Simplicity of groupoid C^*-algebras

56th Meeting of the AustMS,
University of Ballarat

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25 September 2012

Outline

1. Groupoids
2. Groupoid C^*-algebras
3. Simplicity

Groupoids

A *groupoid* is a small category G with inverses: for each $\gamma \in G$ there exists $\gamma^{-1} \in G$ such that $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$.

It's a group with an identity crisis. The set of identity elements ("unit space") is denoted $G^{(0)}$.

In a *topological group*, G is given a locally compact Hausdorff topology.

- Composition is continuous from $G * G \subseteq G \times G$ to G.
- Inversion is continuous from G to G.

G is *étale* if $r, s : G \to G^{(0)}$ are local homeomorphisms. This forces $G^{(0)}$ open in G.

Examples

1. Groups: these are groupoids with one object. Étale means discrete.
2. An $R \subseteq X \times X$ is a groupoid: define $r(x,y) = x$, $s(x,y) = y$, $(x,y)^{-1} = (y,x)$ and $(x,y)(y,z) = (x,z)$.
3. If a group G acts on a space X, then $G \times X$ is a groupoid with $r(g,x) = g \cdot x$, $s(g,x) = x$, $(g,x)^{-1} = (g^{-1}, g \cdot x)$ and $(g, h \cdot x)(h,x) = (gh, x)$; it's étale if G is discrete.

By analogy with the last example, we think of groupoids as "acting" on their unit spaces.

Say G is *topologically principal* if $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. Like a topologically free action.

Bisections

A *bisection* of G is a subset $U \subseteq G$ such that r and s restrict to homeomorphisms on U.

Every étale groupoid has a basis consisting of precompact open bisections.

An étale groupoid is *effective* if $\mathrm{Int}\{g \in G : r(g) = s(g)\} = G^{(0)}$. Like an effective group action.

Theorem (Renault)

Let G be an étale locally compact Hausdorff groupoid. If G is topologically principal then it is effective. If G is second countable then the converse holds.

C^*-algebras

A C^*-algebra is a complete (complex) normed *-algebra satisfying the C^*-identity $\|a^*a\| = \|a\|^2$.

Gelfand-Naimark: every C^*-algebra is isomorphic to a closed C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Key example: if G is a locally compact Hausdorff group, then $C_c(G)$ has a universal C^*-completion $C^*(G)$. If G is amenable, then this is the only completion.

$C^*(G)$ is universal for continuous unitary representations of G.

A C^*-algebra A is simple if every nonzero homomorphism of A is injective. $C^*(G)$ is only simple if $G = \{e\}$ (consider the 1-dimensional representation of G).

Groupoid C^*-algebras

To construct the groupoid C^*-algebra, consider $C_c(G)$. Operations:

$$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta) \qquad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

There is a universal C^*-completion $C^*(G)$ which is essentially unique if G is suitably amenable; Renault's Disintegration Theorem says that representations of $C^*(G)$ correspond precisely to representations (in the appropriate sense) of G.

Question: when is $C^*(G)$ simple?

Simplicity

A groupoid G is *minimal* if $\overline{r(Gu)} = G^{(0)}$ for every $u \in G^{(0)}$. Think of a minimal action: every orbit is dense.

Renault proved (early '80's): if G is amenable, topologically principal and minimal, then $C^*(G)$ is simple; further, minimality is necessary.

The full converse was unknown. Proved in various special cases by: Deaconu-Renault, Kumjian-Pask-Raeburn, Archbold-Spielberg, Exel-Vershik, Robertson-S.

Theorem (Brown-Clark-Farthing-S)

Suppose that G is étale, second-countable and amenable. Then $C^*(G)$ is simple if and only if G is topologically principal and minimal.

Other results

Also obtain nice C^*-algebraic characterisations of when G is (individually) minimal and topologically principal.

There is also a class of abstract algebras, called *Steinberg algebras* associated to étale groupoids with totally disconnected unit space. We obtain a characterisation of simplicity for Steinberg algebras.

Theorem

Suppose that G is étale with totally disconnected unit space. Then $\mathcal{A}(G)$ is simple if and only if G is both effective and minimal.

In this case, "effective" is a strictly weaker hypothesis than "topologically principal."