On the Subdifferential and Recession Function of the Fitzpatrick Function

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It Began with a Question by Jon

- Given that the Fitzpartick function has extended our knowledge on monotone operators and simplified proofs of known facts.
- Are there any similar outcomes in the area of single-valuedness of monotone mapping.
- Recall $M : X \to X^*$ is monotone iff

$$\langle x - y, x^* - y^* \rangle \geqslant 0 \quad \text{for all } (x, x^*),\ (y, y^*) \in M.$$

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Theorem

Suppose X (an Asplund space) and $M : D \to X^*$ is a maximal monotone operator with $\operatorname{int}\operatorname{dom} M \neq \emptyset$. Then there exists a G_δ subset of $\operatorname{int}\operatorname{dom} M$ on which M is single valued.

- The Fitzpatrick function of M is given by

$$\mathcal{F}_M(x, x^*) = \sup_{(u,u^*) \in M} \{\langle (x, x^*), (u, u^*) \rangle - \langle u, u^* \rangle\} = (\langle \cdot, \cdot \rangle + \delta_M(\cdot)$$

- When M is maximal it is a representative function of M in that $\mathcal{F}_M(x, x^*) \geqslant \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and

$$M = \{(x, x^*) \mid \mathcal{F}_M(x, x^*) = \langle x, x^* \rangle\}.$$

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Still Open

Question: Given that $\mathcal{F}_M$ is convex and so differentiable on a G_δ subset of $\operatorname{int}\operatorname{dom}\mathcal{F}_M \supseteq \operatorname{int}\operatorname{co} M$ can we use information about the differentiability properties of $\mathcal{F}_M$ to deduce results about the single valuedness of M? Jon had hoped to be able to say something this for non-maximal representable operators.

To my knowledge this question remains unanswer to date. This talk will contain a number of partial results that are suggestive that it is worth while to study the differentiability properties of M on X an Asplund space for possibly other reasons as well.

Some More Background

- Any convex function $f : X \times X^* \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ is called representative if $f(x, x^*) \geqslant \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ in which case

$$M_f := \{(x, x^*) \mid f(x, x^*) = \langle x, x^* \rangle\}$$

  is a monotone set. Call $R(M)$ all the representative functions f that represent M in that $M \subseteq M_f$.

- When M is not maximal monotone then $\mathcal{F}_M$ is not representative and indeed

$$\begin{aligned}(M_f)^\mu &:= \{(x, x^*) \mid \mathcal{F}_M(x, x^*) \leqslant \langle x, x^* \rangle\} \\ &= \{(x, x^*) \mid \langle u - x, u^* - x^* \rangle \geqslant 0 \text{ for all } (u, u^*) \in M\}\end{aligned}$$

  are the set of monotonically related points i.e. $(M_f)^\mu = \cup \{T \mid T \text{ montone and } T \supseteq M\}$.

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- Even when M is only monotone then we have

$$\mathcal{P}_M(x,x^*) := \mathcal{F}_M^{*\dagger}(x,x^*) = \overline{\mathrm{co}}(\langle\cdot,\cdot\rangle + \delta_M(\cdot))(x,x^*)$$

is representative (here $\mathcal{F}_M^* : X^* \times X^{**} \to \mathbb{R}_{+\infty}$ and on restricting to $X \subseteq X^{**}$ and using the transpose $\dagger : (x^*,x) \to (x,x^*)$ we get $\mathcal{P}_M : X \times X^* \to \mathbb{R}_{+\infty}$).

- Indeed

$$M \subseteq \{(x,x^*) \mid \mathcal{P}_M(x,x^*) = \langle x,x^*\rangle\}.$$

- As $\mathcal{F}_M \leqslant \mathcal{P}_M$, when M is maximal we say $\mathcal{F}_M$ is a bigger conjugate representative and we denote all $f \in R(M)$ with $f \leqslant f^*$ by $bR(M)$. It turns out that $\mathcal{F}_M$ is the minimal element of $bR(M)$ under the partial order $f \leqslant h$ iff $f(y,y^*) \leqslant h(y,y^*)$ for all (y,y^*).
- It has long been recognised that representable monotone operators M_f possess some properties that make them similar to maximal ones. We will discuss this more.

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What can we say about $\partial \mathcal{F}_M$?

- In general not a lot.
- But we really only need to consider the very special case where $(x, x^*) \in (M_h)^\mu$ and $h \in bR(T)$ to have a useful tool.
- Recall the the ε-sub-differential is given by $\partial_\varepsilon f(x, x^*) = \{(y, y^*) \mid f(v, v^*) - f(x, x^*) \geqslant \langle (y, y^*), (v, v^*) - (x, x^*) \rangle - \varepsilon\}$.

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What can we say about $\partial\mathcal{F}_M^*$?

Proposition

Suppose M is monotone and $\varepsilon \geqslant 0$.

1. *If* $\langle x, x^* \rangle \leqslant \mathcal{F}_M(x, x^*) - \delta$ *for some* $\delta \geqslant 0$ *then we have*

$$\begin{aligned}\partial_\varepsilon \mathcal{F}_M(x, x^*) &= \partial_\varepsilon \mathcal{F}_M(x, x^*) \cap \overline{\text{co}}\, M^\dagger \\ &\subseteq \left\{ (z^*, z) \in \overline{\text{co}}\, M^\dagger \mid \langle x - z, x^* - z^* \rangle \leqslant \varepsilon - \delta \right\}.\end{aligned}$$

2. *When* $\langle x, x^* \rangle \geqslant \mathcal{F}_M(x, x^*) - \delta$ *for some* $\delta \geqslant 0$*, we have*

$$\partial_{\varepsilon+\delta} \mathcal{F}_M(x, x^*) \supseteq \overline{\text{co}} \left\{ (z^*, z) \in M^\dagger \mid \langle x - z, x^* - z^* \rangle \leqslant \varepsilon \right\}.$$

3. *Assume* $\mathcal{F}_M(x, x^*) = \langle x, x^* \rangle$ *and* $(x, x^*) \in (M)^\mu$. *Then we have:*

$$\partial \mathcal{F}_{M_h}(x, x^*) \cap M^\dagger = \left\{ (z^*, z) \in M^\dagger \mid \langle x - z, x^* - z^* \rangle = 0 \right\}.$$

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Question: Under the assumptions $\mathcal{F}_M(x, x^*) = \langle x, x^* \rangle$ and $(x, x^*) \in (M)^\mu$ plus (??) can we say that

$$\partial \mathcal{F}_M(x, x^*) \;=\; \overline{\text{co}}\left\{(z^*, z) \in M^\dagger \mid \langle x - z, x^* - z^* \rangle = 0\right\}?$$

What can we say about single valuedness of M?

The first result Jon and I obtained was the following: Let $\mathcal{M}_M(y, y^*) := \{(z^*, z) \in X^* \times X \mid \langle z - y, z^* - y^* \rangle \leqslant 0\}$

Theorem

Suppose $M : X \rightrightarrows X^*$ is monotone. If there exists $(y, y^*), (y, z^*) \in M$ with $y^* \neq z^*$ (i.e. $T(y) \supseteq \{y^*, z^*\}$ is not unique) then (y^*, y), $(z^*, y) \in \partial \mathcal{F}_M(y, y^*)$ and so $\partial \mathcal{F}_T(y, y^*) \cap M^\dagger$ is also not a singleton. Consequently when $(y, y^*) \in M$ and $\nabla \mathcal{F}_M(y, y^*)$ exists then $M(y)$ is a singleton. More generally we have

$$\begin{aligned}
& \operatorname{diam}\{z^* \mid (z^*, y) \in \partial \mathcal{F}_M(y, y^*)\} \leqslant \varepsilon \\
& \quad \Longrightarrow \operatorname{diam}\left\{z^* \mid (z^*, y) \in \mathcal{M}_M(y, y^*) \cap T^\dagger\right\} \\
& \quad \Longleftrightarrow \operatorname{diam}\left\{z^* \mid (z^*, y) \in \partial \mathcal{F}_T(y, y^*) \cap T^\dagger\right\} \leqslant \varepsilon \\
& \quad \Longrightarrow \quad \operatorname{diam}\{z^* \mid (z^*, y) \in \partial \mathcal{F}_T(y, y^*) \cap (T(y), y)\} \leqslant \varepsilon \\
& \quad \Longrightarrow \quad \operatorname{diam} T(y) \leqslant \varepsilon. \qquad (1)
\end{aligned}$$

- When $\operatorname{int}\operatorname{dom} M \neq \emptyset$ we have many example where $\nabla \mathcal{F}_M$ abundantly exists but there are two short coming to this result.

  1. Even if $\nabla \mathcal{F}_M(y, y^*) = (z, z^*)$ how do we know if $(z, z^*) \in M$?
  2. Again: if $\nabla \mathcal{F}_M(y, y^*)$ how do we know that $(y, y^*) \in M$?

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  2. Again: if $\nabla \mathcal{F}_M(y, y^*)$ how do we know that $(y, y^*) \in M$?

The following is well known.

Proposition

Suppose A is a convex set in an Asplund space X. If x is strongly exposed point of $\overline{\text{co}}A$ then $x \in \overline{A}$.

- We now wish to exploit the fact that $\mathcal{P}_M$ is largest closed convex function that interpolates the points $(y, y^*, \langle y, y^* \rangle)$ for $(y, y^*) \in M$.

A fix for the first problem when $X \times X^*$ is Asplund?

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The Fix

Corollary

Suppose $M : X \rightrightarrows X^*$ is a monotone operator with closed graph, $(y, y^*) \in X \times X^*$, an Asplund spaced and $\nabla \mathcal{F}_M(y, y^*) = (a^*, a)$ exists as a Fréchet derivative. Then $(a, a^*) \in M$.

Proof.

As $\nabla \mathcal{F}_T(y, y^*) = (a, a^*)^\dagger$ exists as a Fréchet derivative in the Asplund space $X \times X^*$ then $(a, a^*, \mathcal{F}_M(a, a^*))$ is strongly exposed by $(y, y^*, -1)$ in epi $\mathcal{P}_M$. Now epi $\mathcal{P}_M$ is the closed convex hull of the set

$$A := \{(u, u^*, \langle u, u^* \rangle + \gamma) \mid (u, u^*) \in M, \gamma \geqslant 0\}.$$

Thus by Proposition 4 we have $(a, a^*, \langle a, a^* \rangle) \in \overline{M}$ and so (a, a^*) is in the closure of M so $(a, a^*) \in M$. □

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More on the Fix

- For convenience we will refer to (a, a^*) rather than $(a, a^*, \mathcal{F}_M(a, a^*))$ as a strongly exposed point of epi $\mathcal{F}_M$.

Corollary

Suppose $M : X \rightrightarrows X^*$ a monotone operator with closed graph and $X \times X^*$ is and Asplund space. Then the strongly exposed points of epi $\mathcal{P}_M$ *are all contained in M.*

Proof.

For (a, a^*) to be a strongly exposed point of epi $\mathcal{P}_M$ we need $\nabla \mathcal{F}_M(y, y^*) = (a, a^*)^\dagger$ for some $y \in \operatorname{dom} \mathcal{F}_M$ but then $(a, a^*) \in M$. □

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Recall that a fundamental property of the Fitzpatrick function is

$$M \subseteq \{(z, z^*) \mid (z^*, z) \in \partial \mathcal{F}_M(z, z^*)\} \quad \text{when } M \text{ is monotone}$$
$$\text{and } M = \{(z, z^*) \mid (z^*, z) \in \partial \mathcal{F}_M(z, z^*)\} \quad \text{when } M \text{ is maximal monoton}$$

Corollary

Suppose $M : X \rightrightarrows X^*$ is a monotone operator with a closed graph, $(y, y^*) \in X \times X^*$ and $\nabla \mathcal{F}_M(y, y^*) = (a^*, a)$ exists as a Fréchet derivative with $(y, y^*) \neq (a, a^*)$. Then $\partial \mathcal{P}_M(a, a^*)$ is not a singleton, indeed $(y, y^*), (a, a^*) \in \partial \mathcal{P}_M(a, a^*)$.

Proof.

As $\nabla \mathcal{F}_M(y, y^*) = (a^*, a) \neq (y^*, y)$ by duality $(y^*, y) \in \partial \mathcal{P}_M(a, a^*)$. As $(a, a^*) \in M$ we have $\langle a, a^* \rangle = \mathcal{F}_M(a, a^*) = \mathcal{P}_M(a, a^*)$ and $\mathcal{F}_M \leqslant \mathcal{P}_M$. Thus $(a^*, a) \in \partial \mathcal{F}_M(a, a^*) \subseteq \partial \mathcal{P}_M(a, a^*)$. □

Recall that a fundamental property of the Fitzpatrick function is

$$M \subseteq \{(z, z^*) \mid (z^*, z) \in \partial \mathcal{F}_M(z, z^*)\} \quad \text{when } M \text{ is monotone}$$
$$\text{and } M = \{(z, z^*) \mid (z^*, z) \in \partial \mathcal{F}_M(z, z^*)\} \quad \text{when } M \text{ is maximal monoton}$$

Corollary

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What we have so far

- Thus $\nabla \mathcal{F}_M(y, y^*) = (a^*, a)$ exists and $\nabla \mathcal{P}_M(a, a^*)$ exists implies $\nabla \mathcal{F}_M(y, y^*) = (y^*, y) = \nabla \mathcal{P}_M(y, y^*)$ with $(y, y^*) \in M$.
- When X is reflexive then $X \times X^*$ is reflexive and so $X \times X^*$ Asplund.

Question: Outside of X reflexive is $X \times X^*$ ever Asplund? Has anyone studies the differentiability properties of convex functions $f : X \times X^* \to \mathbb{R}_{+\infty}$ when X is Asplunds?

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More Problems

- The simplest of examples is the mod function $x \to f(x) = |x|$ where we have $f^*(x^*) = \delta_{B_1}(x^*)$ and

$$\mathcal{F}_{\partial f}(x, x^*) = f(x) + f^*(x^*) = \begin{cases} |x| & \text{if } |x^*| \leqslant 1 \\ +\infty & \text{otherwise.} \end{cases}$$

- Here

$$\operatorname{Graph} \partial f(x) \notin \operatorname{int} \operatorname{dom} \mathcal{F}_{\partial f} = \operatorname{int} \operatorname{dom} \mathcal{P}_{\partial f}.$$

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An improved version

It make sense to only look for partial differentiability instead.

Corollary

Suppose $M : X \rightrightarrows X^*$ is monotone and $(y, y^*) \in M$. Suppose in addition $\nabla_x \mathcal{F}_M(y, y^*)$ exists then $M(y) = \{y^*\}$ is a singleton.

Proof.

Suppose $\nabla_x \mathcal{F}_M(y, y^*)$ exists. First note that we always have $(y, y^*) \in \mathcal{M}_M(y, y^*) \cap M \subseteq \partial \mathcal{F}_M(y, y^*)$ and so $y^* \in \partial_x \mathcal{F}_M(y, y^*)$ and $\nabla_x \mathcal{F}_M(y, y^*) = \{y^*\}$. Thus $\partial F_M(y, y^*) = \{y^*\} \times \partial_{x*} F_M(y, y^*)$.

If $M(y)$ is not a singleton then there exists $(y, y^*), (y, z^*) \in \text{Graph}\, M$ with $y^* \neq z^*$ which implies by Theorem 3 that

$$(y^*, y), (z^*, y) \in \partial \mathcal{F}_M(y, y^*) = \{y^*\} \times \partial_{x*} \mathcal{F}_M(y, y^*)$$

in which case $y^* = z^*$, a contradiction. Thus $M(y) = \{y^*\}$. $\square$

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More Improvements?

- We need a replacement of the strong exposure result to now turn the last result into a useful tool.
- This is much more involved, mainly due to the use or partial conjugates. This makes contact with the older approach to representative characterisation of monotone operator due to Krauss.

Krauss, Eckehard (1985) *A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions*. Nonlinear Anal. 9, no. 12, 1381–1399.

- Indeed when we form the (partial) conjugate

$$\mathcal{F}_M^*(\cdot, y^*)\,(a^*)$$

we obtain a skew saddle functions.

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Question: What can be said about the case when we have

$$\nabla \mathcal{F}_M(\cdot, y^*)(y) = a^* \text{ and } \nabla \mathcal{F}_M^*(\cdot, y^*)(a^*) \text{ existing.}$$

Can we have
$\nabla \mathcal{F}_M(\cdot, y^*)(y) = \nabla \mathcal{F}_M^*(\cdot, y^*)(a^*) = a^* = y^* \in M(y)$?

Or something that relates these points to the graph of M? Does this enlighten the structure of $\partial \mathcal{F}_M(y, y^*)$ given it is determined via limits.

Minimality in $bR(M)$ and Maximality?

One of the seminal results on monotone operators in reflexive spaces if due to Burachik and Svaiter that relates maximal monotone representable sets M_f to $f \in bR(M)$. In reflexive spaces all M_f for $f \in bR(M)$ are maximal:

$$\left.\begin{array}{l} T : X \rightrightarrows X^* \text{ maximal and} \\ f \text{ representative for } T \end{array}\right\} \iff \begin{array}{l} f(x, x^*) \geqslant \langle x, x^* \rangle \quad \text{and} \\ f^*(x^*, x) \geqslant \langle x^*, x \rangle, \forall (x, x^*) \end{array}$$

R. S. Burachik and B. F. Svaiter (2003), *Maximal Monotonicity, Conjugation, and the Duality Product*, Proc. Amer. Math. Soc. **132**(8), 2379–2383.

Later Martínez-Legaz and Svaiter pointed out that the existence of a minimal element in $bR(M)$ is a consequence of this result.

J.-E. Martínez-Legaz and B. F. Svaiter (2008), *Minimal convex functions bounded below by the duality product*. Proc. Amer. Math. Soc. **136**(3), 873–878.

Minimality and Maximality?

From Fitzpatrick we know that the minimal element exists, indeed:

Proposition

Let M be a maximal monotone extension of T. Then $\mathcal{F}_M$ is a minimal element of $[bR(T), \leqslant]$. Hence also, $\mathcal{F}_M$ is the unique minimal element of $bR(M)$.

Via Simons we know:

Lemma

Let $T : X \rightrightarrows X^*$ be a monotone operator, let $k, h \in bR(T)$ for which $h \leqslant k$. Then $M_k = M_h \supseteq T$.

It is not immediately clear that within an arbitrary Banach space *all* such minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions. We shall call BMLS (**B**urachik, **M**artínez-**L**egaz, **S**vaiter) spaces to be those Banach spaces X for which (for all monotone operators T on X) all minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions.

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BMLS Spaces

The following are shown to be equivalent to X being a BMLS space in:

A. Eberhard, R. Wenczel, *On the Maximal Extensions of Monotone Operators and Criteria for Maximality*, J. Convex Analysis, Vol 23, no. 4, 2016.

1. The space X has the property that for every monotone T, every minimal element f in $[R(T), \leqslant]$ represents a maximal monotone set M_f.
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It is an open question as to whether exist any non-reflexive BMLS spaces or indeed if all real Banach spaces which are BMLS spaces. All desirable properties of monotone operators hold here and we have the "sum theorem" holding in such spaces. In this paper all conditions have been made necessary and sufficient in order to set up a straw man.

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4. *For any maximal monotone extension $M \supseteq M_f$, we have that* $\mathcal{F}_{M_f} \leqslant \mathcal{F}_M \leqslant f$.
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The Recession Function?

Suppose $h \in bR(T)$. Then

$$\mathcal{F}_{M_h} 0^+ = \delta^{*\dagger}_{\overline{\mathrm{co}} M_h} \tag{2}$$

Furthermore, for any $\alpha > \inf \mathcal{F}_{M_h}$,

$$0^+ [\mathcal{F}_{M_h} \leqslant \alpha] \subseteq \mathrm{dom}\, \delta^{*\dagger}_{\overline{\mathrm{co}} M_h} \subseteq 0^+ \,\mathrm{dom}\, \mathcal{F}_{M_h}. \tag{3}$$

One can use this to show results like the following.

Proposition

Let $h \in bR(T)$, and suppose M_h is not maximal, and that $\mathrm{dom}\, M_h$ *is bounded. Then there exists $(x, x^*) \in (M_h)^{\mu} \cap (M_h)^c$ for which*

$$\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle < h(x, x^*). \tag{4}$$

We observe that ε-subdifferentials of the Fitzpatrick function is then nonempty and meets the graph of the operator.

Proposition (Eberhard and Wenczel)

Suppose $h \in bR(T)$ *and* $(x, x^*) \in (M_h)^\mu$ *with* $\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle$. *Then*

$$\partial_\varepsilon \mathcal{F}_{M_h}(x, x^*) \cap M_h^\dagger \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

Extending this to the following (under the assumption that $\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle$) seems possible

$$\partial \mathcal{F}_{M_h}(x, x^*) = \bigcap_{\epsilon > 0} \overline{\mathrm{co}} \left[\partial_\epsilon \mathcal{F}_{M_h}(x, x^*) \cap M_h^\dagger \right].$$

to go further requires the extraction of bounded nets when approximating.

Subdifferentials and Maximality

In a recent publication we show the following results.

Lemma (Lemma 7, Eberhard & Wenczel)

Suppose $f \in bR(T)$ with M_f not maximal.

1. *There cannot exist $(x, x^*) \in (M_f)^\mu \cap (M_f)^c$ with $\partial f(x, x^*) \cap M_f^\dagger \neq \emptyset$.*
2. *There cannot exist $(x, x^*) \notin M_f$ with $\partial_\varepsilon f(x, x^*) \cap (\{(x^*, x)\})^\mu \neq \emptyset$, for any ε such that $0 < \varepsilon < f(x, x^*) - \langle x, x^* \rangle$.*

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Question: Can we get similar results using the Fitzpatrick function instead?

What we know about these subdifferentials

Proposition

Suppose $h \in bR(T)$, *and* $(x, x^*) \in (M_h)^{\mu}$ *(so* $\langle x, x^* \rangle < h(x, x^*)$*). Then given* $\gamma := h(x, x^*) - \langle x, x^* \rangle$

$$\partial_{\gamma} h(x, x^*) \cap \{(x^*, x)\}^{\mu} = \partial_{\gamma} h(x, x^*) \cap M_h^{\dagger}$$

and when $\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle$, *we also have*

$$\partial_{\varepsilon} \mathcal{F}_{M_h}(x, x^*) \subseteq \partial_{\gamma+\varepsilon} h(x, x^*) \quad \text{for all } \varepsilon \geqslant 0 \text{ and}$$

$$\partial_{\varepsilon} \mathcal{F}_{M_h}(x, x^*) \cap M_h = \partial_{\gamma+\varepsilon} h(x, x^*) \cap M_h^{\dagger} \quad \text{for all } \varepsilon \geqslant 0.$$

In particular, when $\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle$, *we have that* $\partial_{\gamma} h(x, x^*) \cap \{(x^*, x)\}^{\mu} \neq \varnothing$ *iff* $\partial \mathcal{F}_{M_h}(x, x^*) \cap M_h^{\dagger} \neq \varnothing$ *and moreover*

$$\partial_{\gamma} h(x, x^*) \cap \{(x^*, x)\}^{\mu} = \partial \mathcal{F}_{M_h}(x, x^*) \cap M_h^{\dagger} = \partial \mathcal{F}_{M_h}(x, x^*) \cap \{(x^*, x)\}^{\mu}.$$

Conclusion

- The study of the sub-differential of representative functions and the associated Fitzpartick function throws up an entirely different way to pose questions about single valuedness and maximality of representable monotone sets M_h for $h \in bR(T)$. These are essential those monotone sets that extend T in a representable fashion.
- Many related problems remain open regarding maximality and differentiability.

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