Realization of quantum and affine quantum $\mathfrak{gl}_n$

Jie Du

University of New South Wales

AustMS2012, Ballarat

24–27 September 2012

Motivation

1. Two simple questions associated with the structure of algebras.
   - If an algebra is defined by generators and relations, the *realization problem* is to reconstruct the algebra as a vector space with hopefully explicit multiplication formulas on elements of a basis;
   - If an algebra is defined in term of a vector space such as an endomorphism algebra, it is natural to seek their generators and defining relations.
2. Examples:
   - Kac–Moody algebras, quantum enveloping algebras (QEAs) ...
   - Endomorphism algebras such as Iwahori–Hecke algebras, q-Schur algebras and their generalization, degenerate Ringel–Hall algebras, ...
3. Known realizations: Kac–Moody algebras (Kac, Peng–Xiao,...), the $\pm$-part of quantum enveloping algebras (Ringel, ...), QEA of $\mathfrak{gl}_n$ (Beilinson-Lusztig-MacPherson), ...
4. The approaches are all different. I will talk about the BLM approach in this talk.

References

Finite dimensional algebras and quantum groups

Bangming Deng, Jie Du,
Brian Parshall and Jianpan Wang
Mathematical Surveys and Monographs, Volume 150
The American Mathematical Society, 2008
759+ pages

A double Hall algebra approach to affine quantum Schur–Weyl theory

Bangming Deng, Jie Du and Qiang Fu
London Mathematical Society Lecture Note Series, Volume 401
Cambridge University Press, 2012
207+ pages

The (modified) BLM approach

1. By the quantum Schur–Weyl duality, there exist $\mathbb{Q}(v)$-algebra homomorphisms:

$$\xi_r : \mathbf{U}(\mathfrak{gl}_n) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}(r)}(\boldsymbol{\Omega}_n^{\otimes r}) = \boldsymbol{\mathcal{S}}(n,r), \text{the q-Schur algebra}$$

2. Construct spanning set $\{A(\mathbf{j},r)\}_{A\in\Theta^{\pm}(n,r)}$ for $\boldsymbol{\mathcal{S}}(n,r)$ with certain explicit multiplication formulas with structure constants independent of r;
3. Relations similar to the defining relations for $\mathbf{U}(\mathfrak{gl}_n)$ can be derived from these formulas;
4. Consider a quotient $\mathcal{K}(n) \cong \oplus_{r\geqslant 1}\boldsymbol{\mathcal{S}}(n,r)$ of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ and define a completion $\widehat{\mathcal{K}}(n) \cong \prod_{r\geqslant 1}\boldsymbol{\mathcal{S}}(n,r)$;
5. The subspace $\mathfrak{A}(n)$ spanned by all $A(\mathbf{j}) = \sum_{r\geqslant 1} A(\mathbf{j},r)$ is a subalgebra of $\widehat{\mathcal{K}}(n)$ isomorphic to $\mathbf{U}(\mathfrak{gl}_n)$.

The affine analogue

1. By the quantum Schur–Weyl duality, there exist $\mathbb{Q}(\upsilon)$-algebra homomorphisms:

$$\xi_r : \mathbf{U}(\widehat{\mathfrak{gl}_n}) \to \mathrm{End}_{\mathcal{H}_\triangle(r)}(\mathbf{\Omega}_\triangle^{\otimes r}) = \boldsymbol{\mathcal{S}}_\triangle(n, r), \text{ the affine q-Schur algebra}$$

2. Construct spanning set $\{A(\mathbf{j}, r)\}_{A \in \Theta_\triangle^\pm(n,r)}$ for $\boldsymbol{\mathcal{S}}_\triangle(n, r)$ with certain explicit multiplication formulas with structure constants independent of r (not enough!);
3. Relations similar to the defining relations for $\mathbf{U}(\mathfrak{gl}_n)$ for Chevalley generators can be derived from these formulas;
4. Let $\mathcal{K}_\triangle(n) = \oplus_{r \geqslant 1} \boldsymbol{\mathcal{S}}_\triangle(n, r)$ and define a completion $\widehat{\mathcal{K}}_\triangle(n)$;
5. The subspace $\mathfrak{A}(n)$ spanned by all $A(\mathbf{j}) = \sum_{r \geqslant 1} A(\mathbf{j}, r)$ is a conjectural subalgebra of $\widehat{\mathcal{K}}_\triangle(n)$ isomorphic to $\mathbf{U}(\widehat{\mathfrak{gl}_n})$.

Quantum Schur algebras

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in indeterminate v.

- The Hecke algebra $\mathcal{H} = \mathcal{H}(r)$ associated to the symmetric group $\mathfrak{S}_r$ is an associative $\mathcal{Z}$-algebra generated by $T_i, 1 \leqslant i \leqslant r-1$ subject to the relations (where $q = v^2$)

  $$T_i^2 = (q-1)T_i + q, \;\; T_iT_j = T_jT_i, \;\; T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

  Basis $\{T_w\}_{w\in\mathfrak{S}_r}$.
- For each $\lambda \in \Lambda(n,r) := (\mathbb{N}^n)_r$, putting $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$, the algebra

  $$\mathcal{S}(n,r) := \mathrm{End}_{\mathcal{H}}(\oplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H})$$

  is called the quantum Schur algebra.
- This algebra $\mathcal{Z}$-free with a basis $\phi^d_{\lambda,\mu}$ indexed by triples (λ, d, μ), where $\lambda, \mu \in \Lambda(n,r)$ and $d \in \mathfrak{D}_{\lambda,\mu}$. This is seen easily from

  $$\mathcal{S}(n,r) = \bigoplus_{\lambda,\mu} \mathrm{Hom}(x_\lambda\mathcal{H}, x_\mu\mathcal{H}) \cong \bigoplus_{\lambda,\mu} x_\lambda\mathcal{H} \cap \mathcal{H}x_\mu.$$

- There is a bijection

$$\jmath : \{(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r), d \in \mathfrak{D}_{\lambda,\mu}\} \to \Theta(n, r)$$

where $\Theta(n, r)$ denote the set of matrices over $\mathbb{N}$ of size n whose entries sum to r.

- For each $\jmath(\lambda, d, \mu) = A \in \Theta(n, r)$, let $[A] = v^{d_A}\phi^d_{\lambda,\mu}$, where $d_A = \sum_{i\geqslant k, j<l} a_{i,j}a_{k,l}$.
- Let $\Theta^{\pm}(n, r) = \{A \in \Theta(n, r) \mid a_{i,i} = 0 \text{ for all } i\}$.
- For $A \in \Theta^{\pm}(n, r)$ and $\mathbf{j} \in \mathbb{Z}^n$, define

$$A(\mathbf{j}, r) = \sum_{\lambda\in\Lambda(n,r-|A|)} \upsilon^{\lambda\cdot\mathbf{j}}[A + \mathrm{diag}(\lambda)]$$

- The set

$$\{A(\mathbf{j}, r) \mid A \in \Theta^{\pm}(n, r), \mathbf{j} \in \mathbb{N}^n\}$$

forms a spanning set for $\boldsymbol{\mathcal{S}}(n, r)$.

Theorem (BLM '90)

Assume $1 \leqslant h \leqslant n$. *For* $i \in \mathbb{Z}$ $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}^n$ *and* $A \in \Theta^{\pm}(n)$, *if we put* $f(i) = f(i, A) = \sum_{j \geqslant i} a_{h,j} - \sum_{j > i} a_{h+1,j}$, $\alpha_i = e_i - e_{i+1}$, *and* $\beta_i = -e_i - e_{i+1}$, *then the following identities hold in* $\boldsymbol{\mathcal{S}}(n, r)$ *for all* $r \geqslant 0$:

(1) $0(\mathbf{j}, r)A(\mathbf{j}', r) = \upsilon^{\mathbf{j}.\mathrm{ro}(A)} A(\mathbf{j} + \mathbf{j}', r)$, $A(\mathbf{j}', r)0(\mathbf{j}, r) = \upsilon^{\mathbf{j}.\mathrm{co}(A)} A(\mathbf{j} + \mathbf{j}', r)$.

(2)
$$E_{h,h+1}(\mathbf{0}, r)A(\mathbf{j}, r) = \sum_{i<h;\, a_{h+1,i} \geqslant 1} \upsilon^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,i} - E_{h+1,i})(\mathbf{j} + \alpha_h, r)$$
$$+ \sum_{i>h+1;\, a_{h+1,i} \geqslant 1} \upsilon^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,i} - E_{h+1,i})(\mathbf{j}, r)$$
$$+ \upsilon^{f(h) - j_h - 1} \frac{(A - E_{h+1,h})(\mathbf{j} + \alpha_h, r) - (A - E_{h+1,h})(\mathbf{j} + \beta_h, r)}{1 - \upsilon^{-2}}$$
$$+ \upsilon^{f(h+1) + j_{h+1}} \overline{\left[\!\!\left[\begin{matrix} a_{h,h+1}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,h+1})(\mathbf{j}, r).$$

(3) $E^{\triangle}_{h+1,h}(\mathbf{0}, r)A(\mathbf{j}, r) = \cdots$.

- *All formulas are independent of* r.

The affine analogue

Theorem (D–Fu '10)

Assume $1 \leqslant h \leqslant n$. *For* $i \in \mathbb{Z}$ $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}^n_\vartriangle$ *and* $A \in \Theta^\pm_\vartriangle(n)$, *if we put* $f(i) = f(i, A) = \sum_{j \geqslant i} a_{h,j} - \sum_{j>i} a_{h+1,j}$, *then the following identities hold in* $\boldsymbol{\mathcal{S}}_\vartriangle(n, r)$ *for all* $r \geqslant 0$*:*

(1) $0(\mathbf{j}, r)A(\mathbf{j}', r) = \upsilon^{\mathbf{j}\cdot\mathrm{ro}(A)} A(\mathbf{j} + \mathbf{j}', r)$, $A(\mathbf{j}', r)0(\mathbf{j}, r) = \upsilon^{\mathbf{j}\cdot\mathrm{co}(A)} A(\mathbf{j} + \mathbf{j}', r)$.

(2)
$$E^\vartriangle_{h,h+1}(\mathbf{0}, r)A(\mathbf{j}, r) = \sum_{i<h;a_{h+1,i}\geqslant 1} \upsilon^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E^\vartriangle_{h,i} - E^\vartriangle_{h+1,i})(\mathbf{j} + \boldsymbol{\alpha}^\vartriangle_h, r)$$
$$+ \sum_{i>h+1;a_{h+1,i}\geqslant 1} \upsilon^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E^\vartriangle_{h,i} - E^\vartriangle_{h+1,i})(\mathbf{j}, r)$$
$$+ \upsilon^{f(h)-j_h-1} \frac{(A - E^\vartriangle_{h+1,h})(\mathbf{j} + \boldsymbol{\alpha}^\vartriangle_h, r) - (A - E^\vartriangle_{h+1,h})(\mathbf{j} + \boldsymbol{\beta}^\vartriangle_h, r)}{1 - \upsilon^{-2}}$$
$$+ \upsilon^{f(h+1)+j_{h+1}} \overline{\left[\!\!\left[\begin{matrix} a_{h,h+1}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E^\vartriangle_{h,h+1})(\mathbf{j}, r).$$

(3) $E^\vartriangle_{h+1,h}(\mathbf{0}, r)A(\mathbf{j}, r) = \cdots$.

Definition

1. Let $\mathcal{K}_\triangle(n)$ be the algebra which has $\mathcal{Z}$-basis $\{[A]\}_{A\in\Theta_\triangle(n)}$ and multiplication defined by $[A]\cdot[B]=0$ if $\mathrm{co}(A)\neq\mathrm{ro}(B)$, and $[A]\cdot[B]$ as given in $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ if $\mathrm{co}(A)=\mathrm{ro}(B)$ and $r=|A|$.
   - $\mathcal{K}_\triangle(n)\cong\oplus_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$
2. Let $\widehat{\mathcal{K}}_\triangle(n)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A\in\widetilde{\Xi}(\eta)}\beta_A[A]$ which have the following properties: for any $\mathbf{x}\in\mathbb{N}^n_\triangle$,

$$\text{the sets}\quad \begin{matrix}\{A\in\Theta_\triangle(n)\mid \beta_A\neq 0,\ \mathrm{ro}(A)=\mathbf{x}\}\\ \{A\in\Theta_\triangle(n)\mid \beta_A\neq 0,\ \mathrm{co}(A)=\mathbf{x}\}\end{matrix}\quad\text{are finite.}$$

   We obtain an associative algebra with identity element, the sum of all diagonal matrices in $\Theta_\triangle(n)$.
   - $\widehat{\mathcal{K}}_\triangle(n)\cong\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$

BLM type basis

Definition

For $A \in \Theta^{\pm}_{\triangle}(n)$ and $\mathbf{j} \in \mathbb{Z}^n_{\triangle}$, define

$$A(\mathbf{j}) := \sum_{\lambda \in \mathbb{N}^n_{\triangle}} \upsilon^{\lambda \cdot \mathbf{j}}[A + \mathrm{diag}(\lambda)] = \sum_{r \geqslant 0} A(\mathbf{j}, r) \in \widehat{\mathcal{K}}_{\triangle}(n)$$

and let $\mathfrak{A}_{\triangle}(n)$ be the subspace of $\widehat{\mathcal{K}}_{\triangle}(n)$ spanned by

$$\mathscr{B}_{\triangle} = \{A(\mathbf{j}) \mid A \in \Theta^{\pm}_{\triangle}(n), \mathbf{j} \in \mathbb{Z}^n_{\triangle}\}.$$

Lemma

The set $\mathscr{B}_{\triangle} = \{A(\mathbf{j}) \mid A \in \Theta^{\pm}_{\triangle}(n), \mathbf{j} \in \mathbb{Z}^n_{\triangle}\}$ *forms a basis for* $\mathfrak{A}_{\triangle}(n)$.

Call $\mathscr{B}_{\triangle}$ the BLM basis of $\mathfrak{A}_{\triangle}(n)$.

A realization conjecture

Theorem

The algebra homomorphisms $\xi_r : \mathbf{U}(\widehat{\mathfrak{gl}}_n) \to \boldsymbol{\mathcal{S}}_\triangle(n, r)$ *induce an algebra homomorphism* $\xi : \mathbf{U}(\widehat{\mathfrak{gl}}_n) \cong \mathfrak{D}_\triangle(n) \to \widehat{\boldsymbol{\mathcal{K}}}_\triangle(n)$*. (It is constructed by the double Ringel–Hall algebras* $\mathfrak{D}_\triangle(n)$*.) Restriction gives rise to the following algebra homomorphisms:*

- $\xi : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \mathfrak{A}_\triangle(n)$ *(D–Fu);*
- $\xi : \mathfrak{H}_\triangle(n)^{\geqslant 0} \to \mathfrak{A}_\triangle(n)$, $\xi : \mathfrak{H}_\triangle(n)^{\leqslant 0} \to \mathfrak{A}_\triangle(n)$ *([VV]);*

Conjecture

$\mathrm{Im}(\xi) = \mathfrak{A}_\triangle(n)$. Equivalently, the $\mathbb{Q}(v)$-space $\mathfrak{A}_\triangle(n)$ is a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}_\triangle(n)$.

- The conjecture is true in the classical $(v = 1)$ case. We now look at this case.

Loop algebra of $\mathfrak{gl}_n$

- Consider the loop algebra $\widehat{\mathfrak{gl}_n}(\mathbb{Q}) := \mathfrak{gl}_n(\mathbb{Q}) \otimes \mathbb{Q}[t, t^{-1}]$.
- This algebra is identified with the matrix Lie algebra $M_{\triangle,n}(\mathbb{Q})$ of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{Q}$ such that
  - (a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$, and
  - (b) for every $i \in \mathbb{Z}$, the set $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ is finite.
- Thus, we obtain a natural action of $\widehat{\mathfrak{gl}_n}$ on $\Omega_{\mathbb{Q}}$. Hence, an action of $\mathcal{U}(\widehat{\mathfrak{gl}_n})$ on $\Omega_{\mathbb{Q}}^{\otimes r}$.
- This gives algebra homomorphisms

$$\eta_r : \mathcal{U}(\widehat{\mathfrak{gl}_n}) \to \mathcal{S}_\triangle(n, r)_{\mathbb{Q}} = \mathrm{End}_{\mathbb{Q}\mathfrak{S}_r}(\Omega_{\mathbb{Q}}^{\otimes r})$$

  for every $r \geqslant 1$.
- Hence, an algebra homomorphism

$$\eta = \prod_{r\geqslant 0} \eta_r : \mathcal{U}(\widehat{\mathfrak{gl}_n}) \longrightarrow \widehat{\mathcal{K}}_\triangle(n)_{\mathbb{Q}} \cong \prod_{r\geqslant 1} \mathcal{S}_\triangle(n, r)_{\mathbb{Q}}.$$

Realization of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$

Theorem (Deng-D-Fu '11)

The universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ of the loop algebra $\widehat{\mathfrak{gl}}_n(\mathbb{Q})$ has a basis $\{A[\mathbf{j}] \mid A \in \Theta^{\pm}_{\triangle}(n), \mathbf{j} \in \mathbb{N}^n_{\triangle}\}$ which satisfies the following multiplication formulas:

$$(1) \qquad 0[e^{\triangle}_t]A[\mathbf{j}] = A[\mathbf{j} + e^{\triangle}_t] + \Big(\sum_{s\in\mathbb{Z}} a_{t,s}\Big)A[\mathbf{j}];$$

$$\begin{aligned}(2) \qquad E^{\triangle}_{h,h+\varepsilon}[\mathbf{0}]A[\mathbf{j}] = & \sum_{\substack{a_{h+\varepsilon,i}\geqslant 1\\ \forall i\neq h,h+\varepsilon}} (a_{h,i}+1)(A + E^{\triangle}_{h,i} - E^{\triangle}_{h+\varepsilon,i})[\mathbf{j}] \\ & + \sum_{0\leqslant i\leqslant j_h} (-1)^i \binom{j_h}{i} (A - E^{\triangle}_{h+\varepsilon,h})[\mathbf{j} + (1-i)e^{\triangle}_h] \\ & + (a_{h,h+\varepsilon}+1) \sum_{0\leqslant i\leqslant j_{h+\varepsilon}} \binom{j_{h+\varepsilon}}{i} (A + E^{\triangle}_{h,h+\varepsilon})[\mathbf{j} - ie^{\triangle}_{h+\varepsilon}],\end{aligned}$$

Realization of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$

Theorem (Deng-D-Fu)

$$(3) \qquad E^{\triangle}_{h,h+mn}[\mathbf{0}]A[\mathbf{j}] = \sum_{\substack{s\notin\{h,h-mn\}\\ a_{h,s}\geqslant 1}} (a_{h,s+mn}+1)(A+E^{\triangle}_{h,s+mn}-E^{\triangle}_{h,s})[\mathbf{0}]$$
$$+\sum_{0\leqslant t\leqslant j_h}(a_{h,h+mn}+1)\binom{j_h}{t}(A+E^{\triangle}_{h,h+mn})[\mathbf{j}-te^{\triangle}_h]$$
$$+\sum_{0\leqslant t\leqslant j_h}(-1)^t\binom{j_h}{t}(A-E^{\triangle}_{h,h-mn})[\mathbf{j}+(1-t)e^{\triangle}_h].$$

for all $1\leqslant h,t\leqslant n$, $\mathbf{j}=(j_k)\in\mathbb{N}^n_{\triangle}$, $A=(a_{i,j})\in\Theta^{\pm}_{\triangle}(n)$, $\varepsilon\in\{1,-1\}$, *and* $m\in\mathbb{Z}\backslash\{0\}$.

THANK YOU!