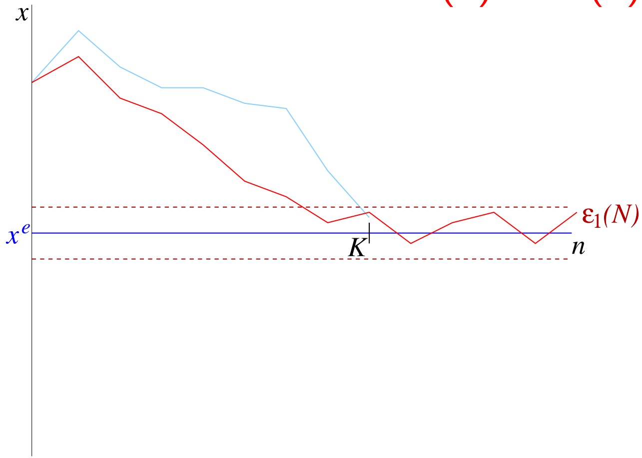

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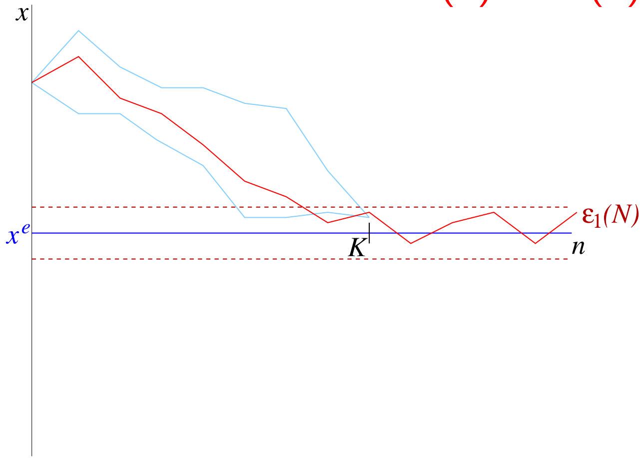

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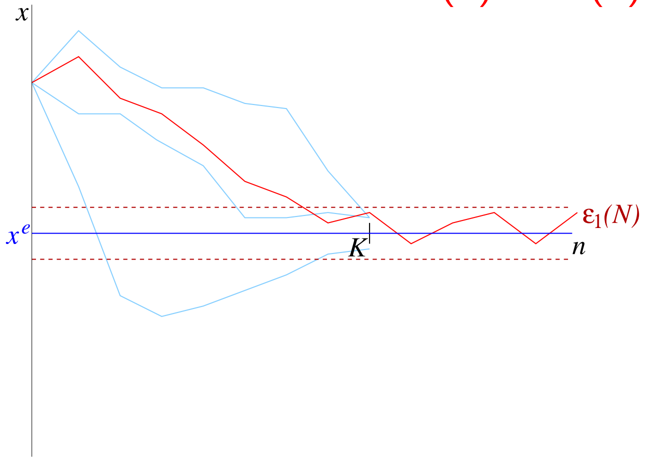

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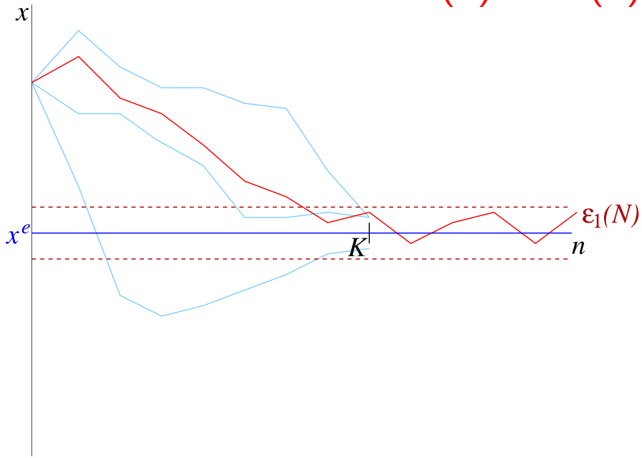

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(3): cost of all other trajectories reaching the ball at time K is higher than that of $x_{\mu_N}(n)$ up to the error $K\varepsilon_1(N) + \varepsilon_2(K)$

Schemes with terminal constraints

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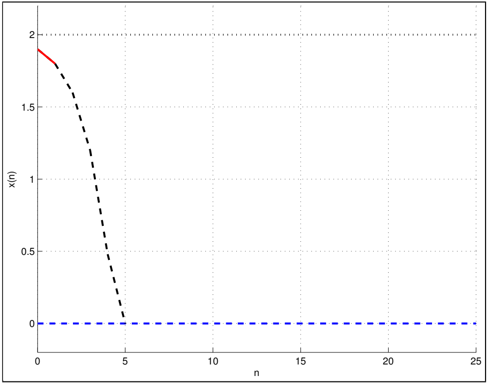

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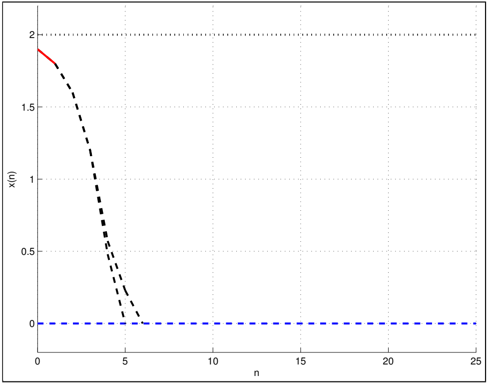

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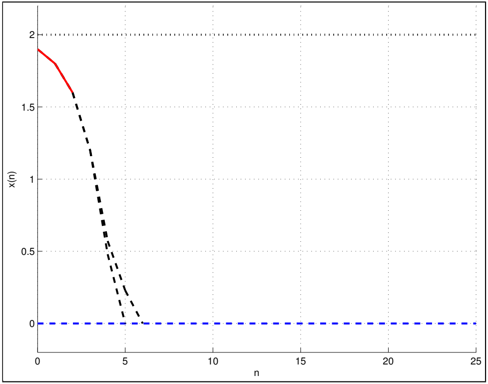

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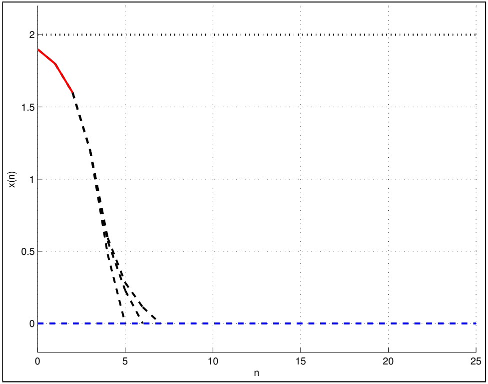

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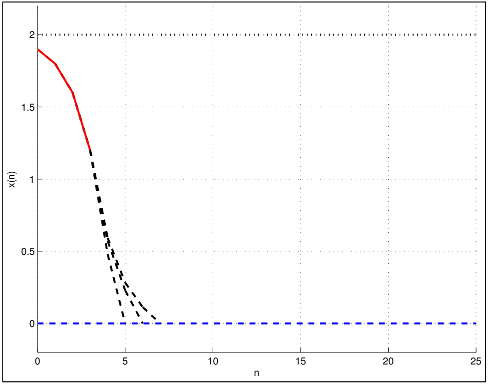

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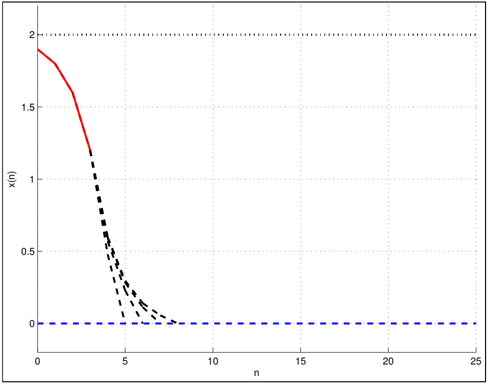

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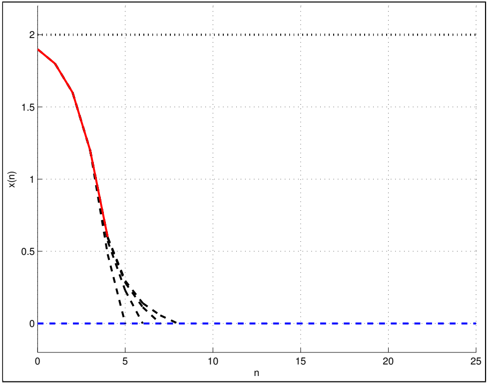

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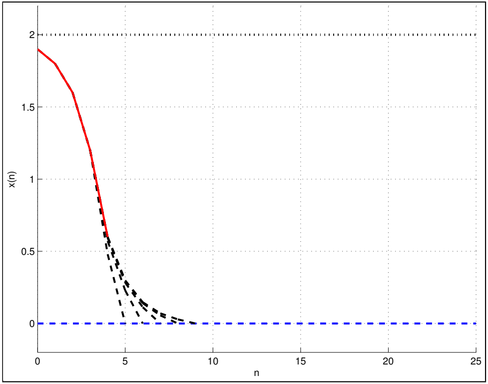

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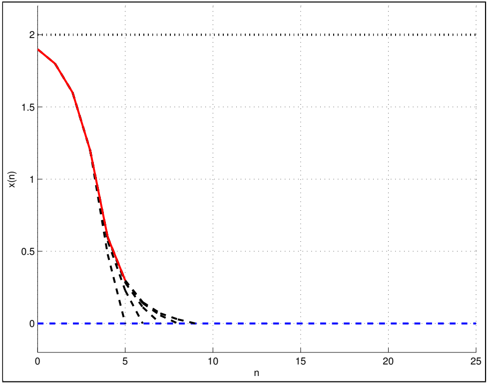

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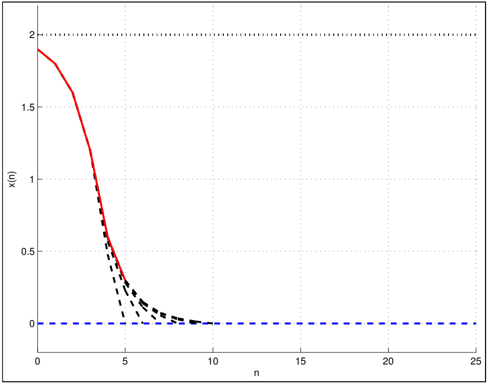

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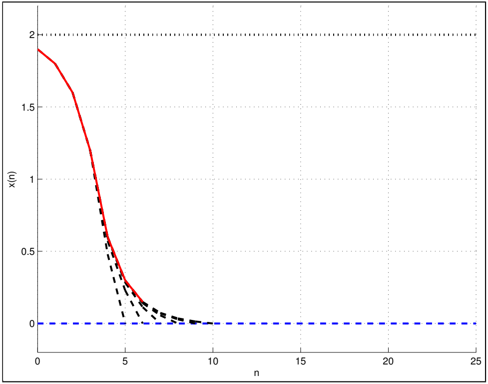

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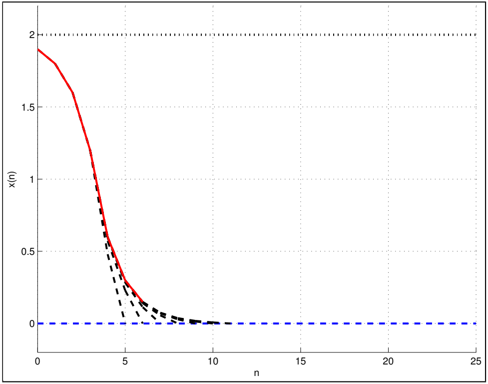

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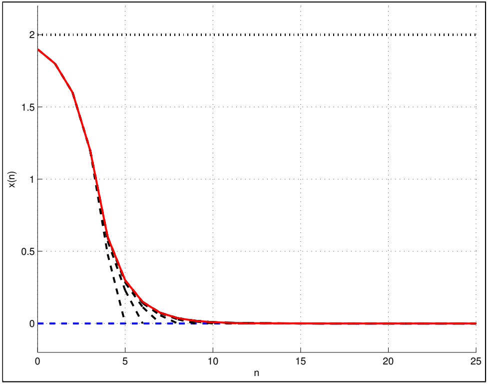

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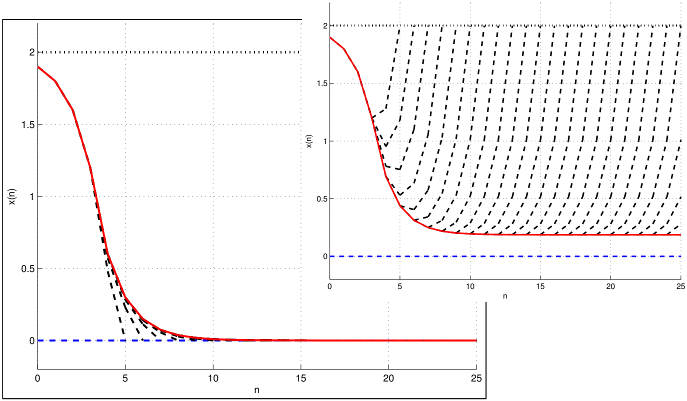

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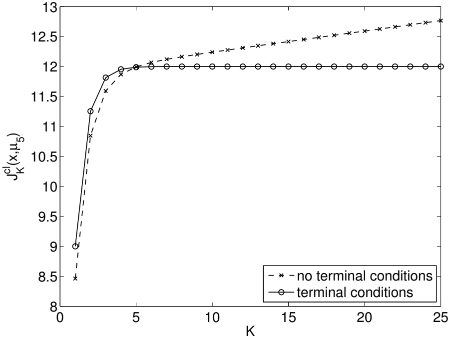

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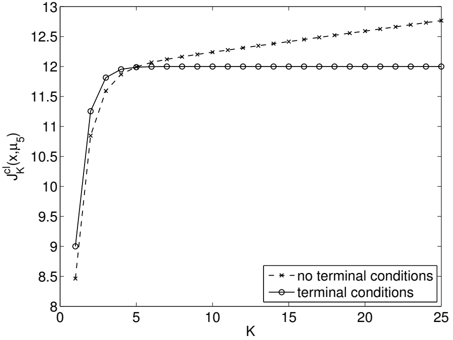

But: terminal constraints can cause infeasibility and severe numerical problems

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- The results can be formulated directly in continuous time [Faulwasser/Bonvin '15, Alessandretti/Aguiar/Jones '15]

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Extensions, further results

- In the affine linear quadratic case our conditions are equivalent to the system being stabilizable [Gr./Stieler '14]
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- First results for discounted optimal control problems [Gr./Semmler/Stieler '15]

Example: Fokker-Planck Equation

Consider a stochastic process governed by a controlled Itô stochastic differential equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X_{t_0} = x_0$$

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Idea: control the statistical properties of X_t by controlling its probability density function $y(x,t)$

The Fokker-Planck Equation

The probability density function (PDF) $y(x,t)$ of X_t solves the Fokker-Planck Equation

$$\begin{aligned} \partial_t y(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\Big(a_{ij}(x,t)y(x,t)\Big) + \sum_{i=1}^{d} \partial_{x_i}\Big(b_i(x,t;u))y(x,t)\Big) &= 0 \\ y(\cdot,0) &= y_0 \end{aligned}$$

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where $y\colon \mathbb{R}^d \times [0,\infty[\to \mathbb{R}_{\geq 0}$ is the PDF

$y_0\colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is the initial PDF

$a = \sigma\sigma^T/2$ is a positive definite symmetric matrix

$b_i\colon \mathbb{R}^d \times [0,\infty[\times U \to \mathbb{R}$, $i = 1,\ldots,d$.

MPC for the Fokker-Planck equation

$$\partial_t y(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\Big(a_{ij}(x,t)y(x,t)\Big) + \sum_{i=1}^{d} \partial_{x_i}\Big(b_i(x,t;u))y(x,t)\Big) \quad = \quad 0$$

Idea: [Annunziato/Borzì '10ff.] Prescribe a desired PDF $y_d(x,t)$ and use MPC for the FP equation in order to track this PDF

$$\rightsquigarrow \quad J_N(y,u) = \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|y(t_{n+1}) - y_d(t_{n+1})\|^2_{L^2(\Omega)} + \lambda\|u(t_n)\|^2\Big)$$

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[Annunziato/Borzì '10ff.] used this idea with $N = 2$ and u independent of the space variable x

MPC for the Fokker-Planck equation

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[Annunziato/Borzì '10ff.] used this idea with $N = 2$ and u independent of the space variable x

We extended this to arbitrary N and u depending on t and x

Numerical Example in 2D

2d Ornstein-Uhlenbeck type process on $\Omega = (-5,5)^2$

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X_{t_0} = x_0$$

with

$$\sigma(x,t) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad b(x,t;u) = \begin{pmatrix} -\mu_1 x_1 + u_1 \\ -\mu_2 x_2 + u_2 \end{pmatrix}$$

$\rightsquigarrow$ Fokker-Planck equation

$$\partial_t y(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\Big(a_{ij}(x,t)y(x,t)\Big) + \sum_{i=1}^{d} \partial_{x_i}\Big(b_i(x,t;u))y(x,t)\Big) \quad = \quad 0$$

with

$$a(x,t) = \begin{pmatrix} 0.32 & 0 \\ 0 & 0.32 \end{pmatrix}, \quad b(x,t;u) = \begin{pmatrix} -\mu_1 x_1 + u_1 \\ -\mu_2 x_2 + u_2 \end{pmatrix}$$

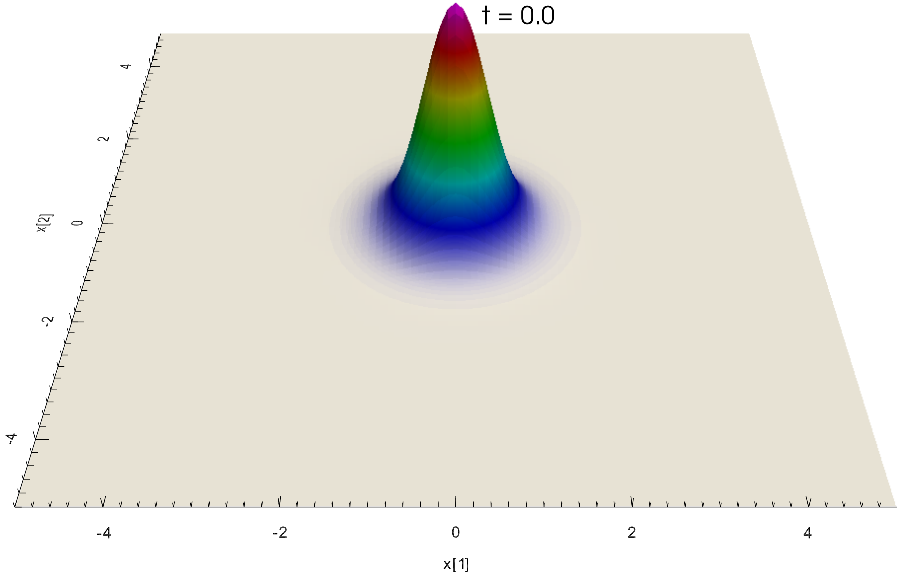
Numerical Example in 2D

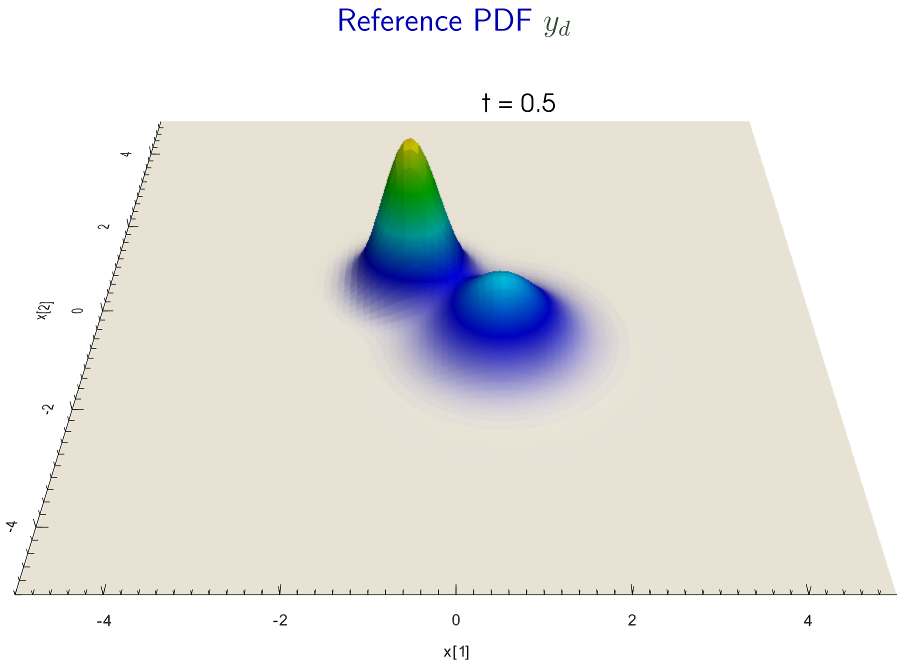
Reference PDF is a bi-modal Gaussian given by

$$y_d(x,t) \;=\; \frac{1}{2}\frac{\exp\left(-\frac{(x_1+\mu(t))^2}{2\sigma_{11}^2}-\frac{(x_2-\mu(t))^2}{2\sigma_{21}^2}\right)}{2\pi\sigma_{11}\sigma_{21}} + \frac{1}{2}\frac{\exp\left(-\frac{(x_1-\mu(t))^2}{2\sigma_{12}^2}-\frac{(x_2+\mu(t))^2}{2\sigma_{22}^2}\right)}{2\pi\sigma_{12}\sigma_{22}}$$

with $\mu(t) = 2\sin(\frac{\pi t}{5}), \sigma_{11} = \sigma_{21} = 0.4, \sigma_{12} = \sigma_{22} = 0.6.$

Reference PDF y_d

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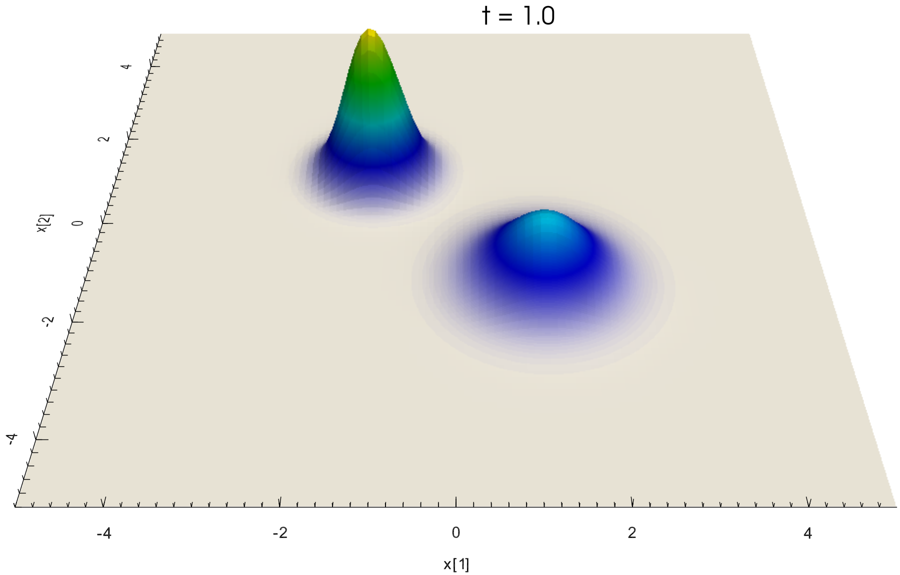
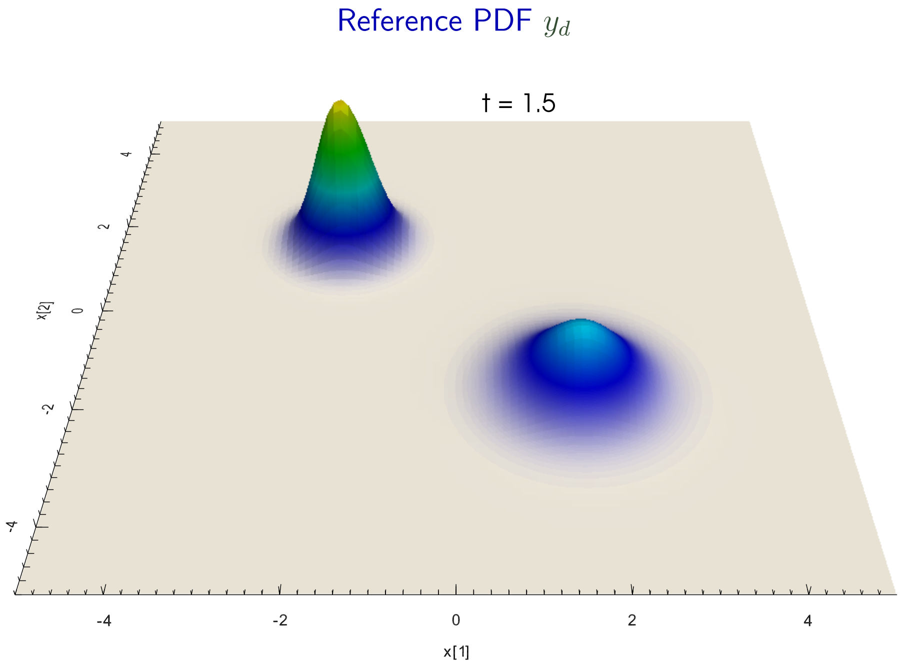
x[1]

Reference PDF y_d

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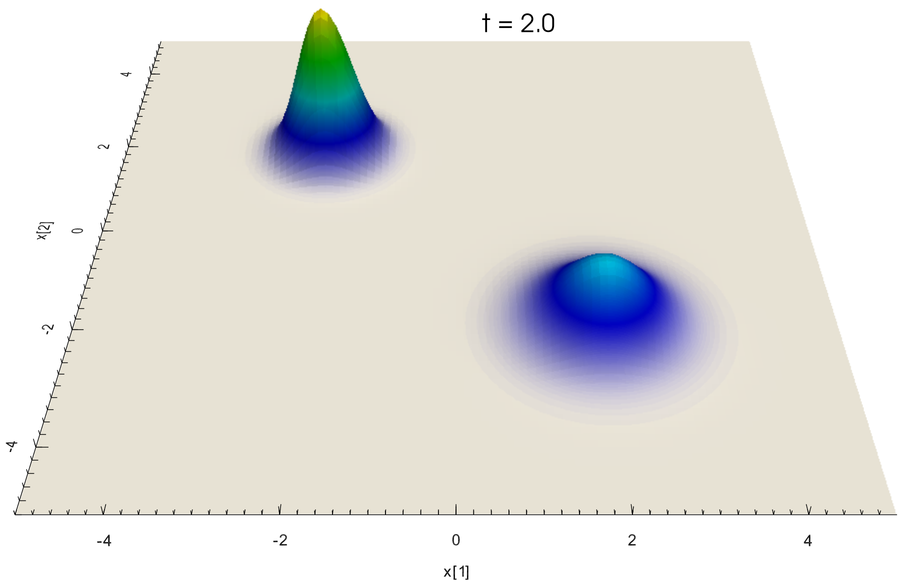
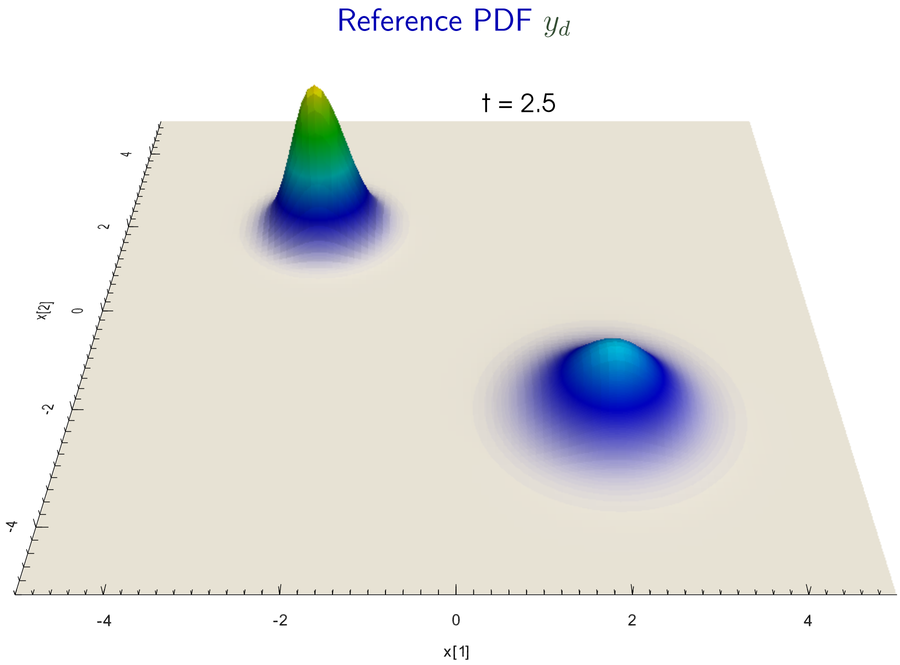
x[1]

Reference PDF y_d

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x[2]
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2
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x[1]

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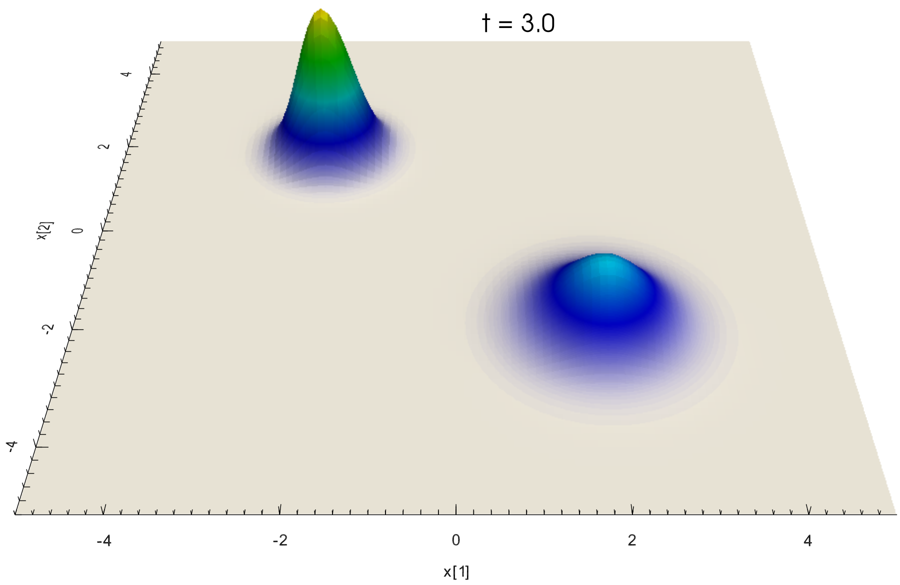

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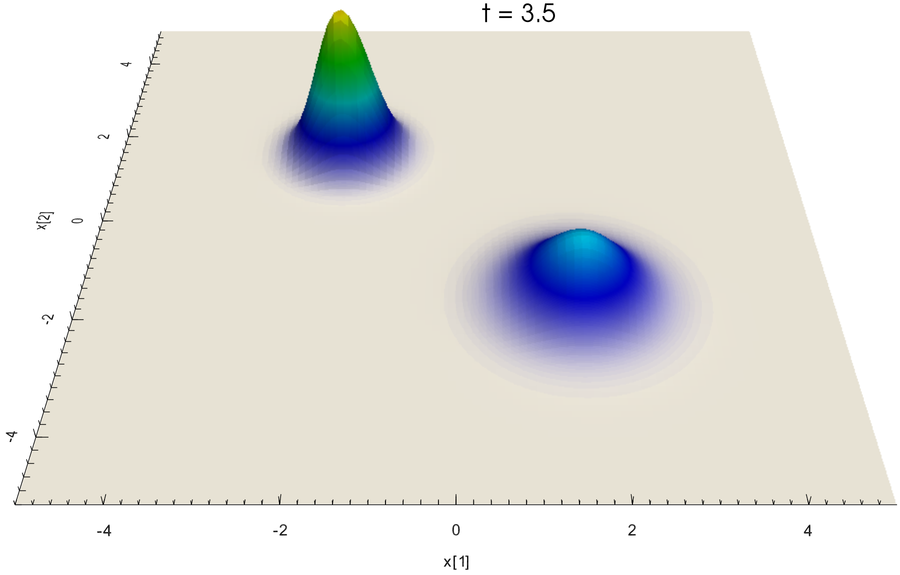

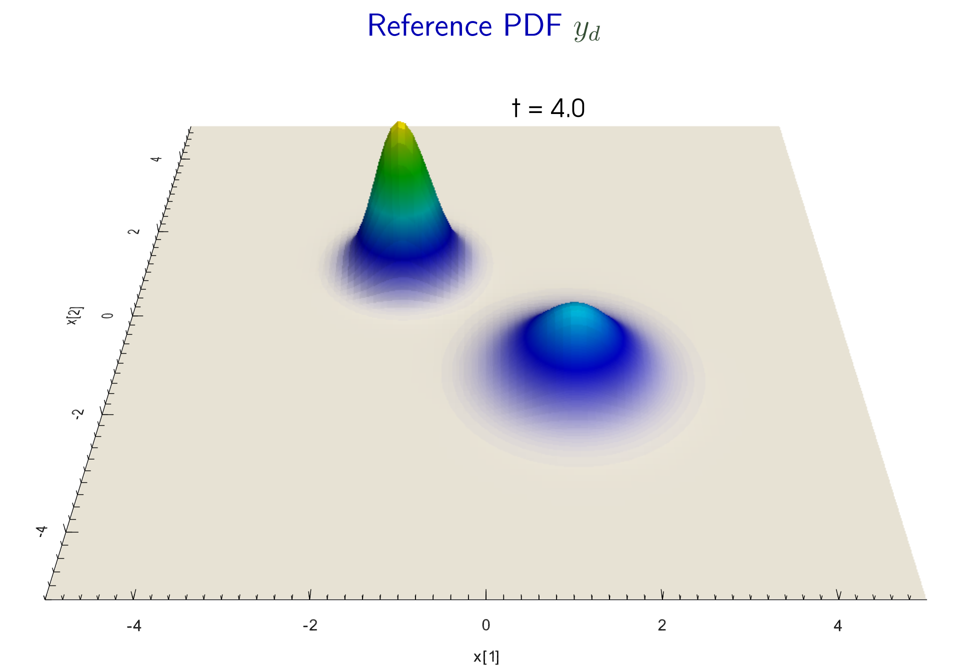
Reference PDF y_d
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x[2]
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2
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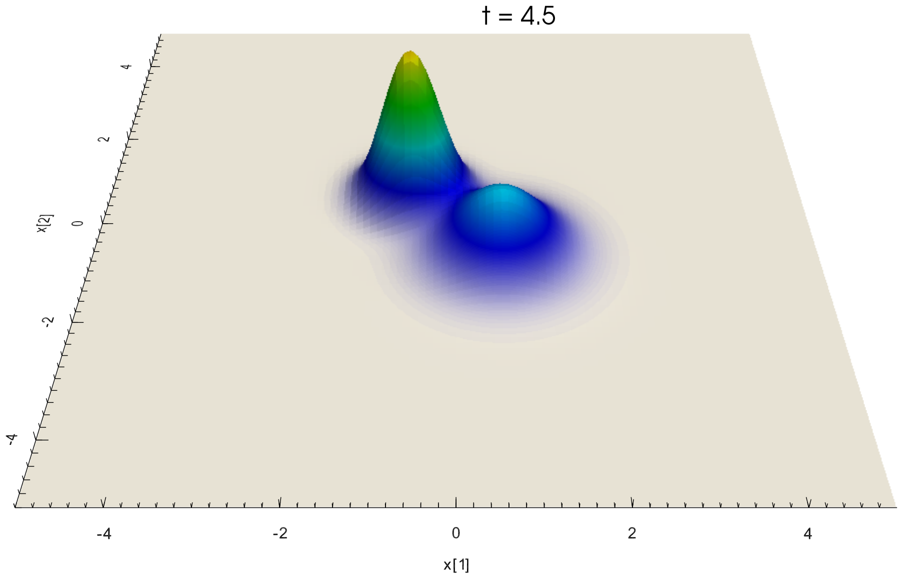
x[1]

Reference PDF y_d



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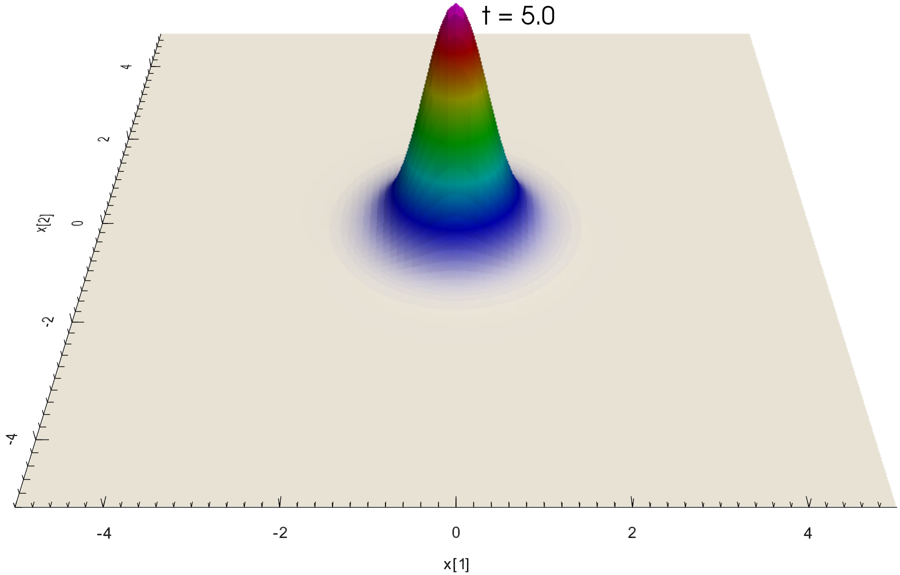

Numerical Example in 2D

Cost functional

$$J(y,u) := \frac{1}{2}\left\|y(t+T) - y_d(t+T)\right\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\left\|u(t)\right\|^2_{L^2(\Omega)}$$

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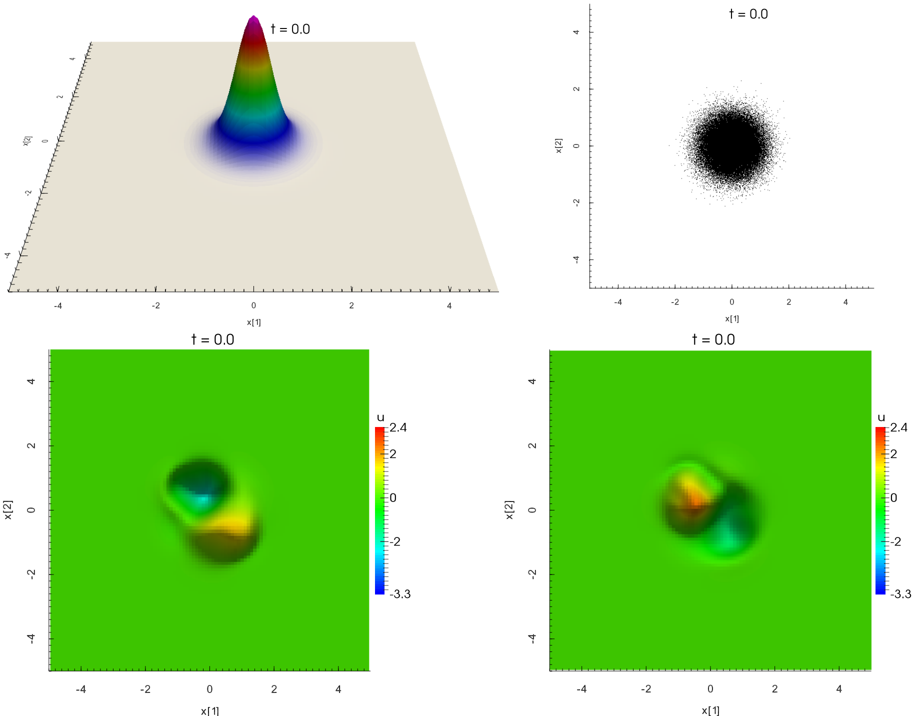
Simulation parameters

- initial distribution $y_0(x) = y_d(x,0)$
- optimization horizon $N = 2$
- sampling time $T = 0.5$
- control penalization $\lambda = 0.001$
- control range $u_{1/2} \in [-10, 10]$



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x[1]
x[2]
t = 0.0
x[1]
x[2]
t = 0.0
u
2.4
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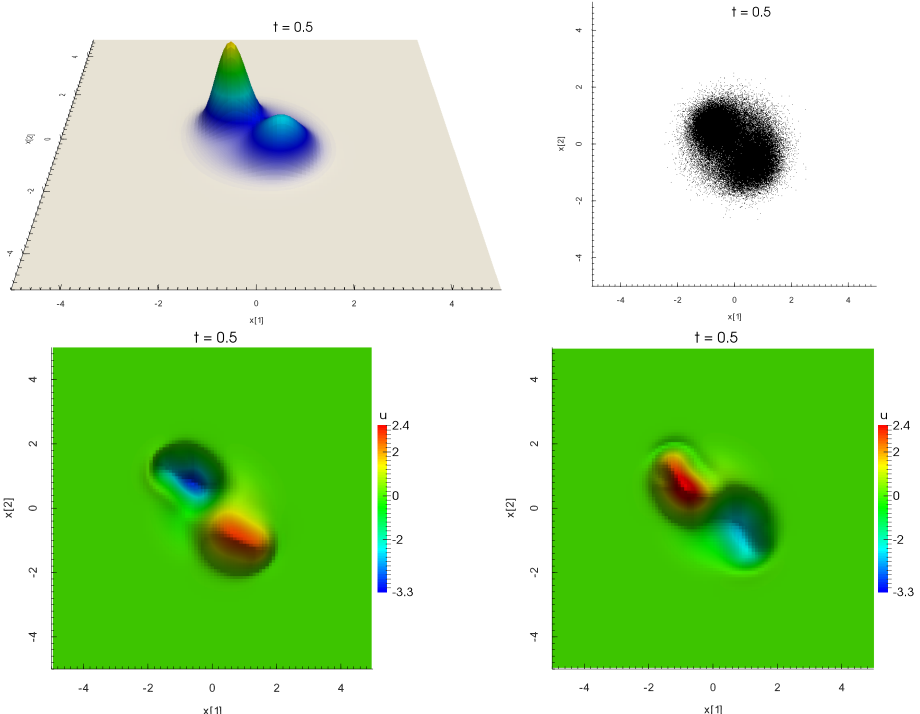
-2
-3.3
x[1]
x[2]



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t = 0.5
u
2.4
2
0
-2
-3.3
x[1]
x[2]
t = 0.5
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2.4
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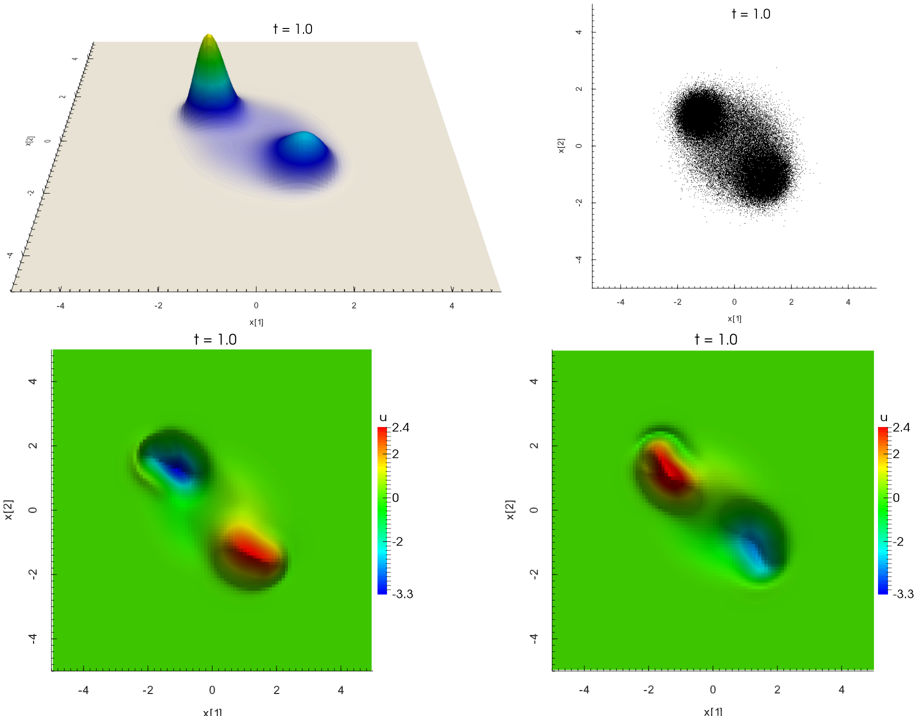
0
-2
-3.3
x[1]
x[2]



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x[2]
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-2
0
2
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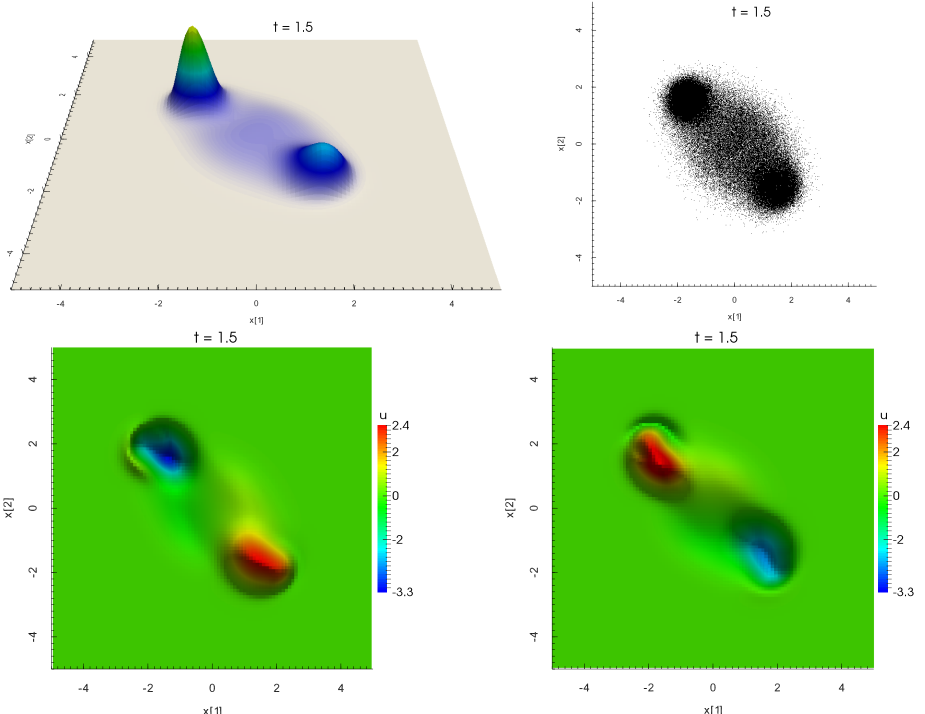
u
2.4
2
0
-2
-3.3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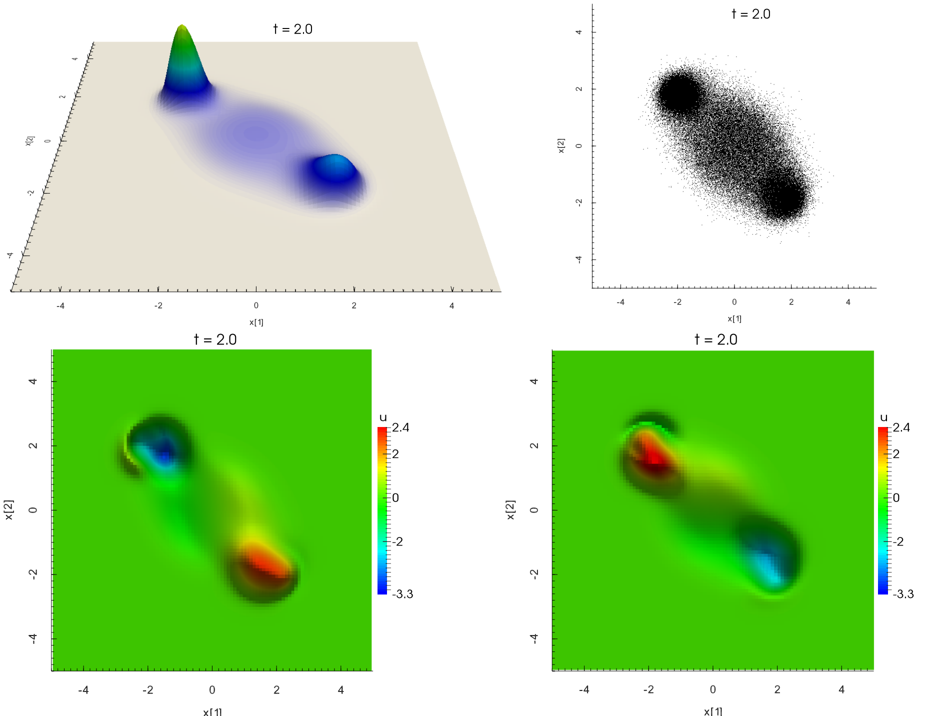

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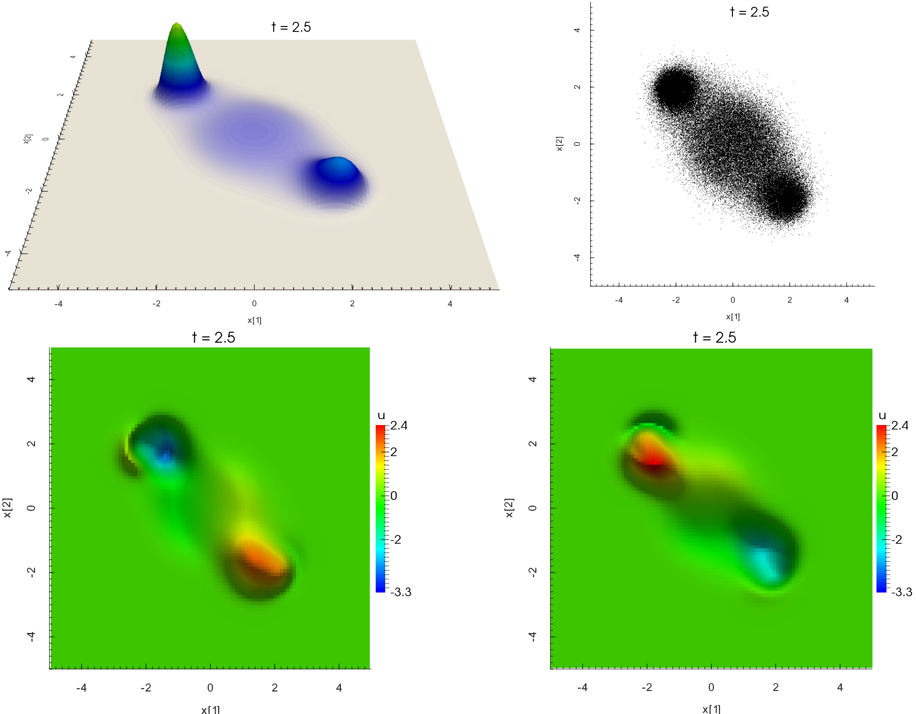
x[2]



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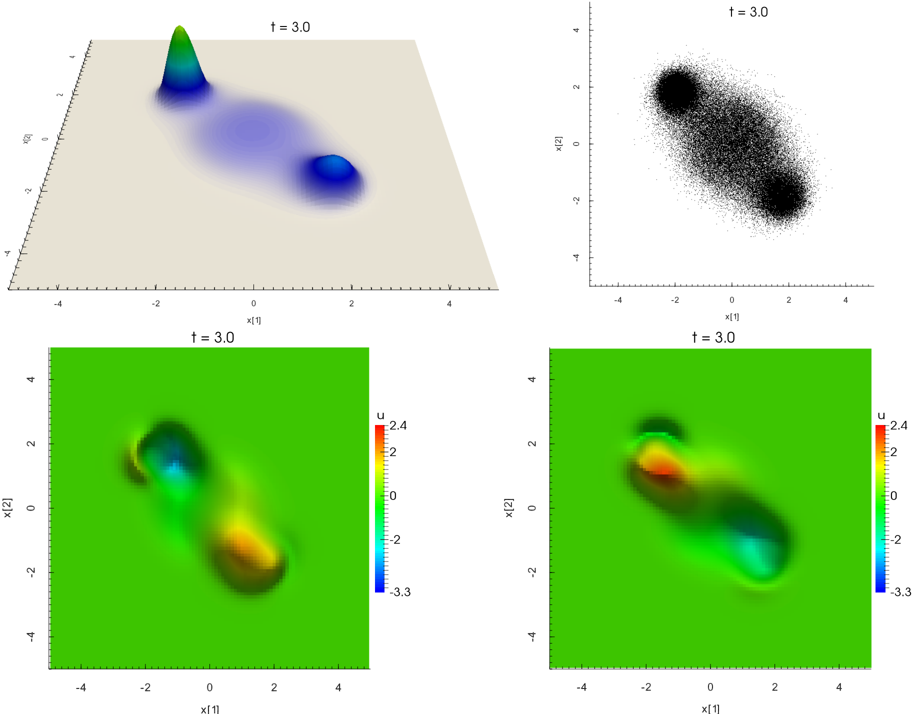
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x[1]
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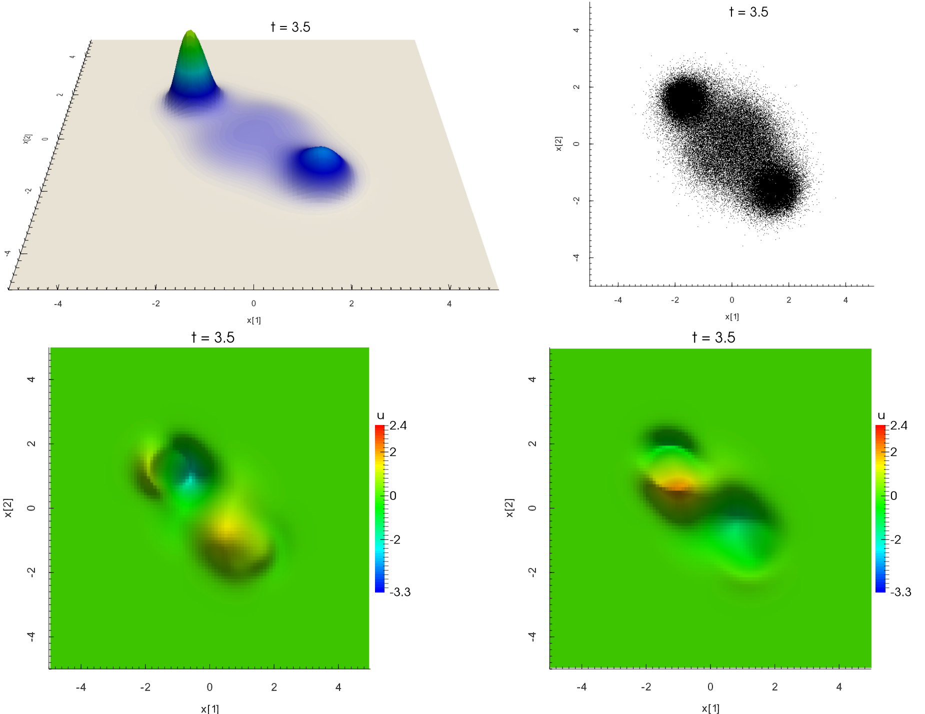

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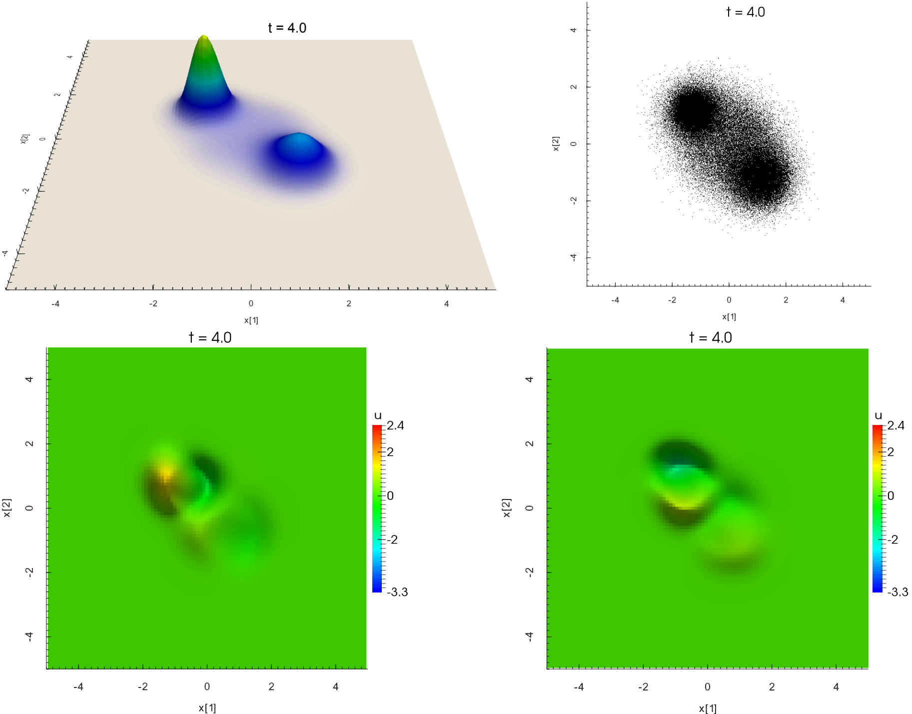

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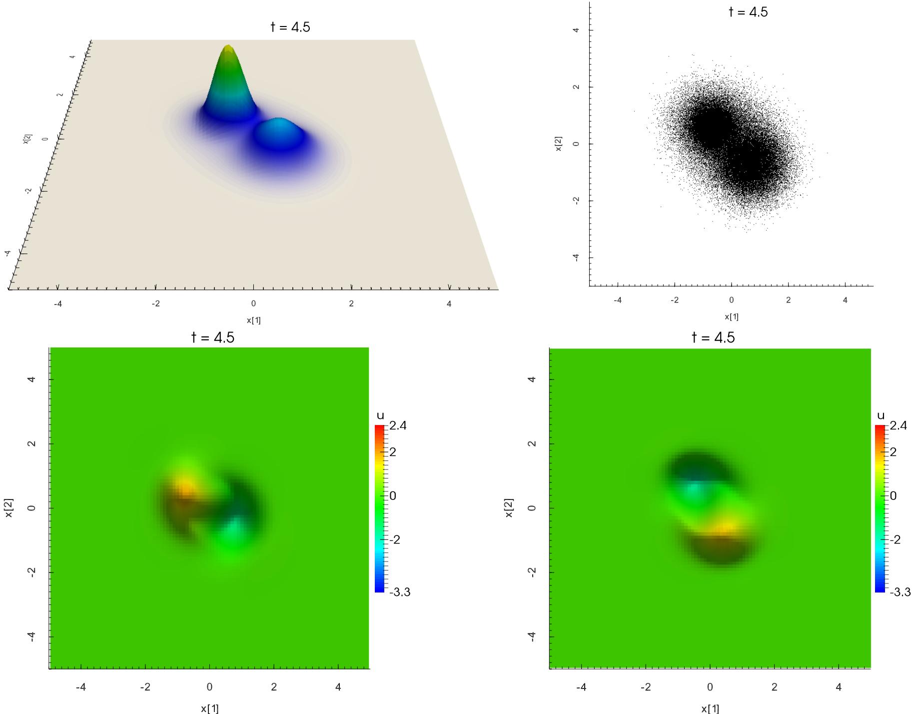
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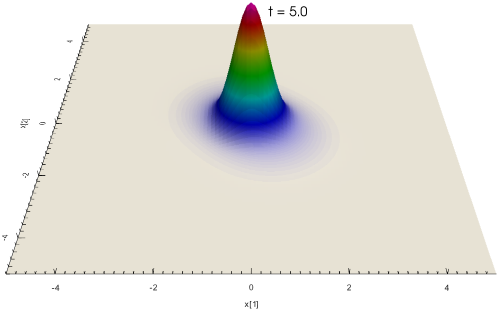
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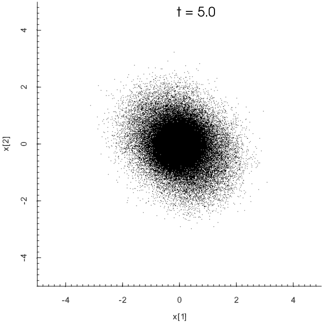
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- For the Fokker-Planck example, this is work in progress

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