KMS states for self-similar actions

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Self-similar actions

- Suppose X is a finite set of cardinality $|X|$;
  - let X^n denote the set of words of length n in X,
  - let $X^* = \bigcup_{n \in \mathbb{N}} X^n$.

Definition

A faithful action of a group G on X^* is *self-similar* if, for all $g \in G$ and $x \in X$, there exist unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \qquad \text{for all finite words } w \in X^*.$$

The pair (G, X) is referred to as a *self-similar action* and the group element $g|_x$ is called the *restriction* of g to x.

Restrictions

- Restrictions extend to words $v \in X^*$ is the natural way:

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all finite words } w \in X^*.$$

Lemma

Suppose (G, X) is a self-similar action. Restrictions satisfy

$$g|_{pq} = (g|_p)|_q, \quad gh|_p = g|_{h \cdot p}\ h|_p, \quad g|_p^{-1} = g^{-1}|_{g \cdot p}$$

for all $g, h \in G$ and $p, q \in X^*$.

Example: the odometer action

- Let $X = \{0, 1\}$ and $G = \mathbb{Z}$
- Let g denote the generator $1 \in \mathbb{Z}$
- $(\mathbb{Z}, X)$ is a self-similar action described by:

$$g \cdot 0w = 1w \qquad g \cdot 1w = 0(g \cdot w)$$

  for every finite word $w \in X^*$
  For example, g^3 denotes $3 \in \mathbb{Z}$ and acts on the word 01100 by

$$g^3 \cdot 01100 = g^2 \cdot 11100 = g \cdot 00010 = 10010.$$

- This defines $(\mathbb{Z}, X)$ as a self-similar group action called the odometer.

The odometer continued

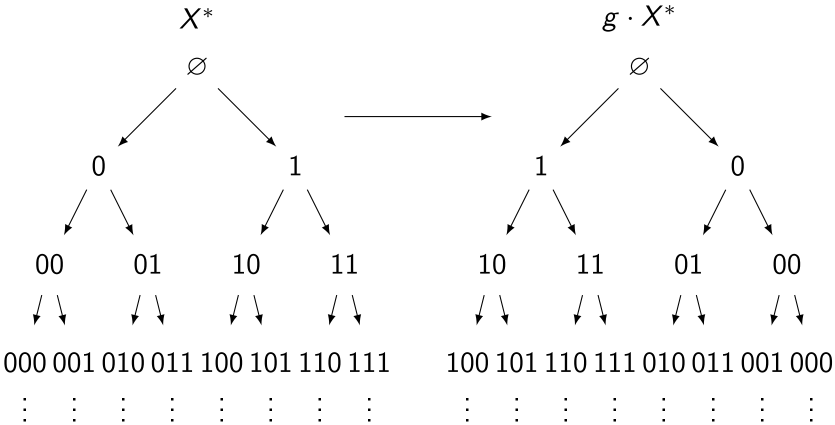


Figure: The action g on the tree associated with the circle

Contracting self-similar actions

Definition

The *nucleus* of a self-similar action (G, X) is the minimal set $\mathcal{N} \subset G$ satisfying the property: For every $g \in G$, there exists $N \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^n$ with $n \geq N$. A self-similar action (G, X) is *contracting* if it has a finite nucleus $\mathcal{N} \subset G$.

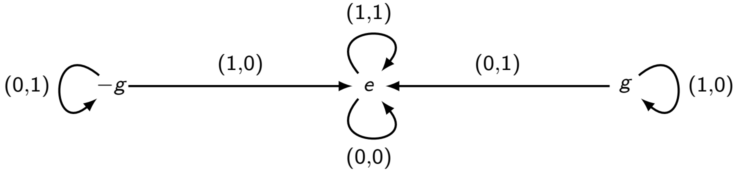
Definition

Let S be a subset of G that is closed under restriction. The *Moore diagram* of S is the labelled directed graph with vertices in S and edges labelled:

$$g \xrightarrow{(x,y)} g|_x$$

The Moore diagram for the nucleus of the odometer

- The nucleus of the odometer action is $\mathcal{N} = \{e, g, g^{-1}\}$.



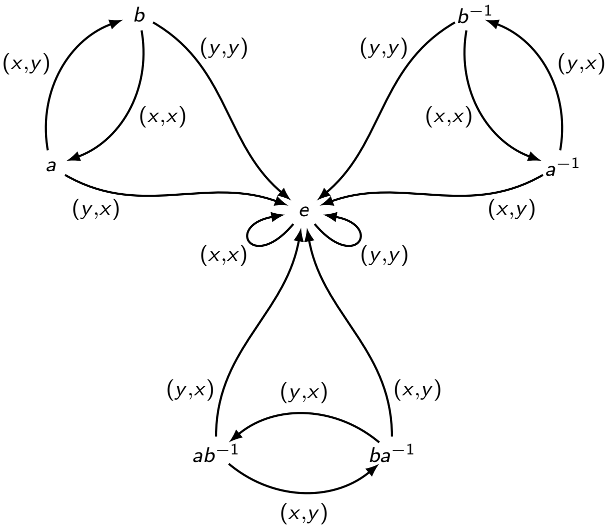
The basilica group [Grigorchuk and Żuk 2003]

- Let $X = \{x, y\}$
- Consider the rooted homogeneous tree T_X with vertex set X^*.
- Two automorphisms a and b of T_X are recursively defined by

$$a \cdot (xw) = y(b \cdot w) \qquad a \cdot (yw) = xw$$
$$b \cdot (xw) = x(a \cdot w) \qquad b \cdot (yw) = yw$$

  for $w \in X^*$.
- The *basilica group* B is the subgroup of Aut T_X generated by $\{a, b\}$. The pair (B, X) is then a self-similar action.
- The nucleus is $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ba^{-1}, ab^{-1}\}$.

The Moore diagram for the nucleus of the basilica group



The Grigorchuk action (1980)

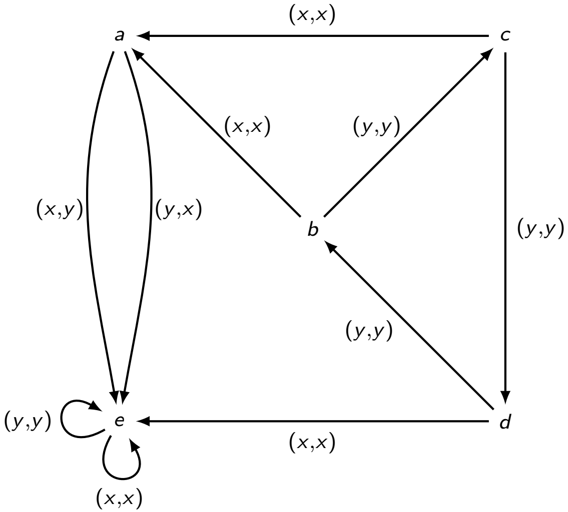
- Let $X = \{x, y\}$
- Consider the rooted homogeneous tree T_X with vertex set X^*.
- Two automorphisms a and b of T_X are recursively defined by
- Grigorchuk group is generated by four automorphisms a, b, c, d of T_X defined recursively by

$$\begin{aligned} a \cdot xw &= yw & \qquad a \cdot yw &= xw \\ b \cdot xw &= x(a \cdot w) & \qquad b \cdot yw &= y(c \cdot w) \\ c \cdot xw &= x(a \cdot w) & \qquad c \cdot yw &= y(d \cdot w) \\ d \cdot xw &= xw & \qquad d \cdot yw &= y(b \cdot w). \end{aligned}$$

The nucleus of the Grigorchuk group is

$$\mathcal{N} = \{e, a, b, c, d\}.$$

The Moore diagram for the nucleus of the Grigorchuk action



C^*-algebras

- A C^*-algebra is a Banach $*$-algebra A such that for all a in A,

$$\|aa^*\| = \|a\|^2$$

- Examples: $M_n(\mathbb{C})$, $C_0(X)$, $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, ...
- An element $u \in A$ such that $u^*u = uu^* = 1$ is called a *unitary*.
- An element $s \in A$ such that $s^*s = 1$ is called an *isometry* and if $ss^* \neq 1$ then s is called a *non-unitary isometry*.

Universal C^*-algebras for self-similar group actions

Theorem

Let (G, X) be a self-similar action. The Cuntz-Pimsner algebra $\mathcal{O}(G, X)$ is the universal C^*-algebra generated by unitaries $\{u_g : g \in G\}$ and a Cuntz family of isometries $\{s_x : x \in X\}$ satisfying

1. $u_g s_x = s_{g \cdot x} u_{g|_x}$
2. $\displaystyle\sum_{x \in X} s_x s_x^* = 1$

for all $g \in G$ and $x \in X$.

Remark: Nekrashevych defined an algebra $\mathcal{O}(M)$ using a particular representation of a self-similar action. In the case that G is amenable Nekrashevych's algebra is the same as $\mathcal{O}(G, X)$.

KMS states

Definition (Haag-Hugenholtz-Winnink 1967)

Given an action $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$ on a C^*-algebras A, a state φ satisfies the KMS condition at inverse temperature $\beta \in [0, \infty)$ if, for all $a, b \in \mathcal{A}$,

$$\varphi(ab) = \varphi(b\ \sigma_{i\beta}(a)).$$

Properties of KMS states:

- Haag-Hugenholtz-Winnink proposed the KMS condition as a definition of equilibrium for quantum systems.
- KMS states are a noncommutative phenomenon, If A has a faithful KMS state and A is commutative, then σ is trivial.
- If $\beta \neq 0$ and φ is a KMS_β state, then φ is σ-invariant.
- KMS states have a natural notion of a phase transition (an abrupt change in the physical properties of a system).

KMS states for self-similar actions

Proposition

Let (G, X) be a self-similar action, then

$$\mathcal{O}(G, X) = \overline{\text{span}}\{s_v u_g s_w^* : v, w \in X^*, g \in G\}.$$

- The action $\sigma : \mathbb{R} \to \text{Aut}(\mathcal{O}(G, X))$ is given by

$$\sigma_t(s_v) = e^{it|v|} s_v \qquad \sigma_t(u_g) = u_g.$$

- On the spanning set $\{s_v u_g s_w^* : v, w \in X^*, g \in G\}$ we have

$$\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)} s_v u_g s_w^*$$

KMS states on $\mathcal{O}(G, X)$

Theorem (Laca-Raeburn-Ramagge-W)

Suppose that (G, X) is a self-similar action.

1. *For $g \in G \setminus \{e\}$, set*

$$F_g^j := \{v \in X^j : g \cdot v = v \text{ and } g|_v = e\}.$$

   Then the sequence $\{|X|^{-j}|F_g^j|\}$ is increasing and converges with limit $c_g \in [0, 1)$.

2. *There is a $KMS_{\log |X|}$ state on $\mathcal{O}(G, X)$ such that*

$$\psi(s_v u_g s_w^*) = \begin{cases} 0 & \text{unless } v = w \\ |X|^{-|w|} c_g & \text{if } v = w. \end{cases}$$

3. *If (G, X) is contracting, then the state in part (2) is the only KMS state of $\mathcal{O}(G, X)$.*

Calculating KMS states using the Moore diagram

- To calculate values of the KMS states explicitly, we need to compute the sizes of the sets F_g^k and evaluate the limit

$$c_g = \lim_{k\to\infty} |X|^{-k}|F_g^k|$$

- For each $v \in F_g^k$ we have $g \cdot v = v$ and $g|_v = e$.
- Each $v \in F_g^k$ corresponds to a path μ_v in the Moore diagram:

$$\mu_v := g \xrightarrow{(v_1,v_1)} g|_{v_1} \xrightarrow{(v_2,v_2)} g|_{v_1 v_2} \xrightarrow{(v_3,v_3)} \cdots \xrightarrow{(v_k,v_k)} g|_v = e$$

- Notice that all the labels have the form (x, x).
- Every path with labels (x, x) arises this way.

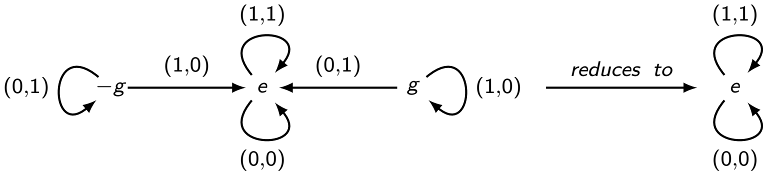
Example: the odometer action

Proposition

The C^*-algebra $\mathcal{O}(\mathbb{Z}, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{e, g, g^{-1}\}$ by

$$\phi(u_n) = \begin{cases} 1 & \text{for } n = e \\ 0 & \text{for } n = g, g^{-1} \end{cases}$$

Sketch of proof.



$$F_g^k = F_{g^{-1}}^k = \varnothing \qquad c_g = c_{g^{-1}} = \lim_{n\to\infty} 2^{-k} \cdot 0 = 0$$

Example: the basilica action

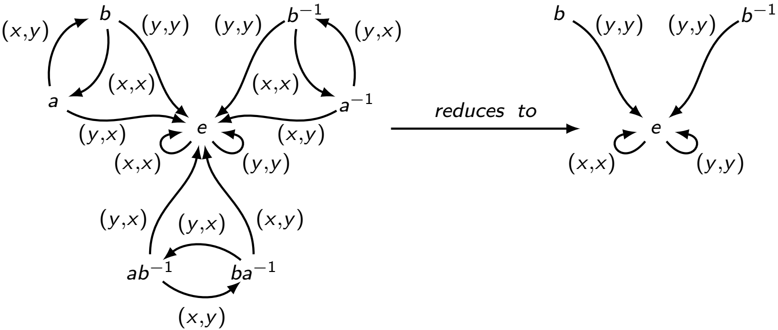
Proposition

The C^*-algebra $\mathcal{O}(B, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ by

$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

Example: the basilica action

Sketch of proof.



Example: the basilica action

$$|F_a^k| = 0 \qquad c_a = \lim_{n\to\infty} 2^{-k}|F_a^k| = 0$$

$$|F_{a^{-1}}^k| = 0 \qquad c_{a^{-1}} = \lim_{n\to\infty} 2^{-k}|F_{a^{-1}}^k| = 0$$

$$|F_{ba^{-1}}^k| = 0 \qquad c_{ba^{-1}} = \lim_{n\to\infty} 2^{-k}|F_{ba^{-1}}^k| = 0$$

$$|F_{ab^{-1}}^k| = 0 \qquad c_{ab^{-1}} = \lim_{n\to\infty} 2^{-k}|F_{ab^{-1}}^k| = 0$$

$$|F_b^k| = 2^{k-1} \qquad c_b = \lim_{n\to\infty} 2^{-k}|F_b^k| = \frac{1}{2}$$

$$|F_{b^{-1}}^k| = 2^{k-1} \qquad c_{b^{-1}} = \lim_{n\to\infty} 2^{-k}|F_{b^{-1}}^k| = \frac{1}{2}$$

Example: the Grigorchuk action

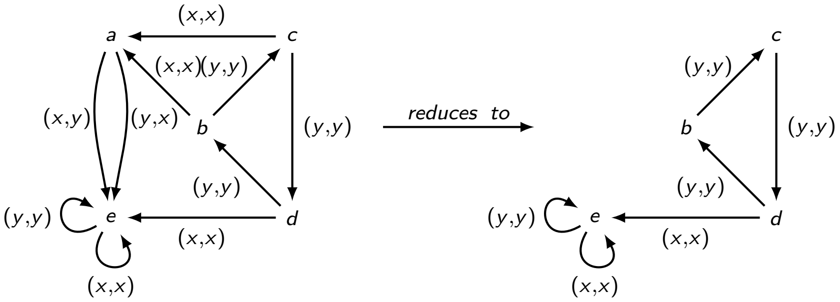
Proposition

Let (G, X) be the self-similar action of the Grigorchuk group. Then $(\mathcal{O}(G, X), \sigma)$ has a unique $KMS_{\log 2}$ state ϕ which is given on the nucleus $\mathcal{N} = \{e, a, b, c, d\}$ by

$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

Example: the Grigorchuk action

Sketch of proof.



Example: the Grigorchuk group

$$F_a^k = \varnothing$$

$$c_a = \lim_{n\to\infty} 2^{-k}|F_a^k| = 0$$

$$|F_b^k| = \frac{2^k - 2^{k-(3j+3)}}{7} \quad \text{where } 3j+3 \leq k \leq 3j+5$$

$$c_b = \lim_{n\to\infty} 2^{-k}|F_b^k| = \frac{1}{7}$$

$$|F_c^k| = \frac{2^{k+1} - 2^{k-(3j+2)}}{7} \quad \text{where } 3j+2 \leq k \leq 3j+4$$

$$c_c = \lim_{n\to\infty} 2^{-k}|F_c^k| = \frac{2}{7}$$

$$|F_d^k| = \frac{2^{k+2} - 2^{k-(3j+1)}}{7} \quad \text{where } 3j+1 \leq k \leq 3j+3$$

$$c_d = \lim_{n\to\infty} 2^{-k}|F_d^k| = \frac{4}{7}$$

Questions

- Is there a general formula for the $\mathrm{KMS}_{\log 2}$ states for the basilica and Grigorchuk actions?
- Do the F_g^k sets appear in other computations associated with self-similar actions?
- Are there new interesting examples of SSAs that we should be looking at?

References:

1. M. Laca and I. Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. **225** (2010), 643–688.
2. M. Laca, I. Raeburn, and J. Ramagge, *Phase transition on Exel crossed products associated to dilation matrices*, J. Funct. Anal. **261** (2011), 3633–3664
3. V. Nekrashevych, *C^*-algebras and self-similar groups*, J. reine angew. Math. **630** (2009), 59–123.
4. V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs **vol. 117**,. Amer. Math. Soc., Providence, RI, 2005.