Automorphisms of Compact Totally Disconnected Groups

George Willis
The University of Newcastle

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

September 27th 2012

CARMA

The basic example

Example

Let

$$G = F^{\mathbb{Z}} = \{(f_n) \mid f_n \in F\},$$

where F is a finite group and define the *shift*

$$\alpha : G \to G \text{ by } \alpha(f)_n = f_{n+1}.$$

Then G is a compact totally disconnected group and α is an automorphism of G.

All (G, α), where G is a compact totally disconnected group and α is an automorphism that acts ergodically, are built up from these shift examples.

'α acts ergodically'

Ergodicity is measure theoretic version of transitivity.

Definition

The measure preserving bijection $\tau : X \to X$, where $(X, \mathcal{M}, \mu)$ is a measure space, is *ergodic* if any measurable subset $E \subseteq X$ with $\tau(E) = E$ has either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

In the above example, $\alpha : F^{\mathbb{Z}} \to F^{\mathbb{Z}}$ (μ the Haar measure) is ergodic because it is *topologically transitive*, that is, is continuous and has a dense orbit.
To see this, note that $\bigcup_{n\in\mathbb{N}} F^n$ is countable and form $f \in F^{\mathbb{Z}}$ such that every element of $\bigcup_{n\in\mathbb{N}} F^n$ appears as a subsequence of f. Then the orbit $\{\alpha^n(f) \mid n \in \mathbb{Z}\}$ is dense in $F^{\mathbb{Z}}$.

G locally compact, the nub of α

Definition

Let G be a t.d.l.c. group and α be an automorphism of G. The *nub* of α is

$$\text{nub}(\alpha) = \bigcap \{U \mid U \text{ is tidy for } \alpha\}\,.$$

Theorem (Jaworski [3], W. [8])

The nub of α is a compact α-stable subgroup of G on which α acts ergodically, and is the largest such subgroup.

nub(α) is thus characterized independently of tidiness.
Every compact, open subgroup of G tidy for α contains nub(α).
In fact, U is tidy below for α if and only if $U \geq \text{nub}(\alpha)$.

Groups with ergodic automorphisms

In his book *Lectures on Ergodic Theory*, Paul Halmos [2] conjectured that any locally compact group for which there is an ergodic automorphism must be compact. This conjecture was:

- proved for connected groups in the 1960's using approximation by Lie groups (Hilbert's 5th problem).
- proved in the 1980's for totally disconnected groups by a topological dynamics argument [1]. A short proof using the scale and tidy subgroups is given in [6].
- already implicit in the claim that the nub of α is compact and is the maximal subgroup on which α is ergodic.

Automorphisms of p-adic linear groups

Let $G = SL(2, \mathbb{Q}_p)$. Let $x = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ and α be the inner automorphism $\alpha : y \mapsto xyx^{-1}$.

Then $\text{nub}(\alpha) = \{1\}$.

Automorphisms of automorphism groups of trees

Let $\mathcal{T}$ be a regular tree and $G = \mathrm{Aut}(\mathcal{T})$. Let $x \in G$ be *translation* along the *axis* ℓ and let α be the inner automorphism $\alpha : y \mapsto xyx^{-1}$.
Then subgroups tidy for α have the form $\mathrm{stab}_G(v_m) \cap \mathrm{stab}_G(v_n)$ for v_m and v_n distinct vertices on ℓ. Hence $\mathrm{nub}(\alpha) = \mathrm{fix}(\ell)$.

Note that

$$\mathrm{fix}(\ell) \cong \mathrm{Aut}(\mathcal{R}_n)^{\mathbb{Z}},$$

where $\mathcal{R}_n$ is the rooted subtree with root v_n on ℓ.
Since $\mathrm{Aut}(\mathcal{R}_n) \cong \varprojlim \mathrm{Aut}(\mathcal{R}_n^{[k]})$, where $R_n^{[k]}$ is the truncation of $R_n^{[k]}$ at level k, is a profinite group, we have

$$\mathrm{fix}(\ell) \cong \varprojlim \mathrm{Aut}(\mathcal{R}_n^{[k]})^{\mathbb{Z}}$$

Finite depth/expansive automorphisms

Definition

Let G be a compact totally disconnected group and α be an automorphism of G. Then (G, α) is *finite depth* if there is an open neighbourhood $\mathcal{U} \supset 1$ such that

$$\bigcap_{n \in \mathbb{Z}} \alpha^n(\mathcal{U}) = \{1\}.$$

Such an automorphism α is called *expansive* in [4,5,7].

Theorem

Let G be a compact totally disconnected group and α be an automorphism of G. Then there is a directed system $\{(G_\iota, \alpha_\iota)\}_{\iota \in \mathcal{I}}$ of finite depth pairs such that

$$(G, \alpha) \cong \varprojlim (G_\iota, \alpha_\iota).$$

Structure of finite depth pairs

Theorem ('Jordan-Holder' for finite depth pairs)

Let G be a compact group and α be an automorphism of G such that (G, α) has finite depth. Then $\mathrm{nub}(\alpha)$ is an open (and hence finite index) subgroup of G that has a composition series

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_{r-1} \lhd G_r = \mathrm{nub}(\alpha)$$

of closed, α-stable subgroups such that, for $i \in \{1, 2, \ldots, r\}$,

$$G_i/G_{i-1} \cong F_i^{\mathbb{Z}} \text{ and } \tilde{\alpha}_i \text{ is the shift,}$$

where F_i is a finite simple group and $\tilde{\alpha}_i$ is the automorphism induced by α.
The composition factors $F_i^{\mathbb{Z}}$ are unique up to permutation.

References

1. N. Aoki, 'Dense orbits of automorphisms and compactness of groups', *Topology Appl.* **20** (1985), 1–15.
2. P. Halmos, *Lectures on Ergodic Theory*, Publ. Math. Soc. Japan, Tokyo (1956).
3. W. Jaworski, 'Contraction groups, ergodicity, and distal properties of automorphisms of compact groups', *preprint*.
4. B. Kitchens, 'Expansive dynamics of zero-dimensional groups', *Ergod. Th. & Dynam. Sys.*, **7** (1987), 249–261.
5. B. Kitchens and K. Schmidt, 'Automorphisms of compact groups', *Ergod. Th. & Dynam. Sys.* **9** (1989), 691–735.
6. W. Previts & T.-S. Wu, 'Dense orbits and compactness of groups', *Bull. Austral. Math. Soc.*, **68(1)** (2003), 155–159.
7. K. Schmidt, *Dynamical Systems of Algebraic Origin*, Progress in Mathematics **128**, Birkhäuser, 1995.
8. G. A. Willis, 'The nub of an automorphism of a totally disconnected locally compact group', *submitted*, `arXiv:1112.4239v1`.

CARMA