Purely infinite C^*-algebras arising from group actions on the Cantor set

joint work with Mikael Rørdam

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Intro

Theorem (Banach-Tarski Paradox, 24')

The unit ball $B \subseteq \mathbb{R}^3$ may be decomposed into 17 disjoint pieces $A_1 \cup \cdots \cup A_{17}$ for which there exist $g_1, \ldots, g_{17} \in$Isom$(\mathbb{R}^3)$ such that the sets $g_i A_i$ are disjoint and their union is two copies of B.

Paradoxical groups

Theorem (Tarski, 49')

Let G be a discrete group. Then G is either amenable or G-paradoxical.

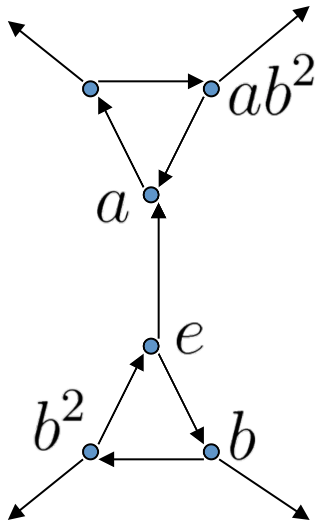
- G amenable $=$ there exist a finitely additive G-invariant probability measure (also called a *mean*) $\mu \colon P(G) \to [0,1]$.
- G-paradoxical $=$ there exist a finite number of disjoint subsets of G that can be translated to cover G two times.

Example

Example

- The group $G = \langle a, b : a^2 = e, b^3 = e \rangle$
- Elements: $e, a, b, b^2, ab, ab^2, \ldots$
- Multiplication: $(aba)(ab^2a) = e$

Example



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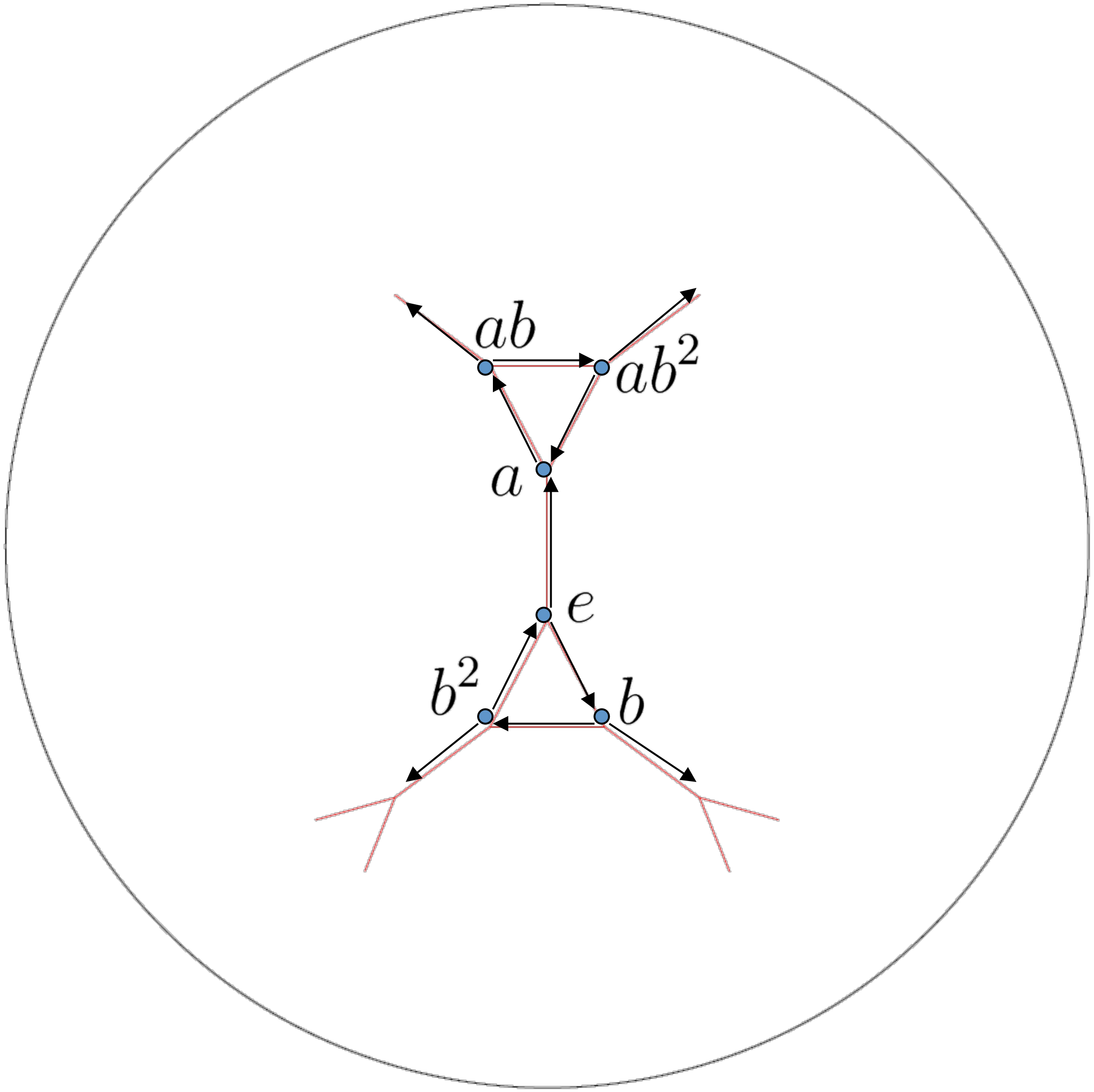

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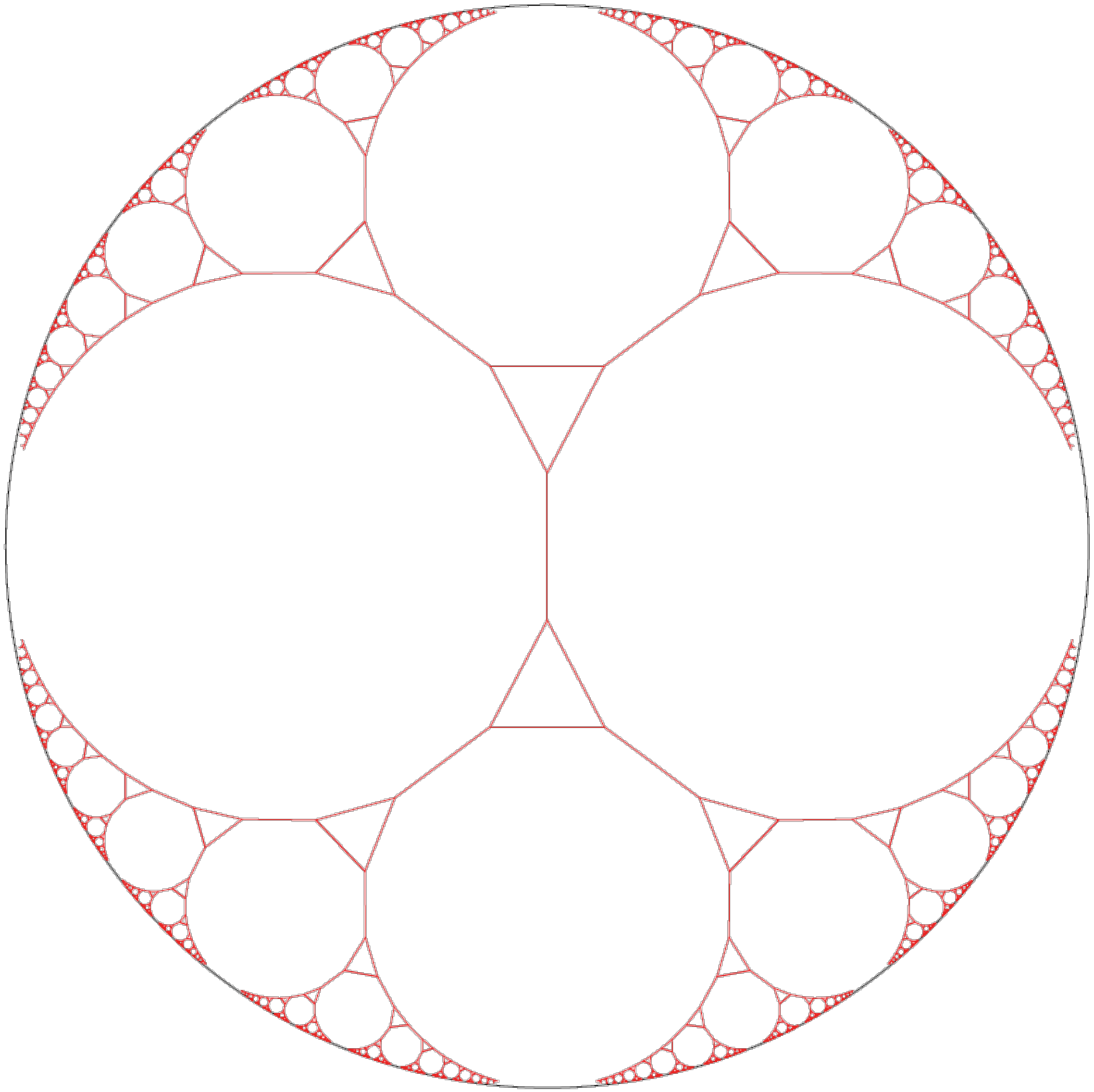

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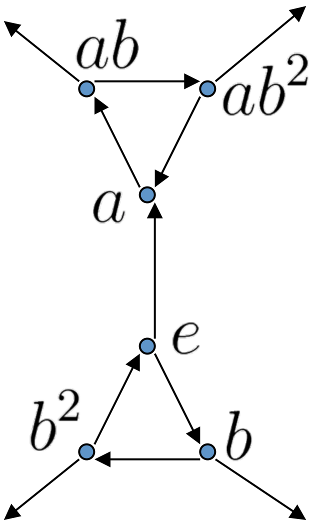

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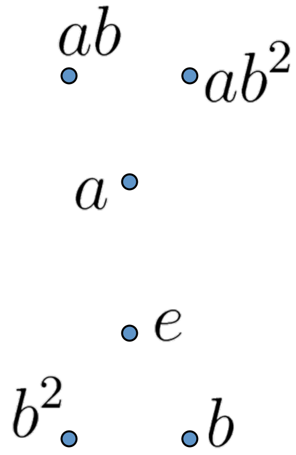

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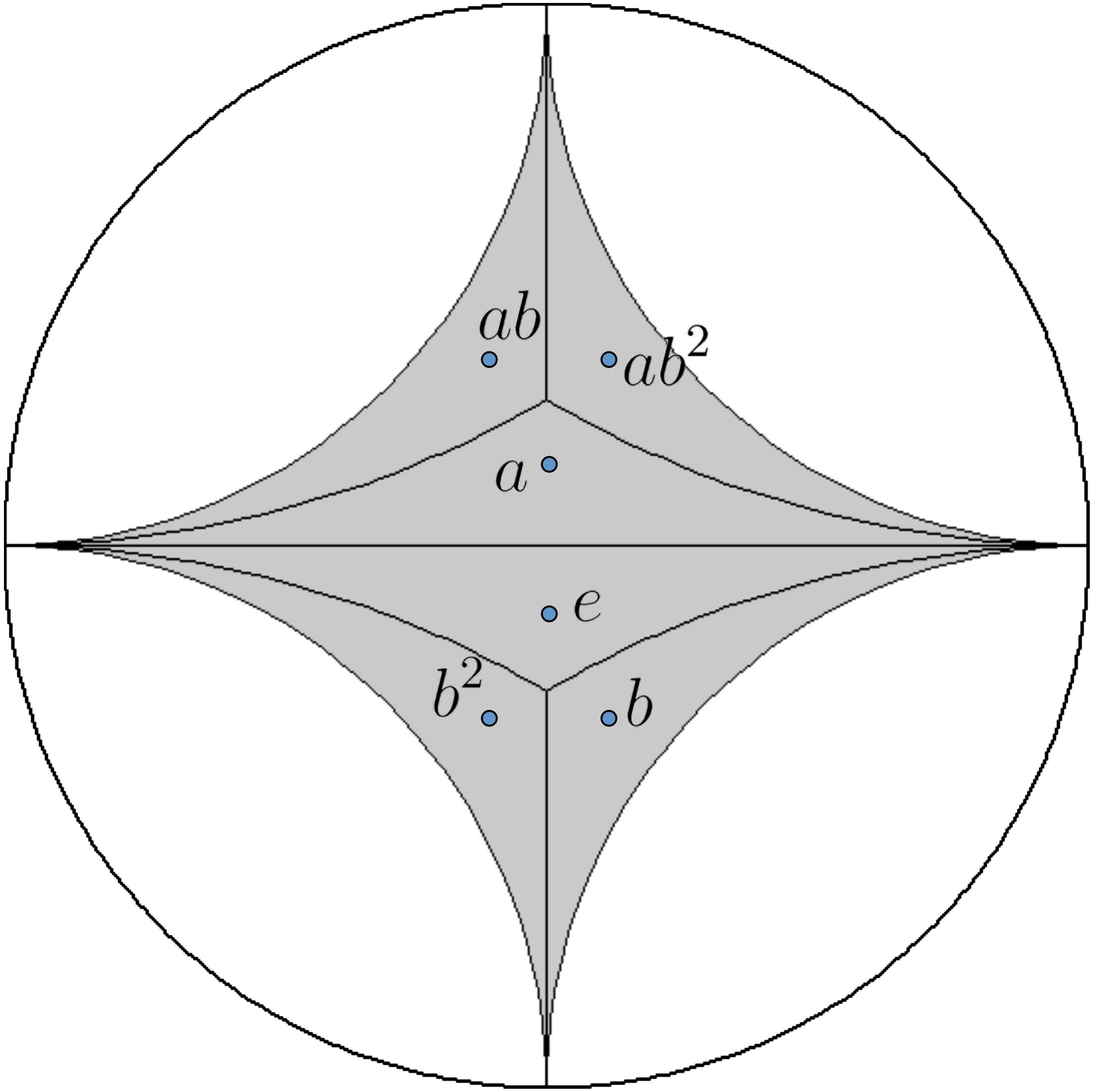

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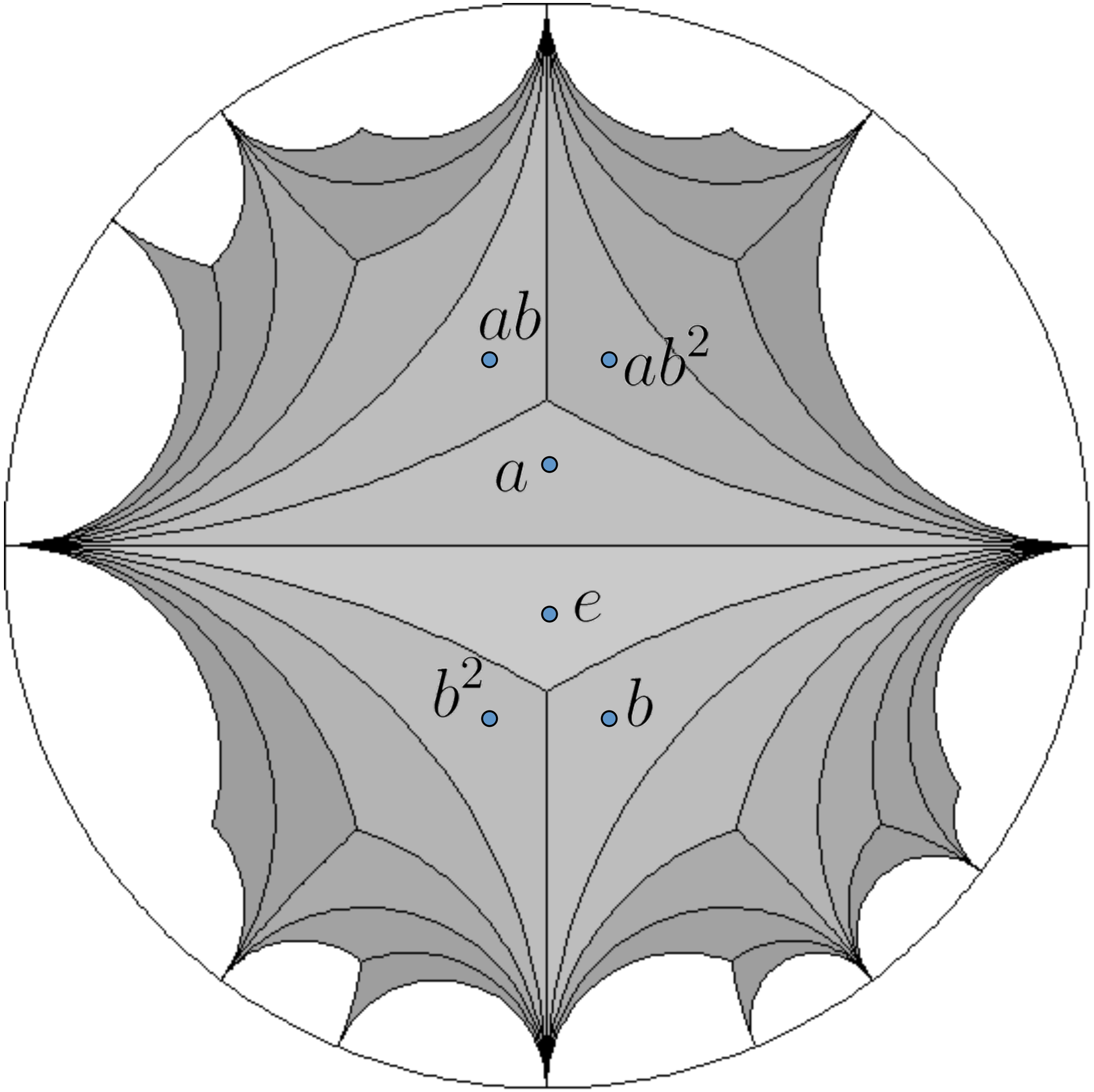
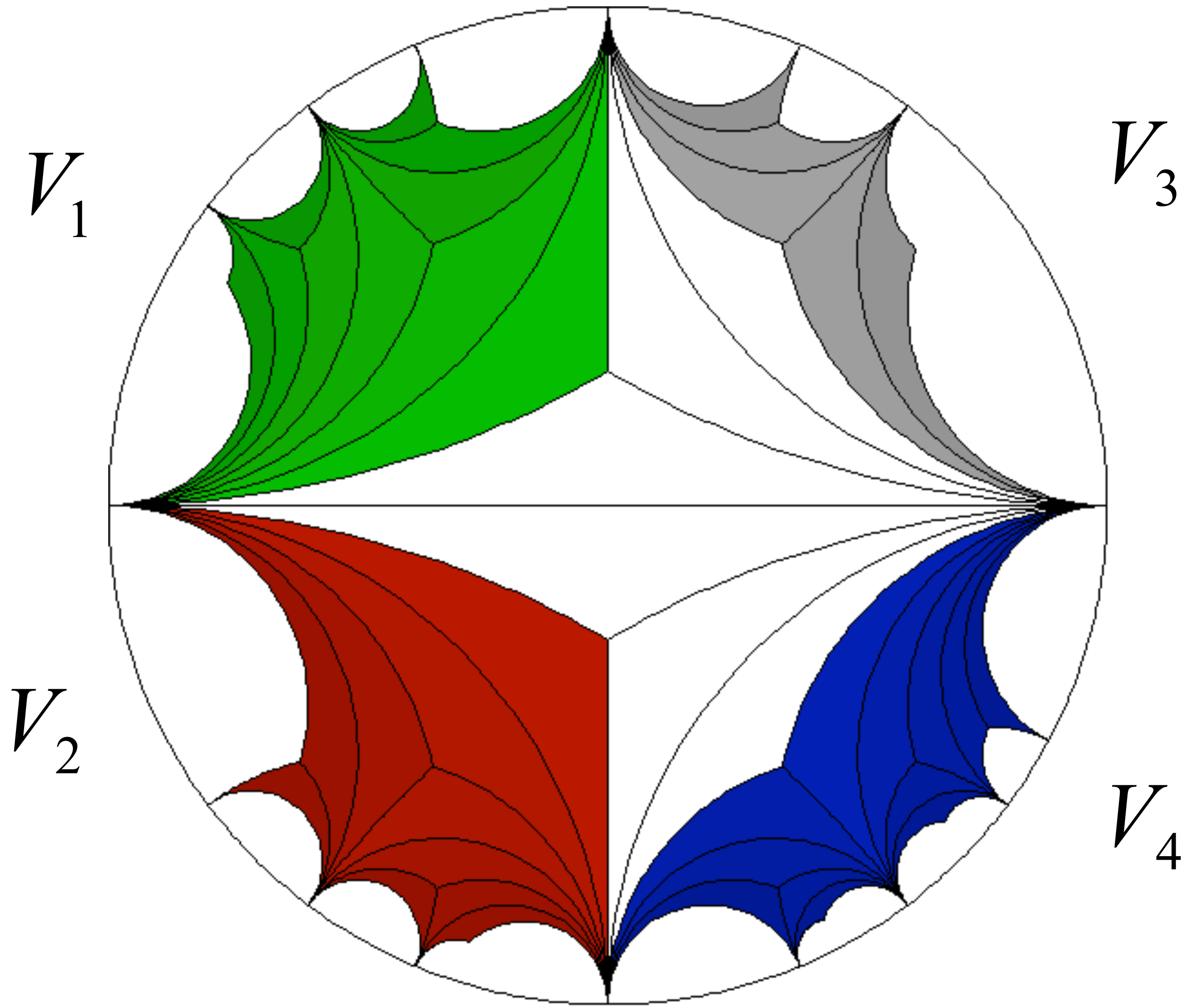
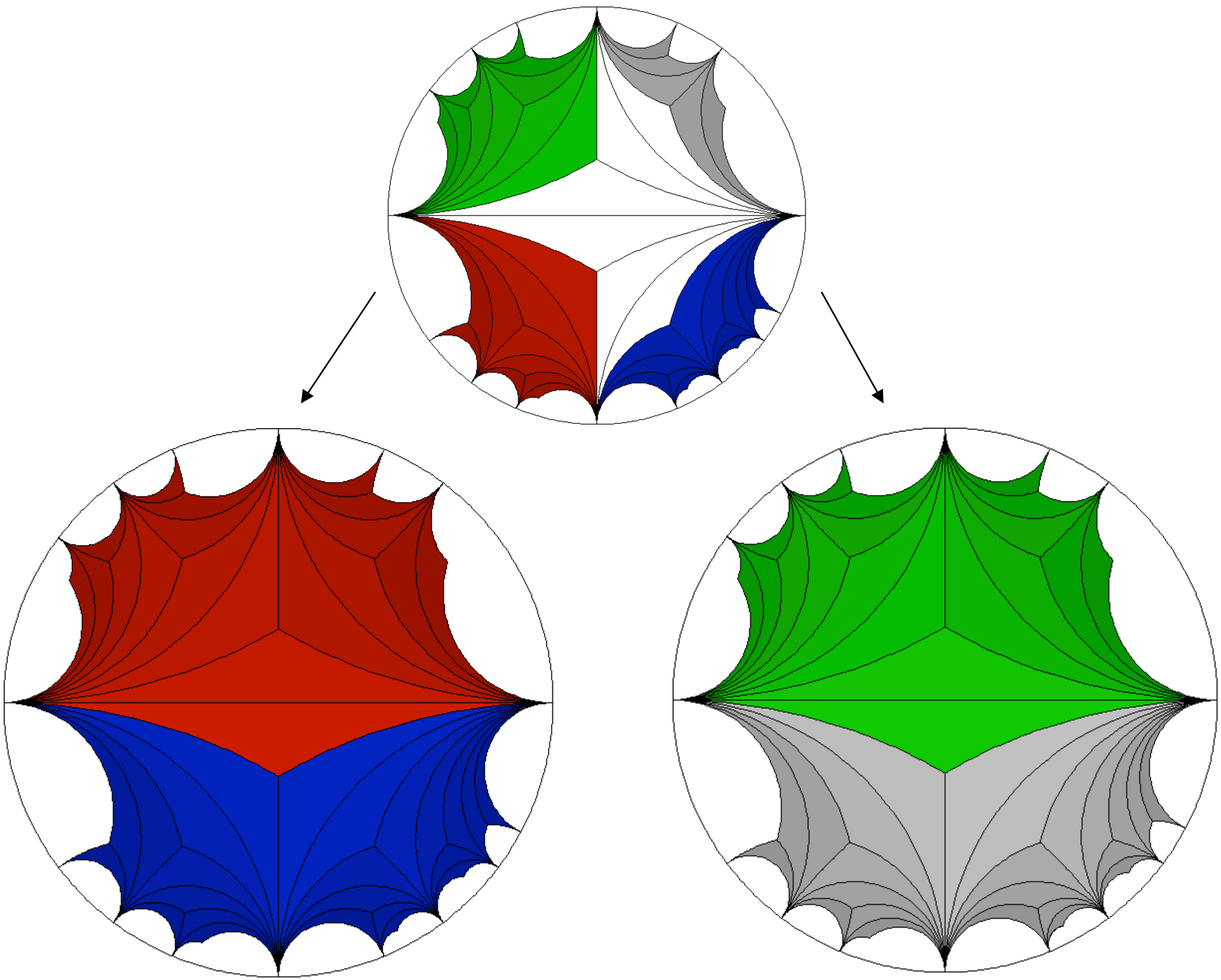




V_1
V_3
V_2
V_4

Example



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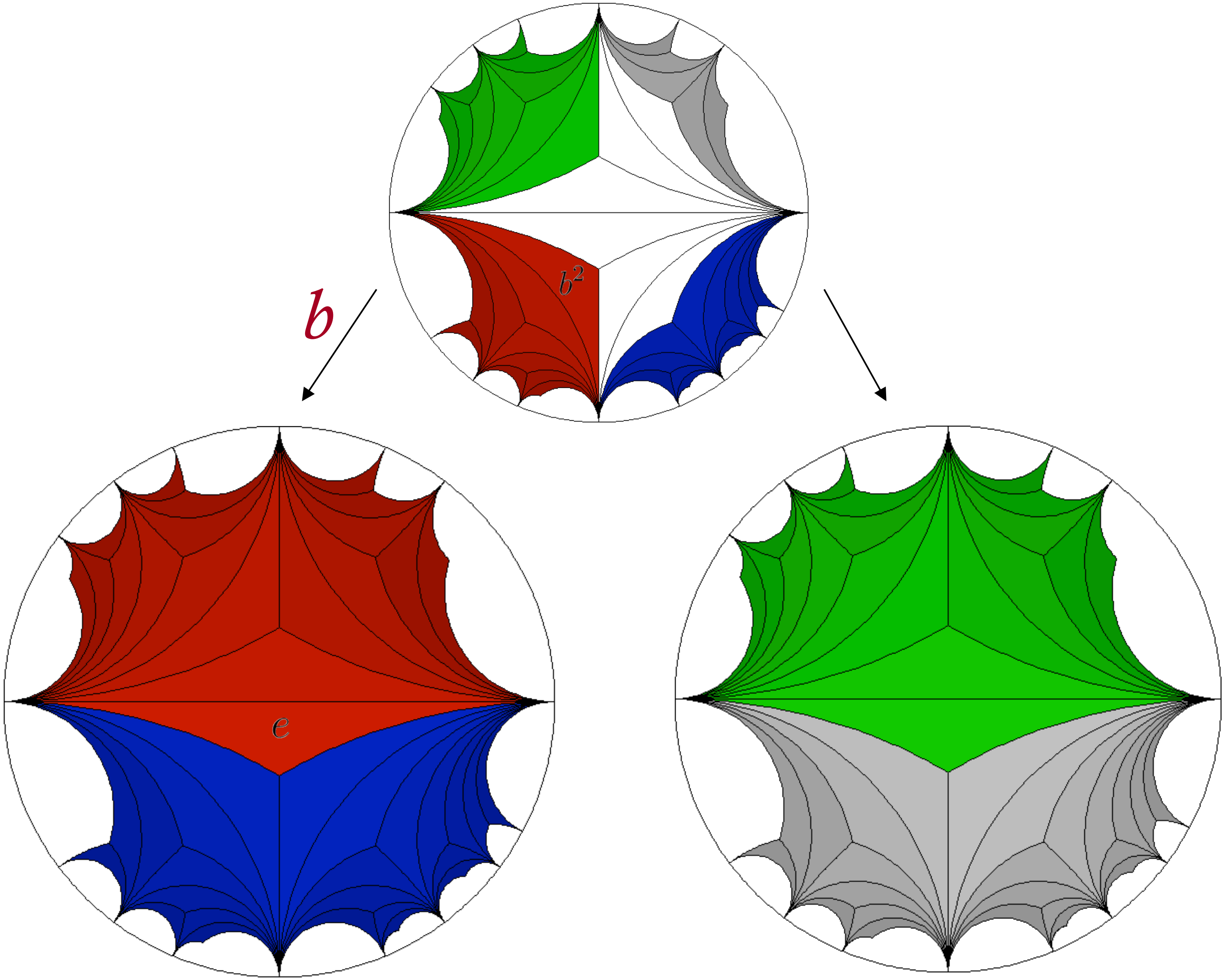


Paradoxical sets

Theorem

Let G be countable group acting on a compact space X. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$:

(i) X *is* (G, τ_X)*-paradoxical.*

(ii) $C(X) \rtimes_r G$ *is properly infinite* $(1 = uu^* = vv^*, u^*u \perp v^*v)$.

(iii) X *admits no* G*-invariant Borel probability measure.*

- τ_X = open subsets of X
- $E \subseteq X$ is $(G, \mathbb{E})$-*paradoxical* = there exist disjoint subsets $V_1, V_2, \ldots, V_{n+m} \in \mathbb{E}$ of E and $t_1, t_2, \ldots, t_{n+m} \in G$ st

$$\bigcup_{j=1}^{n} t_i.V_j = \bigcup_{j=n+1}^{n+m} t_j.V_j = E$$

- (iii) $\Rightarrow$ (i)???

Partial results

Theorem (Becker, Kechris, 96')

Let G be a countable group acting on a standard Borel space X. Then either there exist a G-invariant Borel probability measure on X or X is countably $(G, \mathbb{B}(X))$-paradoxical.

Theorem

Let G be countable group acting on $X = \beta G$. TFAE

(i) *X is (G, τ_X)-paradoxical.*

(ii) *$C(X) \rtimes_r G$ is properly infinite ($1 = uu^* = vv^*, u^*u \perp v^*v$).*

(iii) *X admits no G-invariant Borel probability measure.*

Classification of C^*-algebras

Theorem (Kirchberg, Phillips, 94')

Unital Kirchberg algebras in UCT are classifiable by $(K_0, K_1, [1])$.

Theorem

Let G *be a countable group acting on the Cantor set* X. *TFAE*

- $C(X) \rtimes_r G$ *is a Kirchberg algebra in UCT*
- *action is*
  - *topologically free*
  - *amenable*
  - *minimal*

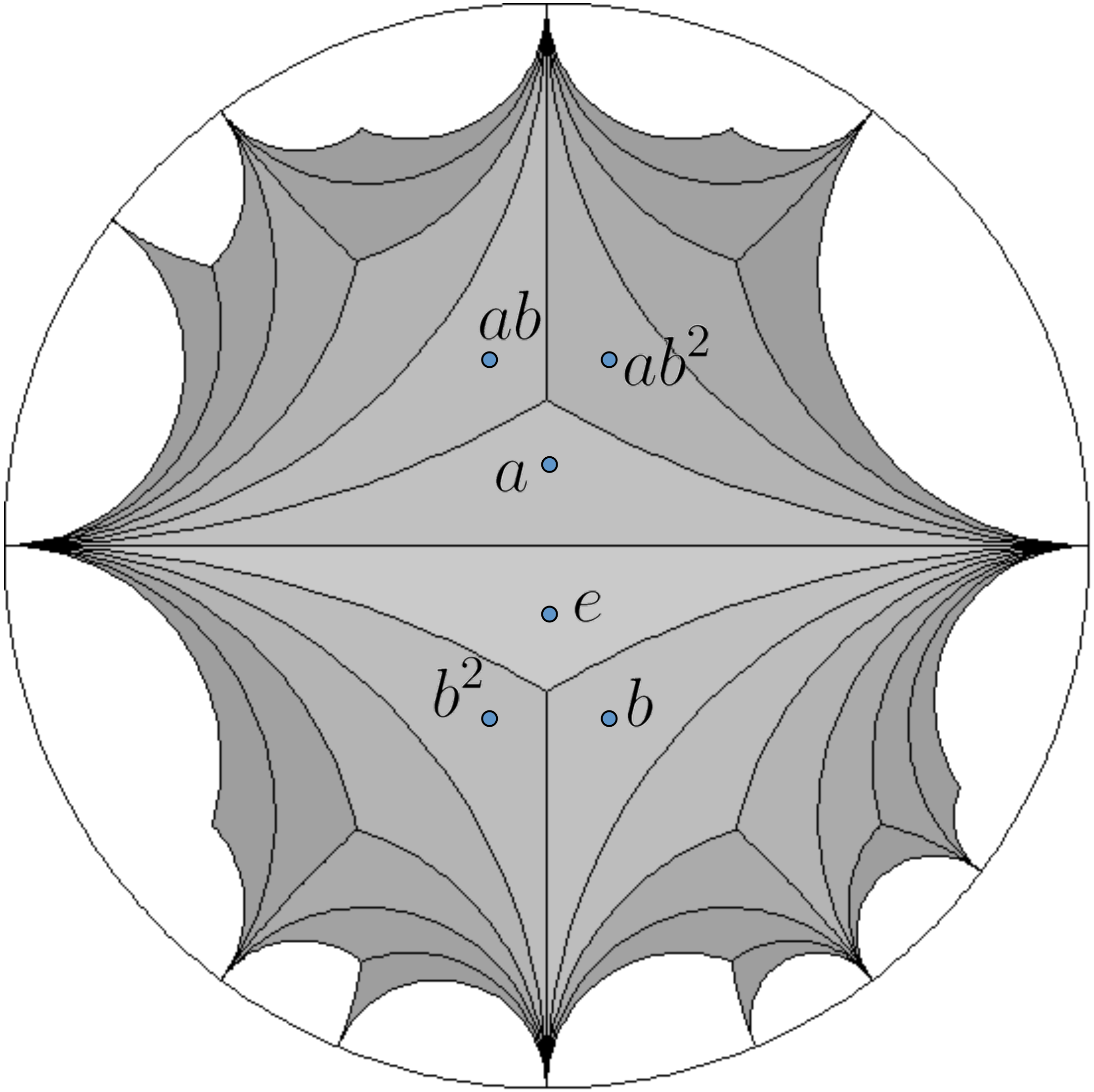
  and nonzero projections in $C(X)$ *are properly infinite in* $C(X) \rtimes_r G$

Example

Example

- The group $G = \langle a, b : a^2 = e, b^3 = e \rangle$
- The space $X =$ infinite word space for G
- Action of G on X: $(aba)(ab^2ab^2ab^2\ldots) = b^2ab^2\ldots$

Example



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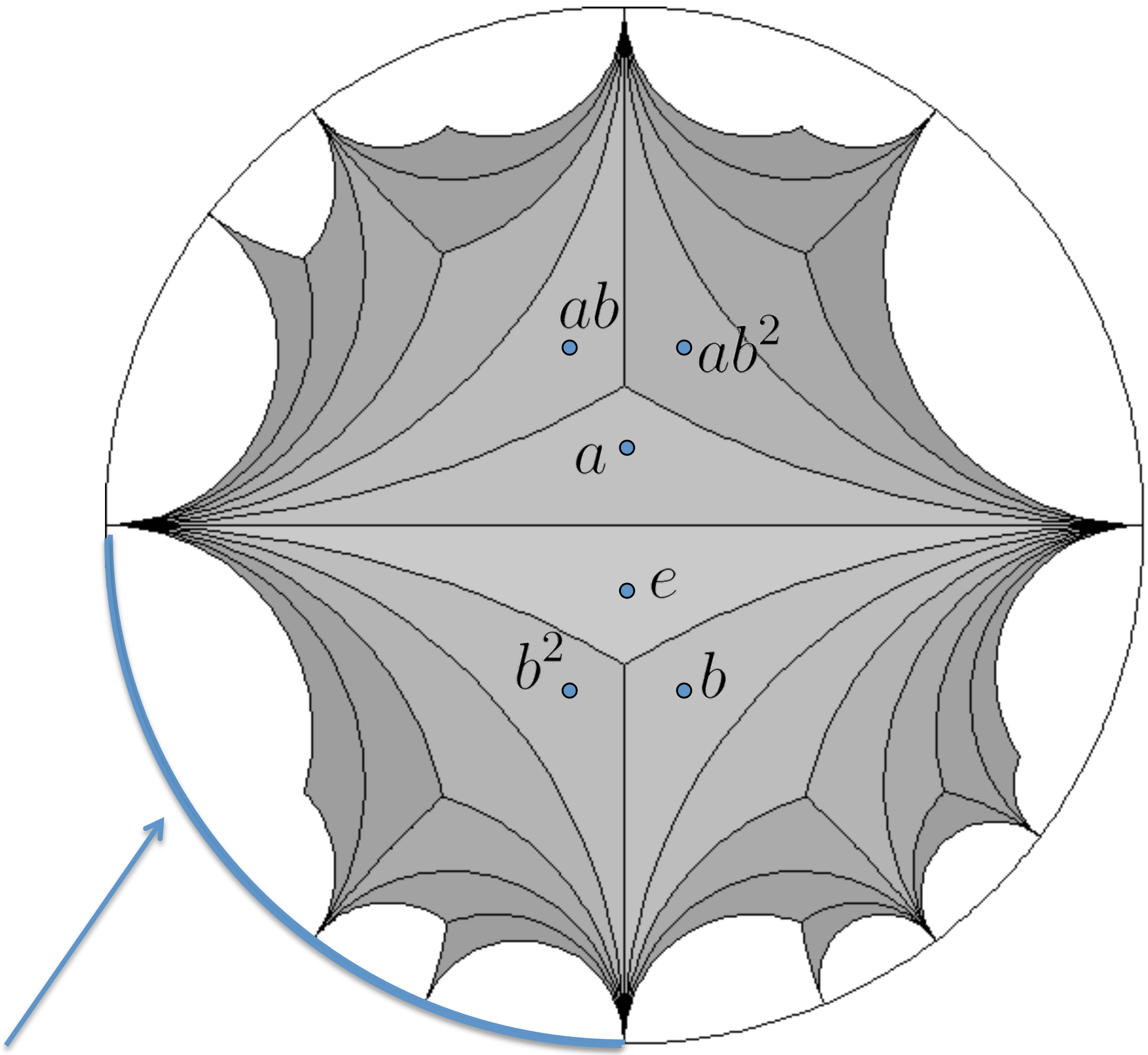
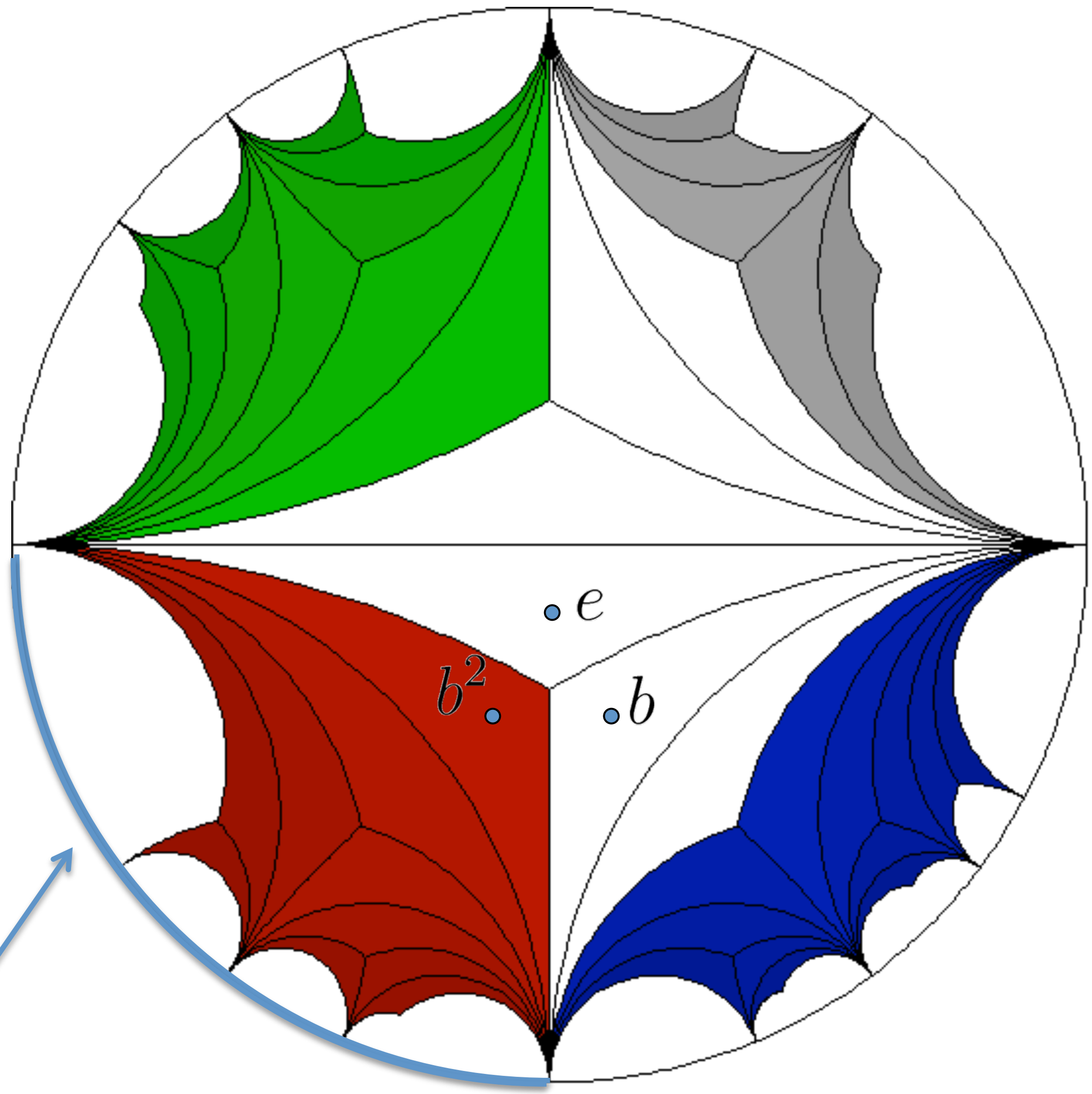


Words beginning with b^2

Example



Words beginning with b^2

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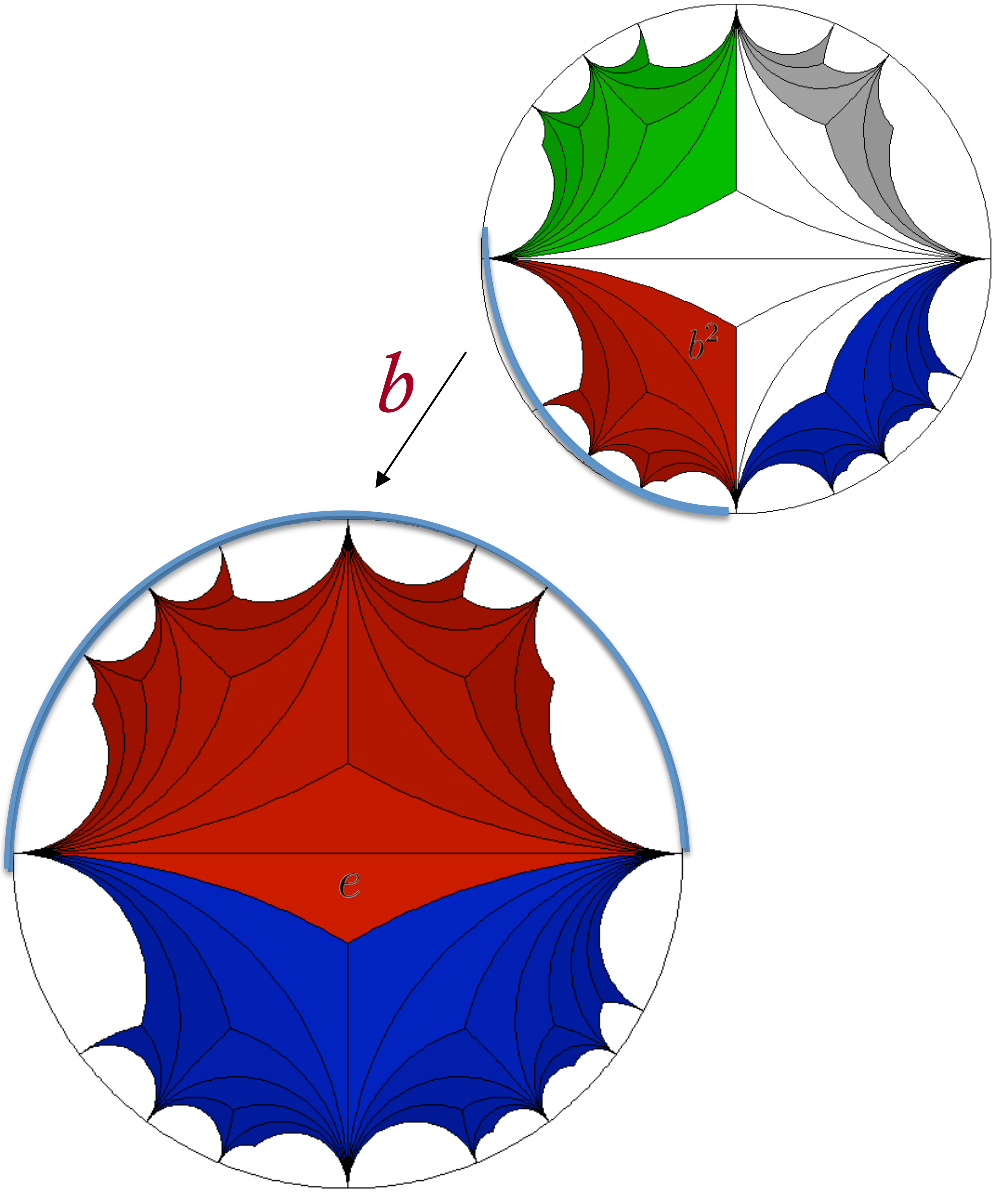

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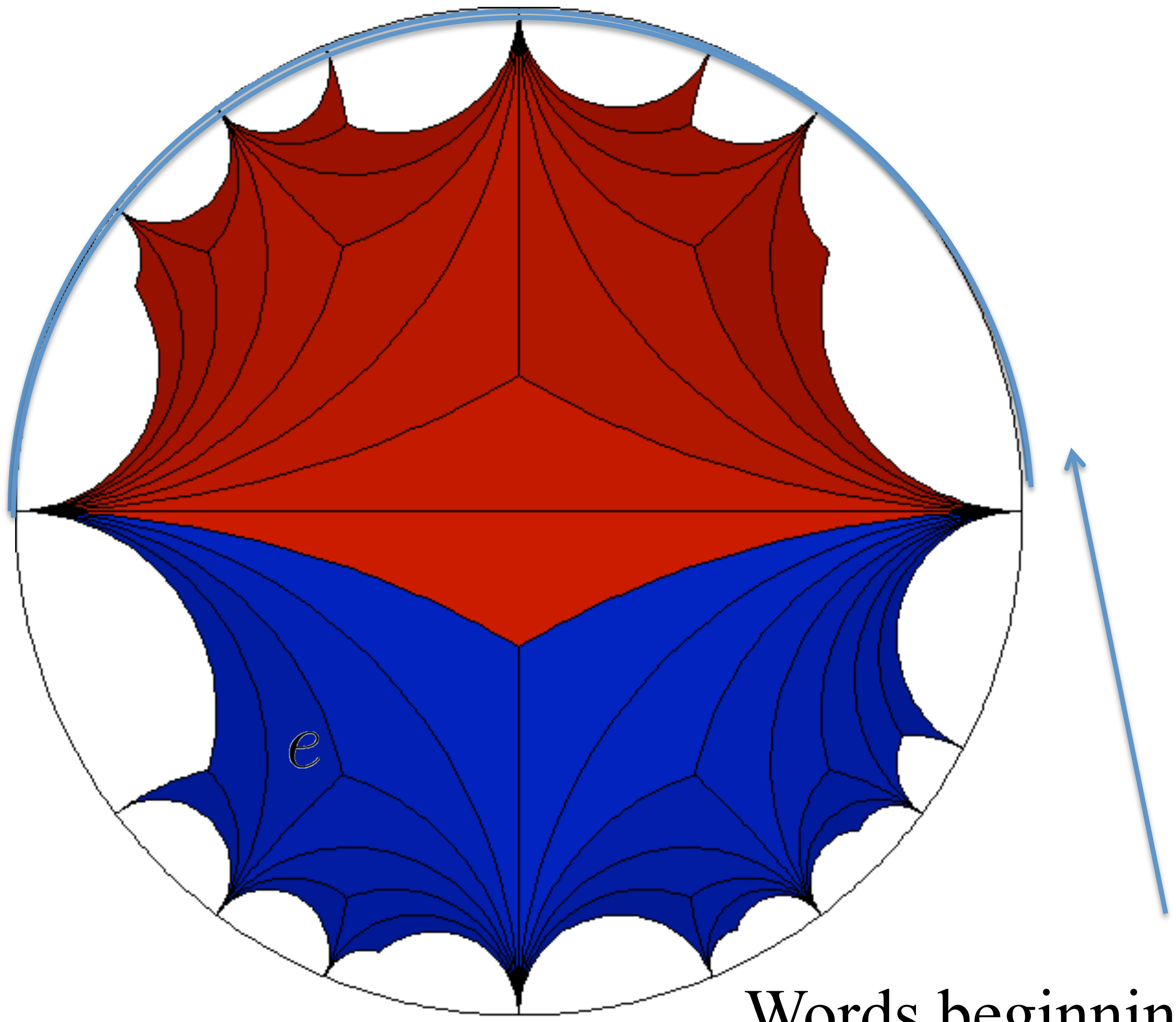

Words beginning with a

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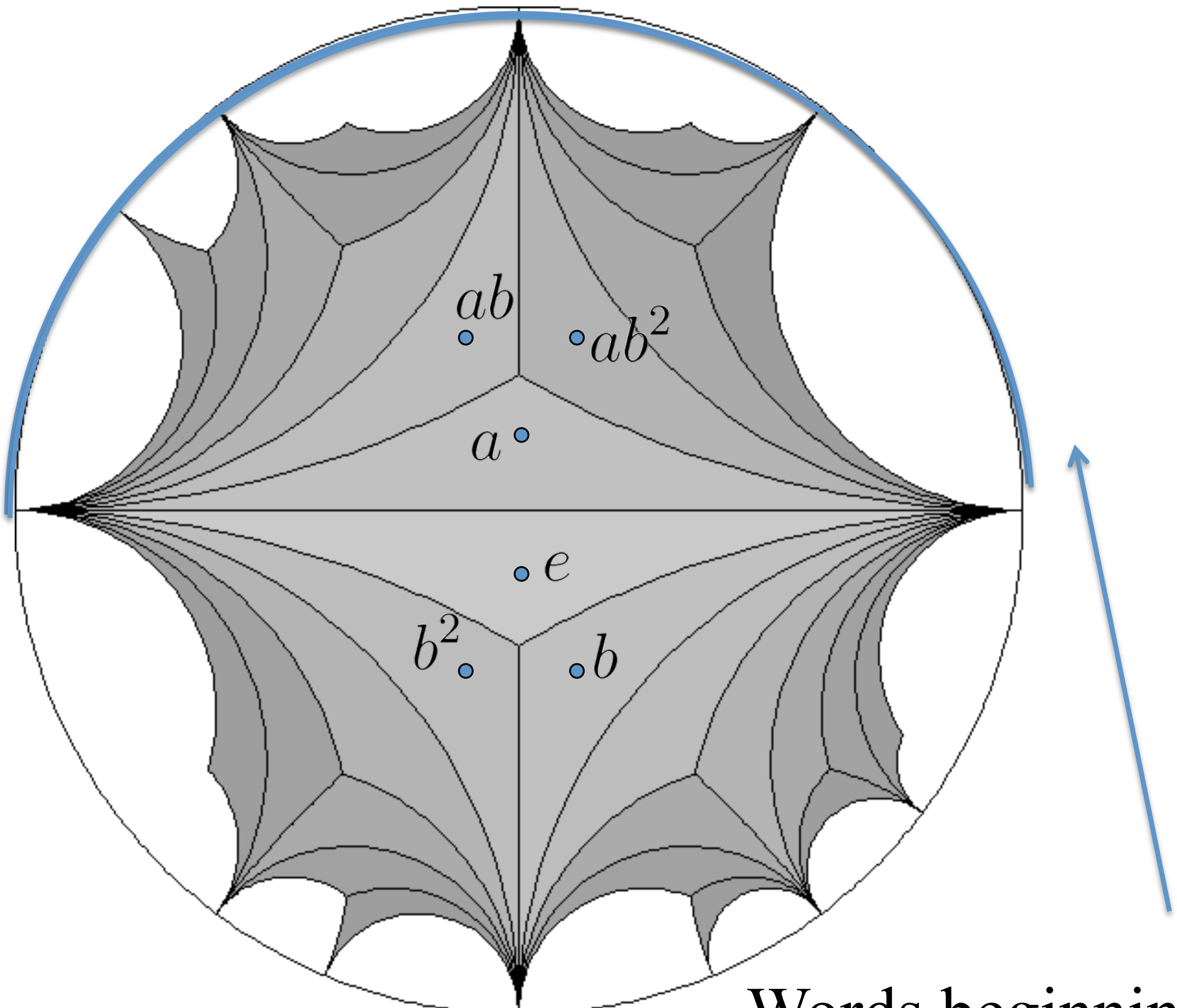

Words beginning with a

Example

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- The group $G = \langle a, b : a^2 = e, b^3 = e \rangle$
- The space $X =$ infinite word space for G
- Action of G on X: $(b)(b^2ab^2ab^2\dots) = ab^2ab^2\dots$

Theorem (Archbold, Kumjian, Spielberg, 91')

There is an action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor set X st

$$C(X) \rtimes_r \mathbb{Z}_2 * \mathbb{Z}_3 \cong \mathcal{O}_2.$$

Main result

We show here the following:

Theorem

Let G be a countable group. Then G admits a free action on the Cantor set X such that $C(X) \rtimes_r G$ is a Kirchberg algebra in UCT if and only if G is exact and non-amenable.

Strategy

Theorem

Let G be a countable group acting on the Cantor set X. TFAE

- $C(X) \rtimes_r G$ *is a Kirchberg algebra in UCT*
- *action is*
    - *topologically free*
    - *amenable*
    - *minimal*

  and nonzero projections in $C(X)$ are properly infinite in $C(X) \rtimes_r G$

Proper infinite projections

Theorem

Let G be a countable discrete group acting on a set X (eg. X could be G itself). The following are equivalent for every $E \subseteq X$:

(i) E *is* $(G, P(X))$*-paradoxical.*

(ii) $1_E \in \ell^\infty(X) \cong C(\beta X)$ *is properly infinite in* $C(\beta X) \rtimes_r G$.

(iii) *The n-fold direct sum* $1_E \oplus \cdots \oplus 1_E$ *is properly infinite in* $C(\beta X) \rtimes_r G$ *for some n.*

Corollary

Suppose that G is non-amenable. Then every projection in $C(\beta G)$ which is G-full (not contained in a proper G-invariant ideal of $C(\beta G)$), is properly infinite in $C(\beta G) \rtimes_r G$.

Proof

- Choose any separable G-invariant sub-C^*-algebra $A \subseteq C(\beta G)$ such that:
  - *G acts freely and amenably on A*
  - *Every projection in A that is properly infinite in $C(\beta G) \rtimes_r G$ is also properly infinite in $A \rtimes_r G$.*
  - *A is generated by projections.*
- Choose any maximal G-invariant ideal I in A.
- With $A/I = C(X)$: X is the Cantor set, and the action on X is free, amenable, minimal.
- It suffices to check that every non-zero projection in $C(X)$ is properly infinite in $C(X) \rtimes_r G$

$$\begin{array}{ccc} A \rtimes_r G & \hookrightarrow & C(\beta G) \rtimes_r G \\ \downarrow \pi & & \\ C(X) \rtimes_r G & & \end{array}$$

Freeness

Theorem

Let G be a discrete group acting on the Cantor set X. TFAE

- *action is free (hence topologically free)*
- *each $e \neq t \in G$ there exists a finite partition $\{p_{i,t}\}_{i \in F}$ of 1, st $p_{i,t} \perp t.p_{i,t}$.*

Theorem

Let G be a discrete group. The action of G on βG is free.

Corollary

Let G be a countable group. There is a countable subset M of $C(\beta G)$ such that if $A := C(X)$ is any G-invariant sub-C^*-algebra of $C(\beta G)$ containing M, then the action of G on X is free.

Amenability

Theorem

Let G be a discrete group acting on a compact Hausdorff space X. TFAE

- *the action is amenable*
- *for each $i \in \mathbb{N}$ there exists a family $\{m_{i,t}\}_{t\in G}$ in $C(X)_+$ st $\sum_{t\in G} m_{i,t} = 1$ and $\lim_i(\sup_{x\in X} \sum_{t\in G} \|m_{i,st}(x) - s.m_{i,t}\|) = 0$*

Theorem (Ozawa, Anantharman-Delaroche, 06')

Let G be a discrete exact group. Then action of G on βG is amenable

Corollary

Let G be a countable exact group. There is a countable subset M' of $C(\beta G)$ such that if $A := C(X)$ is any G-invariant sub-C^*-algebra of $C(\beta G)$ containing M, then the action of G on X is amenable.

Thank you

Thank you for your attention