Locally normal subgroups of totally disconnected groups

(paper in preparation)

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Topological groups

A **topological group** is a group that is also a topological space, such that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

Any topological group G has a largest connected subgroup G_0. G is **totally disconnected** if $G_0 = \{1\}$.

G is **locally compact** if there is a compact neighbourhood of 1.

Totally disconnected, locally compact (t.d.l.c.) groups arise in several contexts, e.g.

- Automorphism groups of locally finite graphs
- Galois groups (compact)
- Linear algebraic groups over local fields

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van Dantzig's theorem

Special case: compact + totally disconnected = **profinite**
Profinite groups are inverse limits of finite groups and can be well understood in terms of asymptotic properties of finite groups.
In particular they are **residually finite**, so have many normal subgroups.

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Lemma

Let G be a topological group and let U and V be open compact subgroups of G. Then $U \cap V$ is open, so has finite index in both U and V. In other words U and V are **commensurate**.

Corollary

Every non-discrete t.d.l.c. group has a distinguished **commensurability class** of infinite residually finite subgroups, namely its open compact subgroups.

Conversely, given any group Γ with a residually finite subgroup Δ, if all conjugates of Δ are commensurate to Δ then Γ can be embedded densely in a t.d.l.c. group G so that $\overline{\Delta}$ is open and compact in G. (Belyaev)

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Every finitely generated group $\Gamma \neq 1$ has a **simple** quotient.

Analogue of finitely generated for t.d.l.c. groups is **compactly generated** (= generated by a compact subset).

G is **topologically simple** if there are no proper non-trivial **closed** normal subgroups.

Theorem (Caprace-Monod 2011)

Let G be a compactly generated t.d.l.c. group. Then exactly one of the following holds.

(i) G has an infinite discrete quotient.

(ii) G has a cocompact closed normal subgroup N such that N has no infinite discrete quotient, but N has exactly n non-compact topologically simple quotients, where $0 < n < \infty$.

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Say a subgroup K of a t.d.l.c. group G is **locally normal** if K is compact and $\mathrm{N}_G(K)$ is open.

General idea:

- Study t.d.l.c. groups via their locally normal subgroups
- Use structures that are invariant under commensurability
- Special interest in compactly generated, topologically simple groups

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The structure lattice

Definition

Let G be a t.d.l.c. group. Given a subgroup K of G, write $[K]$ for the set of compact subgroups L of G such that $K \cap L$ is open in K and L ($\Rightarrow$ finite index if K is compact). The **structure lattice** of G is the set

$$\mathcal{LN}(G) := \{[K] \mid K \text{ is locally normal in } G\},$$

equipped with a partial order: $[K] \leq [L]$ if $K \cap L$ is open in K.

This is a **lattice** in the sense that any pair of elements has a least upper bound and greatest lower bound.

Fixed points in $\mathcal{LN}(G)$

- Two 'trivial' elements of $\mathcal{LN}(G)$:
  $0 := [\{1\}]$ $\quad \infty := \{\text{open compact subgroups}\}$.
- G acts on $\mathcal{LN}(G)$ by conjugation, fixing 0 and ∞.
- It can happen that $|\mathcal{LN}(G)| = 2$, e.g. $G = \mathrm{PSL}_n(\mathbb{Q}_p)$.

Proposition

If G is compactly generated and **abstractly** simple, then G has no fixed points in $\mathcal{LN}(G)$ other than 0 and ∞.

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Boolean algebras

A **Boolean algebra** is a uniquely complemented distributive lattice. It can be thought of as a collection of subsets of an ambient set, ordered by inclusion, that is closed under complementation and under pairwise unions and intersections.

For every Boolean algebra $\mathcal{A}$ there is a corresponding compact topological space $\mathfrak{S}$, the **Stone space** of $\mathcal{A}$.
elements of $\mathcal{A}$ $\leftrightarrow$ clopen subsets of $\mathfrak{S}$
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Local decomposition lattice

A **local factor** of G is a direct factor of an open compact subgroup of G.

The **quasi-centre** $\mathrm{QZ}(G)$ of G is the set of elements that centralise an open subgroup of G.

Proposition

Let G be a t.d.l.c. group such that $\mathrm{QZ}(G) = 1$. Then

$$\mathcal{LD}(G) := \{[K] \mid K \text{ is a local factor of } G\}$$

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Centraliser lattice

With an extra condition we can obtain another Boolean algebra that accounts for all direct decompositions of locally normal subgroups into locally normal factors.

Proposition

Let G be a t.d.l.c. group such that $\mathrm{QZ}(G) = 1$ and suppose G has no non-trivial abelian locally normal subgroups. Then

$$\mathcal{LC}(G) := \{[\mathrm{C}_G(K)] \mid K \text{ is locally normal in } G\}$$

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Overview

$$\{0, \infty\} \subseteq \mathcal{LD}(G) \subseteq \mathcal{LC}(G) \subseteq \mathcal{LN}(G)$$

From now on G is a compactly generated, topologically simple t.d.l.c. group.

For some of our results we need no further assumptions. But we can show more in the case that $|\mathcal{LC}(G)| > 2$.

Most known examples have $|\mathcal{LC}(G)| > 2$ (excluding linear algebraic groups over local fields).

Quasi-centre and abelian subgroups

To define $\mathcal{LD}(G)$ and $\mathcal{LC}(G)$ we needed to impose some conditions. Fortunately:

Theorem (Barnea–Ershov–Weigel; CRW)

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Dynamics on the centraliser lattice

Theorem (CRW)

Let $\mathcal{A}$ be a G-invariant subalgebra of $\mathcal{LC}(G)$. Suppose $|\mathcal{A}| > 2$.

(i) Every orbit of G on $\mathfrak{S}(\mathcal{A})$ is dense. ($\Rightarrow$ faithful action)
(ii) There exists $\alpha \in \mathcal{A} \setminus \{0\}$ such that for all $\beta \in \mathcal{A} \setminus \{0\}$ there is some $g \in G$ such that $g\alpha < \beta$.
(iii) $\mathcal{A}$ is infinite and has no atoms.
(iv) G is not amenable.
(v) There exists $g \in G$ with non-trivial contraction group.
(vi) There exist $g, h \in G$ such that the submonoid of G generated by g and h is free on $\{g, h\}$.

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Dense normal subgroups

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Last slide

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