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On nonlinear wave interaction with deformable ice sheets

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Abstract

Diffraction of nonlinear water waves by floating ice sheets in marginal ice zones is studied by use of the Level I Green-Naghdi equations and by Reynolds-Averaged Navier-Stokes equations. In general, there are N number of compact and deformable ice sheets of arbitrary size. Particular attention is given to the wave-induced loads on the ice sheets and their deformation, and the wave energy attenuation and dissipation.

Introduction

This study is concerned with the interaction of nonlinear waves of solitary and cnoidal types with deformable ice sheets floating on water surface in marginal ice zones. The wave-ice interaction problem is studied by use of the Level I Green-Naghdi equations for the fluid field, coupled with the elastic plate equations for the deformation of the ice sheet, and by the Reynolds-Averaged Navier-Stokes (RANS) equations. There are N number of compact, but deformable ice sheets, i.e., we do not consider ice breaking in this study. That is, attention is confined to the loads on the ice sheets and their deformation, and attenuation of the wave field. The ice sheets may have arbitrary sizes.

The Level I Green-Naghdi Equations

The Green-Naghdi (GN) equations are nonlinear partial differential equations used in describing the nonlinear wave mechanics. In the general form, incompressibility is the only assumption made about the medium; the GN equations can be applied to homogenous or inhomogeneous matters, whether viscous or inviscid. They may be applied to nonlinear wave propagation in deep- or shallow-waters. Irrotationality of the flow is not required, although the equations can also be derived for irrotational flows. In the absence of any perturbation and scaling restrictions, the GN equations satisfy the nonlinear boundary conditions exactly, and the integrated conservation laws are postulated.

The GN equations are originally derived by Green & Naghdi (1976*b*) in a direct approach and by use of the Cosserat surfaces. A Cosserat surface is a model that idealizes a three-dimensional deformable body by a deformable surface, to every point of which, a deformable vector is assigned. The Cosserat theories are approximate solutions of the exact three-dimensional theory. The Cosserat theory of shells (Naghdi (1973)), for example, is used to model the response of a thin shell-like body, in which one of its dimensions (namely the thickness) is smaller than the other two. Hence, the shell theory equations depend on only two spatial dimensions and time. In this direct approach, the conservation laws of mass, linear momentum, director momentum and angular momentum are satisfied exactly on the two-dimensional surfaces. By use of the Cosserat surfaces, the balance laws are then integrated along the thickness of the shell. Similar to the shells, in fluid sheets the water depth (corresponding to the shell thickness) is smaller than the other two dimensions, allowing for the same approach to be applied to water wave problems.

Green & Naghdi (1976*a*) showed that the GN equations can also be obtained from the exact three-dimensional equations of a homogeneous, incompressible and inviscid fluid by making a single assumption about the distribution of vertical velocity over the fluid thickness. The GN equations are classified based on the level of the functions used to prescribe the distribution of the vertical velocity along the water column. The Level I GN equations, also known as *the restricted theory*, assume a linear distribution of the vertical velocity along the water column. This assumption, which is the only one made about the kinematics of the fluid sheet, results in the horizontal velocities being invariant in the vertical direction for an incompressible fluid. The resultant Level I GN equations, hence, are mostly applicable to the propagation of fairly long waves in shallow water.

In this study, a two dimensional right-handed Cartesian coordinate system, whose origin is at the still-water level (SWL) is used. The incident waves propagate in the positive x_1-direction, and the gravitational acceleration is pointing opposite to x_2 direction. The free surface, $\eta(x_1,t)$, is measured from the SWL. The Level I GN equations as used here are given by the conservation of mass and momentum as

$$\eta_{,t} + \{(1+\eta-\alpha)u_1\}_{,x_1} = \alpha_{,t}\,, \tag{1a}$$

$$\dot{u_1} + \eta_{,x_1} + \hat{p}_{,x_1} = -\frac{1}{6}\{[2\eta+\alpha]_{,x_1}\ddot{\alpha} + [4\eta-\alpha]_{,x_1}\ddot{\eta} + (1+\eta-\alpha)[\ddot{\alpha}+2\ddot{\eta}]_{,x_1}\}\,, \tag{1b}$$

$$u_2(x_1,t) = \dot{\alpha} + \frac{(x_2+1-\alpha)}{(\eta+1-\alpha)}\,(\dot{\eta}-\dot{\alpha})\ , \tag{1c}$$

$$\bar{p}(x_1,t) = \left(\frac{1}{2}\right)(1+\eta-\alpha)\,(\ddot{\alpha}+\ddot{\eta}+2) + \hat{p}\,, \tag{1d}$$

where $\alpha(x_1,t)$ is the elevation of the bottom of the fluid sheet. $\bar{p}$ is the pressure on the bottom curve (α), and $\hat{p}(x_1,t)$ is the pressure on the top curve of the fluid sheet. The velocity field is defined by $\boldsymbol{V} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2}$, where $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ are the unit base vectors in the x_1 and x_2 directions, respectively. The superposed dot is the two-dimensional material time derivative and double superposed dot denotes the second material time derivative. The subscripts after comma indicate partial differentiation with respect to the variables. The equations are nondimesnionalized by use of ρ, g and h, where ρ is the mass density of the fluid, g is the gravitational acceleration and h is the water depth.

Nonlinear Wave Interaction With Deformable Ice Sheets

We consider N number of two dimensional floating, ice sheets of arbitrary lengths and variable thickness of δ_i where $1 \le i \le N$. The translational motion of the ice sheets with the wave motion is negligible, i.e. the ice sheets are fixed in space between $x_1 = X_{Li}$ and $x_1 = X_{Ti}$, where X_{Li} and X_{Ti} are the coordinate of the leading edge and trailing edges of the $i-$th ice sheet. The ice sheets may deform about their initial position due to the wave loads. Water depth, h, is constant throughout the domain and the seafloor is flat and stationary, although these are not required in general. A schematic of the problem is shown in Fig. 1.

The problem of nonlinear wave interaction with floating ice sheets by the GN equations is best studied by splitting the domain into two types of regions, shown in Fig. 1. In type RI Regions, the top surface, $\eta(x_1,t)$, is free, and pressure on this surface is continuous and atmospheric. We assume $\hat{p} = 0$ in RI Regions without loss in generality. The bottom surface is flat and stationary, i.e., $\alpha(x_1,t) = 0$. Hence, the Level I GN equations, Eqs. (1a) - (1d), are simplified to:

$$\eta_{,t} + \{(1+\eta)u_1\}_{,x_1} = 0\,, \tag{2a}$$

$$\dot{u}_1 + \eta_{,x_1} = -\frac{1}{3}\{(2\eta_{,x_1}\ddot{\eta}) + (1+\eta)\,\ddot{\eta}_{,x_1}\}\,, \tag{2b}$$

$$u_2(x_1,t) = \frac{(x_2+1)}{(\eta+1)}\dot{\eta}\,, \tag{2c}$$

$$\bar{p}(x_1,t) = \left(\frac{1}{2}\right)(1+\eta)\,(\ddot{\eta}+2)\ . \tag{2d}$$

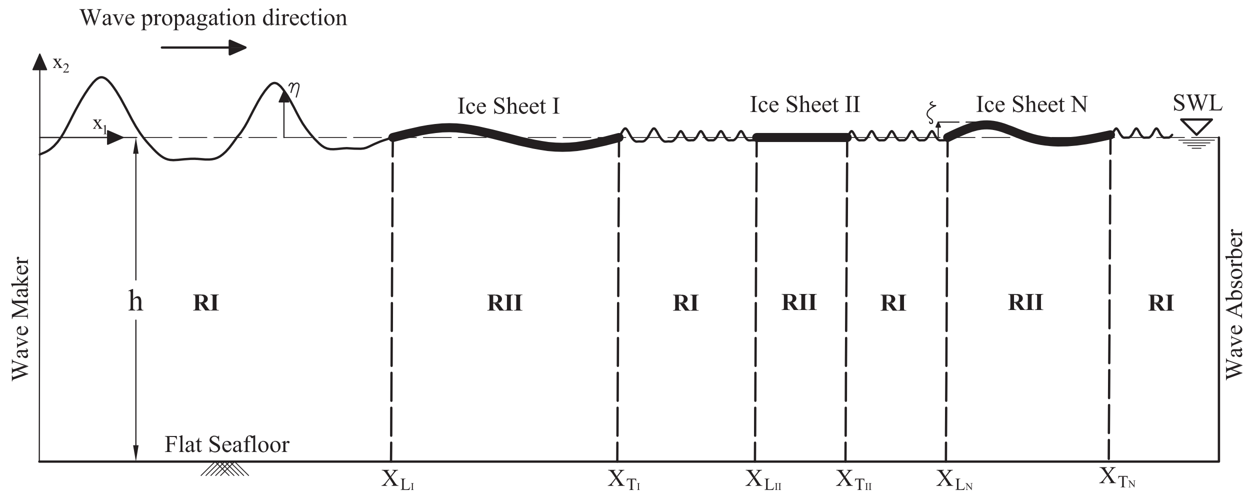

Figure 1: Schematic of the theoretical wave tank of two-dimensional wave interaction with floating, deformable ice sheets. Only three ice sheets are shown here, and these may be N in general. Also shown, are the two types of regions discussed within the text. Figure not to scale.

Equations (2a)-(2d) are solved for surface elevation (η), velocity field (u_1 and u_2) and the bottom pressure $\bar{p}$, subject to the solution of the equations in other regions.

In type RII Regions, the top surface is confined by the floating, deformable ice sheets. The ice is compact and homogeneous at all times, and the fluid and ice sheets are always in contact, i.e. we do not consider ice cracks or breaking, or formation of air pockets and voids. Hence, in RII Regions, $\eta(x_1,t) = \zeta(x_1,t)$, where $\zeta(x_1,t)$ is the deformation of the ice sheet from the initial horizontal position. We assume flat and stationary seafloor under the ice sheets, $\alpha(x_1,t) = 0$, and hence the water depth is constant h, i.e. δ_i is small. The top pressure in RII Regions, $\hat{p}_{II}$, is unknown and it is the pressure under the ice sheets. Therefore, in RII Regions, the GN equations, Eqs. (1a) - (1d), are simplified to:

$$\zeta_{,t} + \{(1+\zeta)u_1\}_{,x_1} = 0\,, \tag{3a}$$

$$\dot{u}_1 + \zeta_{,x_1} + (\hat{p}_{II})_{,x_1} = -\frac{1}{3}\{\left(2\zeta_{,x_1}\ddot{\zeta}\right) + (1+\zeta)\,\ddot{\zeta}_{,x_1}\}\,, \tag{3b}$$

$$u_2(x_1,t) = \frac{(x_2+1)}{(\zeta+1)}\dot{\zeta}\,, \tag{3c}$$

$$\bar{p}(x_1,t) = \left(\frac{1}{2}\right)(1+\zeta)\left(\ddot{\zeta}+2\right) + \hat{p}_{II}\,. \tag{3d}$$

Unknowns in RII Regions are the surface elevation (η), velocity field (u_1 and u_2), the bottom pressure ($\bar{p}$), and the top pressure $\hat{p}_{II}$. That is, number of unknowns is one more than the number of equations.

To close the system of equations in RII Regions, the additional equation comes from the deformation of the ice sheet. In this study, we confine our attention to the interaction of nonlinear long waves with deformable ice sheets with finite length. The ice deformation is defined by use of the thin, elastic plate theory given in dimensionless form as

$$m\zeta_{,tt} + D\zeta_{,x_1x_1x_1x_1} + m = \hat{p}_{II}\ \ , \tag{4}$$

where m is the mass of the ice sheet, D is the flexural rigidity of the ice sheet, and it is defined by (see, *e.g.*, Timoshenko & Woinowsky-Krieger (1959))

$$D = E\delta^3/[12(1-\nu^2)]\,, \tag{5}$$

where E and ν are the corresponding Young's modulus and Poisson's ratio of the ice sheet, respectively, and δ is the thickness of the sheet. See e.g. Kohout & Meylan (2008) for discussion on the use of elastic plate models for wave-ice interaction.

At the leading and trailing edges of the ice sheets, the bending moments and shear stresses must vanish due to the free-free end boundary condition. Therefore $\zeta_{,x_1x_1} = \zeta_{,x_1x_1x_1} = 0$ at $x_1 = X_{Li}$ and $x_1 = X_{Ti}$. The fluid is always in contact with the ice sheets, and hence this boundary condition must also be satisfied by the fluid at the leading and trailing edges of the ice sheets. This is enforced by taking the second and third derivatives of the conservation of mass equation in RII Regions, Eq. (3a), and setting $\zeta_{,x_1x_1} = \zeta_{,x_1x_1x_1} = \zeta_{,x_1x_1t} = \zeta_{,x_1x_1x_1t} = 0$, see e.g. Ertekin & Xia (2014). Hence,

$$3\zeta_{,x_1}u_{1,x_1x_1} + (1+\zeta)u_{1,x_1x_1x_1} = 0\,, \tag{6a}$$

$$4\zeta_{,x_1}u_{1,x_1x_1x_1} + (1+\zeta)u_{1,x_1x_1x_1x_1} + \zeta_{,x_1x_1x_1x_1}u_1 = 0\,, \tag{6b}$$

at $x_1 = X_{Li}$ and $x_1 = X_{Ti}$.

Matching and jump conditions are used to obtain a continuous solution throughout the domain. The jump conditions are demanded by the theory and ensure conservation of mass and momentum across the discontinuity curves, see e.g. Hayatdavoodi & Ertekin (2015). Continuous bottom pressure is used as the matching condition, demanded by the physics of the problem.

On the left side of the domain, a numerical wavemaker capable of generating GN solitary and cnoidal waves is installed. On the right side of the domain, Orlanski's condition is used to minimise the wave reflection back to the tank.

The system of equations of the entire domain, subject to appropriate boundary conditions in each region, along with the matching and jump conditions, are solved simultaneously for the unknowns. Spatial discretization of the equations is carried out by a central-difference method, second order in space, and time marching is obtained by use of the modified Euler's method. The system of equations are solved by use of a Guassian Elimination method. Discussion on the numerical solution and results will be presented at the workshop. We also use the RANS equations to study the problem of wave interaction with floating ice sheets. Computations of RANS equations are carried out by OpenFOAM.

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