# A Linear Elasticity Model for Ice Shelf Vibrations

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## 1 Governing Equations

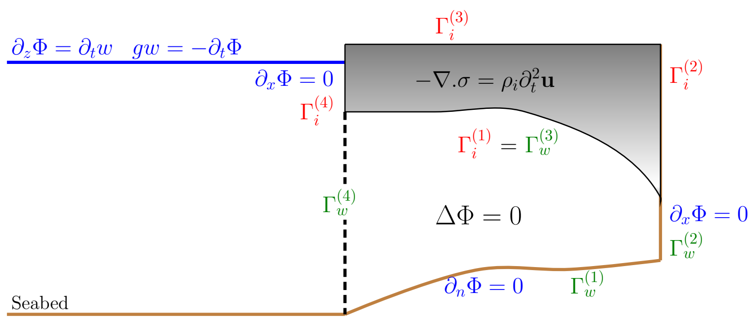


Figure 1: Geometry and governing equations

Consider the schematic shown in Figure 1. The ice shelf is modelled as a two dimensional elastic body. The open water region is denoted as $\Omega_o$ and the sub-shelf cavity region is denoted as $\Omega_w$. The ice shelf is denoted by $\Omega_i$ and the boundaries of the ice-shelf and the cavity are shown in Figure 1.

The seabed under the ice-shelf is assumed to be non-uniform and is given by the function $z = -H(x)$. The set of governing equations for the fluid motion in the ocean and the shelf/cavity region is given by

$$\begin{aligned} \Delta\Phi &= 0 && \text{in} \quad \Omega_o \cup \Omega_w, \\ \partial_t w &= \partial_z\Phi, && x < 0, \\ \rho_w g w &= -\rho_w \partial_t \Phi, && x < 0, \end{aligned} \tag{1}$$

with the appropriate boundary conditions shown in Figure 1. Here $w = w(x,t)$ is displacement of the free surface. The ice shelf is modelled as a 2D elastic body assuming plane strain conditions. The strain and stress tensors are given by the relation

$$\boldsymbol{\epsilon}(\boldsymbol{\eta}) = \frac{1}{2}\left(\nabla\boldsymbol{\eta} + \nabla\boldsymbol{\eta}^{\mathrm{T}}\right), \quad \boldsymbol{\sigma}(\boldsymbol{u}) = \lambda\nabla.\boldsymbol{u}\ \mathbf{I} + 2\mu\,\boldsymbol{\epsilon}(\boldsymbol{u})$$

where $\mathbf{I}$ denotes the identity tensor and $\lambda$, $\mu$ are the Lamé parameters. The governing equations for the displacement of the ice shelf are given as

$$
\begin{aligned}
\nabla.\boldsymbol{\sigma} &= -\rho_i\, \partial_t^2 \boldsymbol{u} && \text{in } \Omega_i, \\
\boldsymbol{\sigma}.\mathbf{n} &= -\rho_w g\, \boldsymbol{u} + \rho_w \partial_t \Phi\, \mathbf{n}, && \text{on } \Gamma_i^{(1)}, \\
\boldsymbol{\sigma}.\mathbf{n} &= \mathbf{0}, && \text{on } \Gamma_i^{(3)} \cup \Gamma_i^{(4)}, \\
\boldsymbol{u} &= \mathbf{0}, && \text{on } \Gamma_i^{(2)},
\end{aligned}
\tag{2}
$$

where $\boldsymbol{u} = \boldsymbol{u}(x,z,t)$ is the displacement of the ice shelf. We non-dimensionalise the problem by defining the following scalings

$$
\hat{x} \to \frac{x}{L_c},\ \hat{x} \to \frac{z}{L_c},\ \hat{t} \to \frac{t}{t_c}, \quad \text{with} \quad L_c = \sqrt[4]{\frac{D}{\rho_w g}}, \quad t_c = \sqrt{\frac{\rho_w L_c^6}{DH}}.
$$

We obtain the time harmonic motions at an angular frequency $\omega$ by setting $\Phi(x,z,t) = \mathrm{Re}\{\phi(x,z)\ e^{-i\omega t}\}$ and $\boldsymbol{u}(x,z,t) = \mathrm{Re}\{\mathbf{u}(x,z)\ e^{-i\omega t}\}$. The time-harmonic, non dimensional versions of (1) is

$$
\begin{aligned}
\Delta\phi &= 0 && \text{in } \Omega_o \cup \Omega_w, \\
-i\omega w &= \frac{1}{L_c}\partial_z \phi, && x < 0, \\
\rho_w g w &= i\omega\rho_w\, \phi, && x < 0, \\
\partial_n \phi &= 0, && \text{on } \Gamma_w^{(1)}, \Gamma_w^{(3)} \text{ and } z = -H\ (x<0), \\
\partial_n \phi &= -i\omega L_c\, \mathbf{u}.\mathbf{n}, && \text{on } \Gamma_w^{(3)}.
\end{aligned}
\tag{3}
$$

and (2) is

$$
\begin{aligned}
\frac{1}{L_c^2}\nabla.\boldsymbol{\sigma} &= -\rho_i \omega^2\, \mathbf{u} && \text{in } \Omega_i, \\
\boldsymbol{\sigma}.\mathbf{n} &= -\rho_w g L_c\, \mathbf{u} - i\omega\rho_w L_c\, \phi\, \mathbf{n}, && \text{on } \Gamma_i^{(1)}, \\
\boldsymbol{\sigma}.\mathbf{n} &= \mathbf{0}, && \text{on } \Gamma_i^{(3)} \cup \Gamma_i^{(4)}, \\
\mathbf{u} &= \mathbf{0}, && \text{on } \Gamma_i^{(2)}.
\end{aligned}
\tag{4}
$$

We solve the coupled shelf–cavity system (3– 4) using the finite element method, which can handle arbitrary shelf and cavity geometries. The solution method is based on the methods used to solve hydroelastic problems for modelling very large container ships.

The boundary condition at $x = 0$, i.e. at the interface between the open water and the shelf/cavity regions is expressed as

$$
\partial_x \phi(0,z) = \mathrm{Q}\phi(0,z) + \chi(z), \quad \text{on} \quad \Gamma_w^{(4)}.
$$

The operator Q and the function $\chi(z)$ are constructed using analytical expressions for the potential in the open ocean region, as discussed by Ilyas et al. [1]. The weak formulation of the boundary value problem (4) is

$$
\frac{1}{L_c^2}\int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}_h) : \boldsymbol{\epsilon}(\mathbf{v}_h)\, dx = \rho_i \omega^2 \int_{\Omega_i} \mathbf{u}_h.\mathbf{v}_h\, dx + \frac{-i\omega\rho_w}{L_c}\int_{\Gamma_i^{(1)}} \phi_h\, \mathbf{v}_h.\mathbf{n}\, ds \ + \frac{-\rho_w g}{L_c}\int_{\Gamma_i^{(1)}} \mathbf{u}_h.\mathbf{v}_h ds, \tag{5}
$$

where $\boldsymbol{u}_h$ and $\phi_h$ are the solutions to the displacement of the ice shelf and the potential in the appropriate finite element space. The displacement of the ice-shelf and the potential in the shelf/cavity region is expressed as

$$\mathbf{u}_h(x,z) \;=\; \sum_{j=1}^{\infty} \lambda_j \boldsymbol{u}_j(x,z), \qquad \phi_h(x,z) \;=\; \phi_0(x,z) + \sum_{j=1}^{\infty} \lambda_j \phi_j(x,z), \tag{6}$$

where $\boldsymbol{u}_j(x,z)$ are eigenmodes, which are solutions to the eigenvalue problem

$$\begin{aligned} \frac{1}{L_c^2}\nabla.\boldsymbol{\sigma}_j &= \Omega_j \rho_i \, \boldsymbol{u}_j \\ \boldsymbol{\sigma}_j.\mathbf{n} = \mathbf{0} &\quad \text{on} \;\; \Gamma_i^{(1)}, \Gamma_i^{(3)}, \Gamma_i^{(4)}, \\ \boldsymbol{u}_j = 0 &\quad \text{on} \;\; \Gamma_i^{(2)}. \end{aligned} \tag{7}$$

They are found using the finite element method. The diffraction potential $\phi_0$ and the radiation potential $\phi_j$ are the solutions to Laplace's equation in the cavity region, which is also found using the finite element method. We then substitute (6) into (5) and set $\boldsymbol{v}_h = \boldsymbol{u}_i$ to obtain the linear system

$$\left(\mathbf{K} - \omega^2\mathbf{M} - \mathrm{i}\omega\mathbf{B}\right)\boldsymbol{\lambda} = \mathbf{f}$$

which is solved to obtain the coefficients $\lambda_j$.

# 2 Results

In the numerical examples, the length of the ice-shelf is taken as $L = 10\,\mathrm{km}$ and the density of the ice-shelf as $\rho_i = 922.5$ kg m$^{-3}$. For a non-uniform ice-shelf, the thickness is taken to be 200 m at the seaward end and 600 m at the landward end. The base of the ice shelf was constructed using cubic spline interpolation. Figure 2 shows the different mode shapes and the corresponding eigenvalues for the eigenvalue problem in (7).

The left-hand panel of Figure 3 shows the forced displacement of the ice shelf for an incident wave-period of $T = 100\,\mathrm{s}$. The right-hand panel compares the solution of the linear-elasticity model with the solution of the thin-beam model for two different incident wave periods. The linear-elasticity model agrees well with the thin-beam model for the uniform thickness case, since the thickness to length ratio is small for the choice of parameters which are examples of typical ice-shelf geometries.

# 3 Conclusions

We have shown how vibrations of the ice-shelf can be modelled using the linear elasticity equations and how the finite element method can be used to solve the problem for arbitrary ice-shelf and cavity geometries. We have also compared the solution of the linear elasticity model to the Euler-Bernoulli model and we have observed that the two solutions agree well with each other for uniform ice-shelf thickness.

# References

[1] M. Ilyas, M. H. Meylan, B. Lamichhane, and L. G. Bennetts. Time-domain and modal response of ice shelves to wave forcing using the finite element method. *J. Fluids Struct.*, 80, 2018. doi: 10.1016/j.jfluidstructs.2018.03.010.

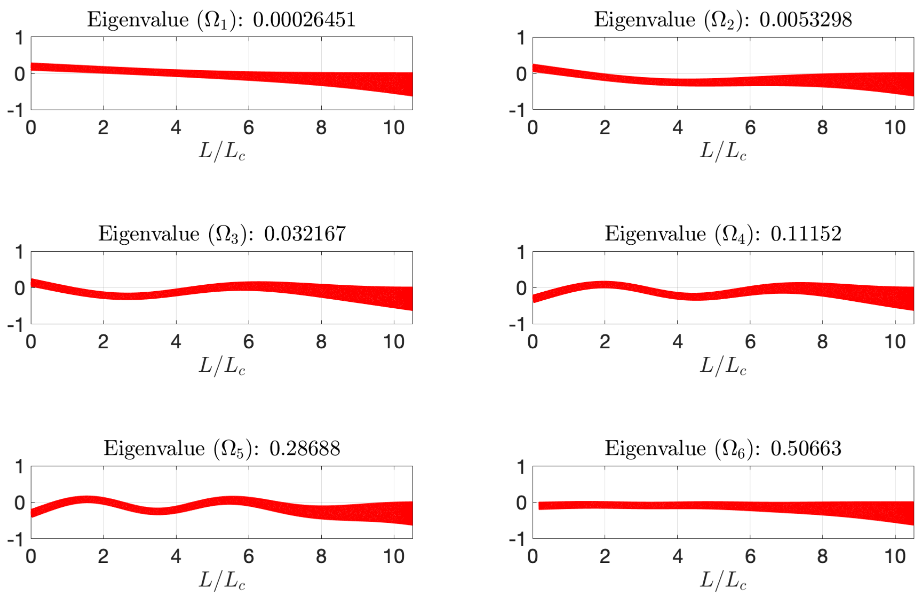


Figure 2: Figure showing the first five eigenfunctions $\boldsymbol{u}_j$ and the corresponding eigenvalues $\Omega_j$ for the eigenvalue problem in (7).

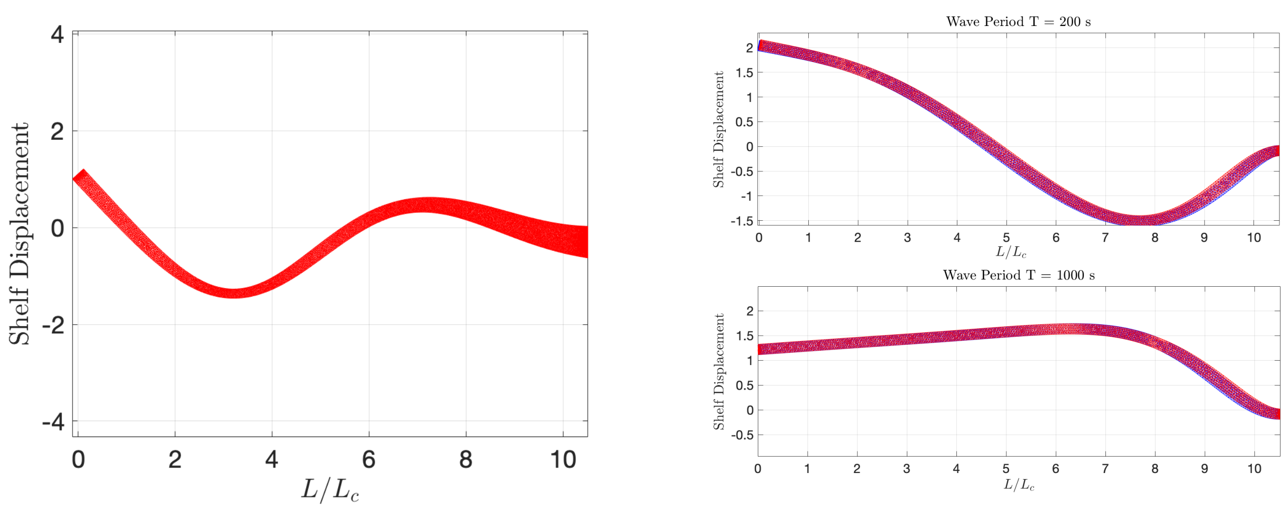


Figure 3: (Left) Figure showing the forced displacement of the ice shelf for an incident wave period $T = 100\,\mathrm{s}$. (Right) Figure comparing the Euler-Bernoulli solution (blue) with the solution of the linear-elasticity model (red) for two incident wave periods and a uniform thickness of $200\,\mathrm{m}$.