Meta-optimisation: Lower bounds for higher faces

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In a standard optimisation problem, we have a domain P (possibly a polytope), a reasonable function $g : P \to \mathbb{R}$ (possibly convex), and we wish to find

$$\min_{x \in P} f(x)$$

or perhaps

$$\max_{x \in P} f(x).$$

We will be interested in another optimisation problem; our domain $\mathcal{P}$ will be a collection of polytopes (of the same dimension), and for some natural functions $f : \mathcal{P} \to \mathbb{R}$ we want to find

$$\min_{P \in \mathcal{P}} f(P).$$

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Precise upper bounds for the numbers of m-dimensional faces were obtained in 1970 by McMullen and Shephard, so we will concentrate on lower bounds.

Barnette (1973) established a precise lower bound for *simplicial* polytopes, but for general polytopes, lower bounds are not so easy to obtain.

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McMullen (1971) proved this conjecture for *facets*, i.e. for the case $m = d-1$ and for all $v \leq 2d$; he actually calculated $\min F_{d-1}(v,d)$ for all $v \leq 2d + \frac{1}{4}d^2$.

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Until 2014, no further progress had been made on this problem.

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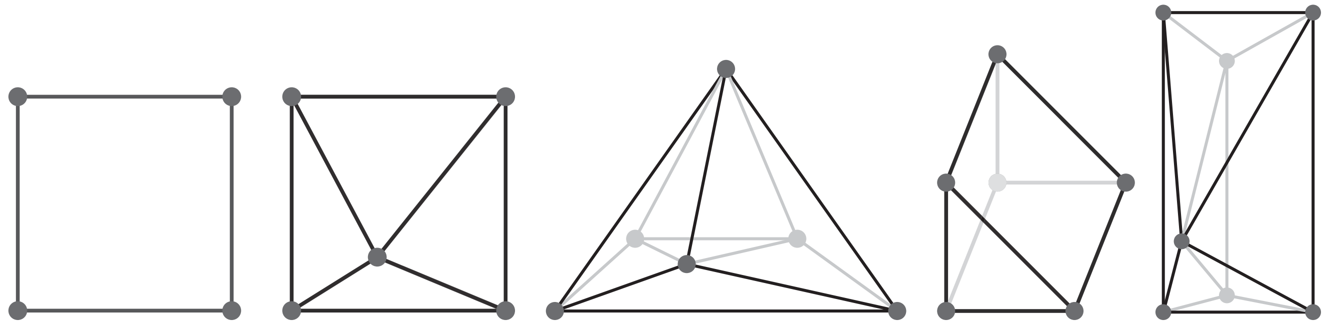
Let us remark that for all d, and all sufficiently large v, we have $\min F_1(v, d) = \frac{1}{2}vd$ if either v or d is even (known), and

$\min F_1(v, d) = \frac{1}{2}(v + 1)d - 1$ if both v and d are odd (new).

Theorem

Let P be a d-dimensional polytope with $d + k$ vertices, where $0 < k \leq d$.
(i) If P is a $(d - k)$-fold pyramid over the k-dimensional prism based on a simplex, then P has $\phi_1(d + k, d)$ edges.
(ii) Otherwise P has $> \phi_1(d + k, d)$ edges.



(a) $P_2 = M(2,0)$ (b) $M(2,1)$ (c) $M(2,2)$ (d) P_3 (e) $M(3,1)$

FIGURE 1. Triplices

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The polytope described in (i) will be called a triplex, and denoted $M_{k,d-k}$.
In fact, the set $F_1(d + k, d)$ contains gaps if $k \geq 4$; the number of edges of a non-minimising polytope is at least

$$\phi_1(d + k, d) + \max\{2, k - 3\}.$$

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provided $d \leq 15$, or $d = 16$ is we drop the uniqueness claim.

For the case $m = d - 1$, i.e. for facets, we recall the results of McMullen:

Theorem

Fix k with $2 \leq k \leq d$. Then
(i) $\min F_{d-1}(d+k, d) = \phi_{d-1}(d+k, d) = d+2$;
(ii) the minimum is attained by $M_{k,d-k}$;
(iii) the minimiser is unique, i.e. there is only one polytope with $d+k$ vertices and $d+2$ facets, if and only if $k-1$ is not composite (i.e. $k = 2$ or $k-1$ is a prime number).

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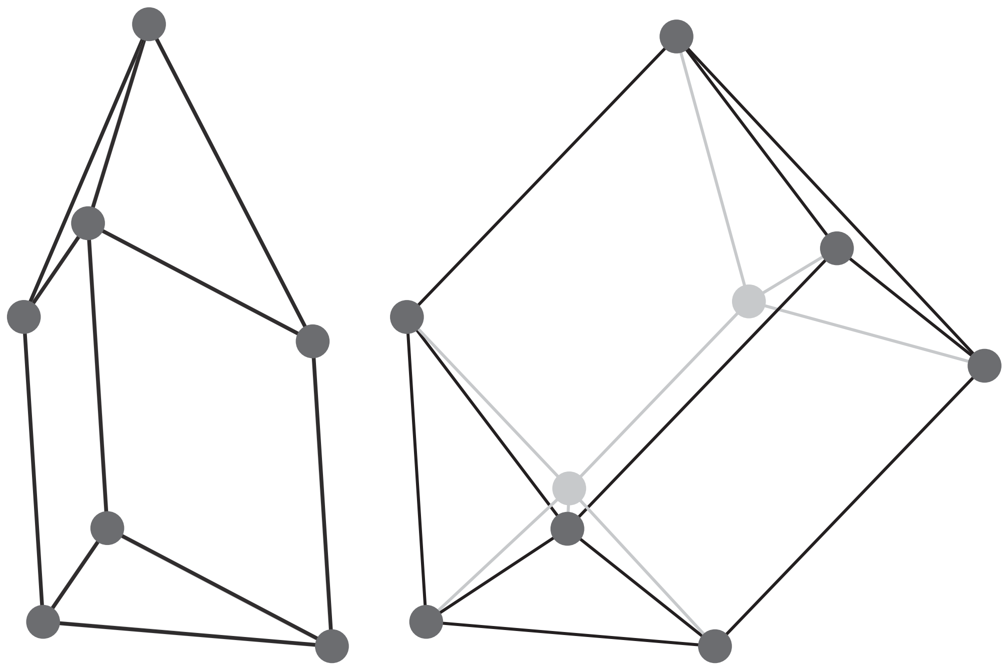
And for more than $2d$ vertices:

Theorem

Fix $k > d$. Then there is a polytope P with $d+k$ vertices and $d+2$ facets if, and only if, $k-1$ is a product of integers, say mn, with $m+n \leq d$. Different decompositions of $k-1$ give rise to combinatorially distinct polytopes.

And now, $2d + 1$ vertices: we can also calculate $\min F_m(2d + 1, d)$ for $m = 1$, $m = d - 1$ and $m = d - 2$. The answer depends on some number theory.
Slicing one corner from the base of a square pyramid yields a polyhedron with 7 vertices and 6 faces, one of them a pentagon. We call this a *pentasm*.



(a) *Pentasm3* (b) *Pentasm4*

FIGURE 2. Pentasms

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We will use the same name for the higher-dimensional version, obtained by slicing one corner from the quadrilateral base of a $(d-2)$-fold pyramid. It has $2d+1$ vertices and can also be represented as the Minkowski sum of a d-dimensional simplex, and a line segment which lies in the affine span of one 2-face but is not parallel to any edge.

First, edges:

Theorem

Let P be a d-dimensional polytope with $2d+1$ vertices.
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(i) If P is d-dimensional pentasm, then P has d^2+d-1 edges.
(ii) Otherwise the numbers of edges is $> d^2+d-1$, or P is the sum of two triangles.

This shows that the pentasm is the unique minimiser of the number of edges if $d \geq 5$.
If $d=4$, the sum of two triangles has 9 vertices, and is the unique minimiser, with only 18 edges.
If $d=3$, the sum of two triangles can have 7, 8 or 9 vertices; the example with $v=7$ has 11 edges, the same as the pentasm.
Summarising, $\min F_1(9,4)=18$, and
$\min F_1(2d+1,d)=d^2+d-1$ for all $d \neq 4$.

Then, facets (McMullen):

Theorem

Consider the class of d-polytopes with $2d + 1$ vertices.
(i) If d is a prime, then the pentasm has the minimal number of facets, namely $d + 3$, but it is not the unique minimiser.
(ii) If d is a product of 2 primes, the minimal number of facets is $d + 2$, and the minimiser is unique.
(iii) If d is a product of 3 or more primes, the minimal number of facets is $d + 2$, and the minimiser is not unique.

Finally, ridges:

Theorem

Consider the class of d-polytopes with $2d+1$ vertices.
(i) If d is a prime, the minimal number of ridges is $\frac{1}{2}(d^2+5d-2)$, and the pentasm is the unique minimiser.
(ii) If d is a product of two primes, the minimal number of ridges is $\frac{1}{2}(d^2+3d+2)$, and the minimiser is unique.
(iii) If d is a product of three or more primes, the minimal number of ridges is $\frac{1}{2}(d^2+3d+2)$, and the minimiser is not unique.

Theorem

Let P be a d-dimensional polytope with $2d + 2$ vertices, where $d \geq 8$, $d = 6$ or $d = 3$.

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If $d = 7$, there is a third minimising polytope with 16 vertices and 60 edges.
If $d = 4$, there two more minimising polytopes with 10 vertices and 21 edges.
If $d = 5$, the unique minimiser is the sum of a tetrahedron and triangle; this clearly has 12 vertices and 30 edges; $30 < 32$.
Summarising, $\min F_1(12, 5) = 30$, and
$\min F_1(2d+2, d) = d^2 + 2d - 3$ for all $d \neq 5$.
The case of $2d + 3$ vertices appears to be difficult.

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There are 708 polyhedra with 16 or fewer edges; with D. Briggs, we have classified 703 of them as decomposable or indecomposable.

Some discussion of methods?

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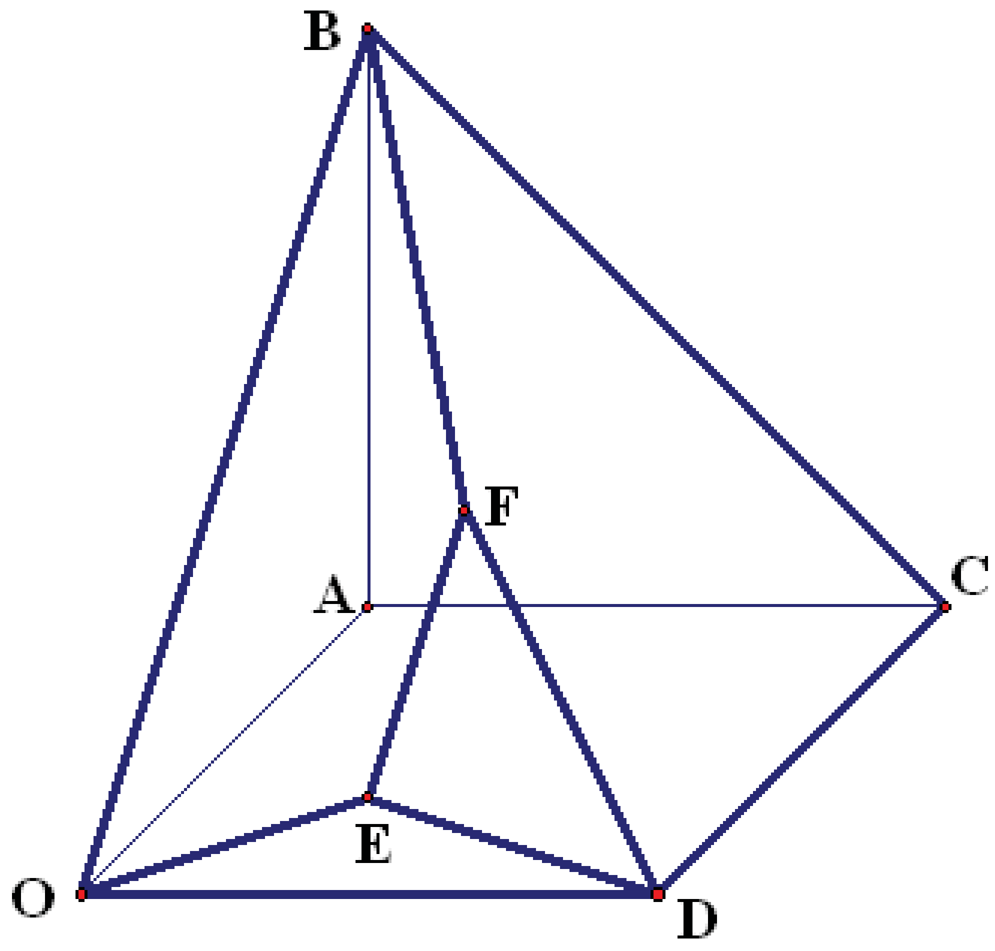
More generally, affinely independent cycles are (with a suitable definition) indecomposable geometric graphs.

In particular, if a polytope contains an affinely independent cycle, which touches every maximal face, then it is indecomposable.

Some examples: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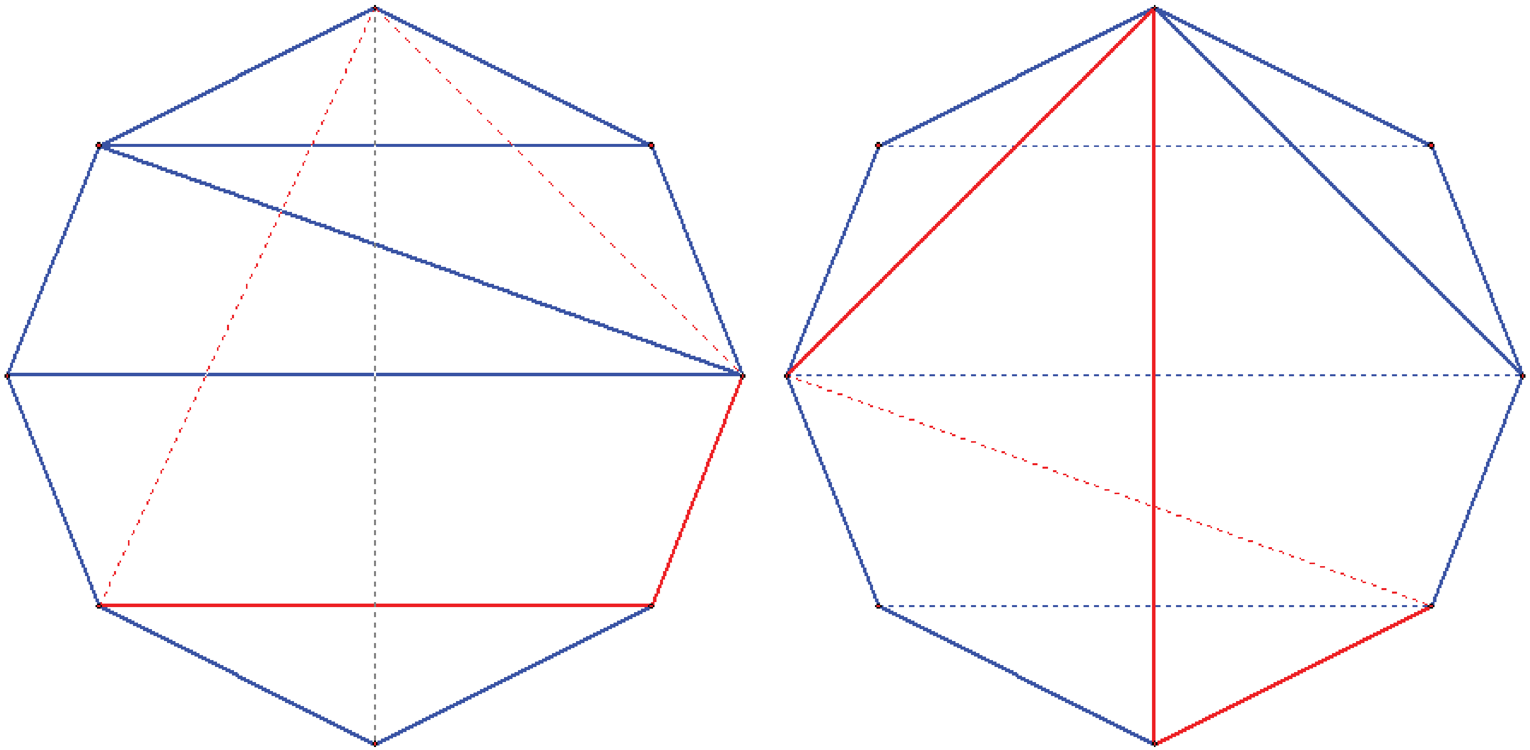

Figure 2: BD173 and BD179

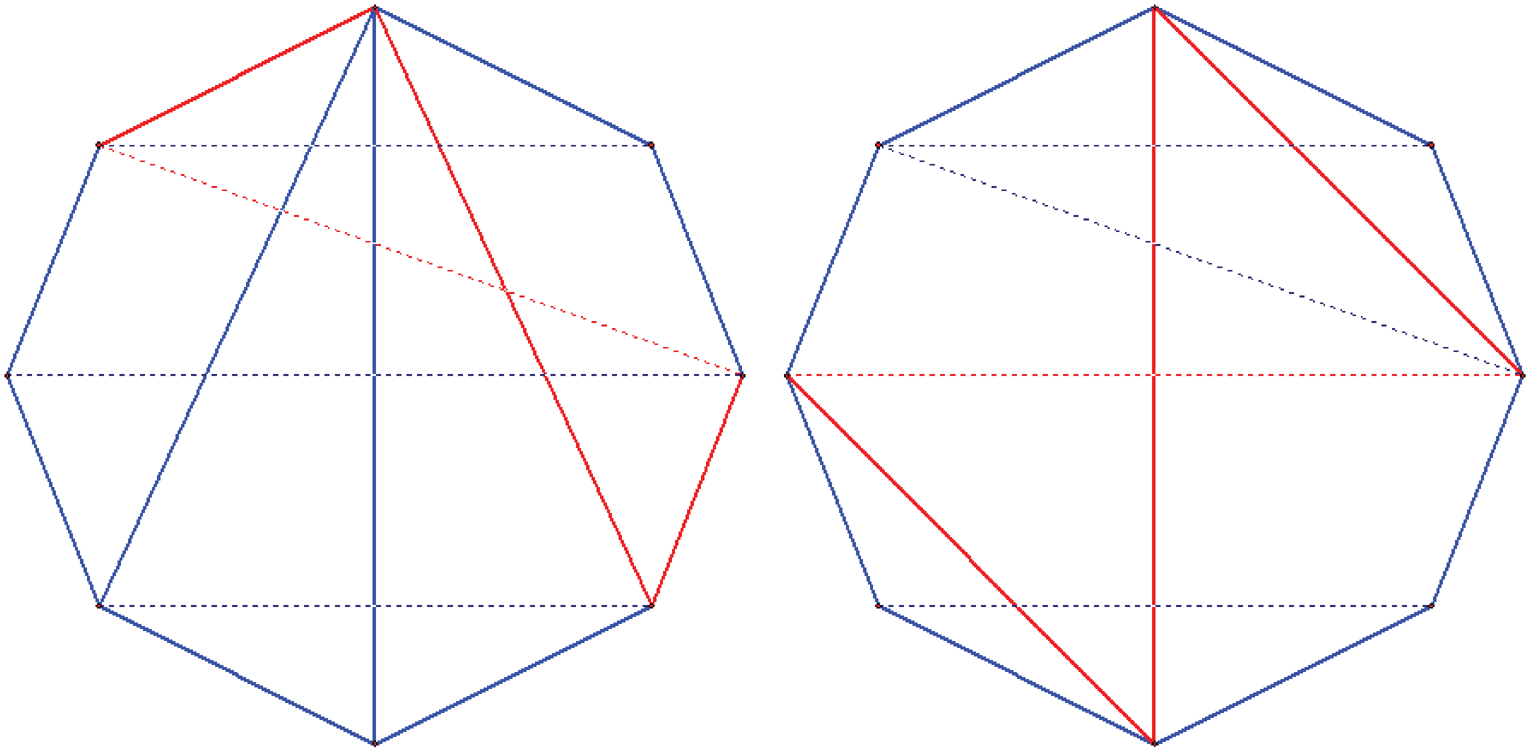

Figure 3: BD187 and BD190

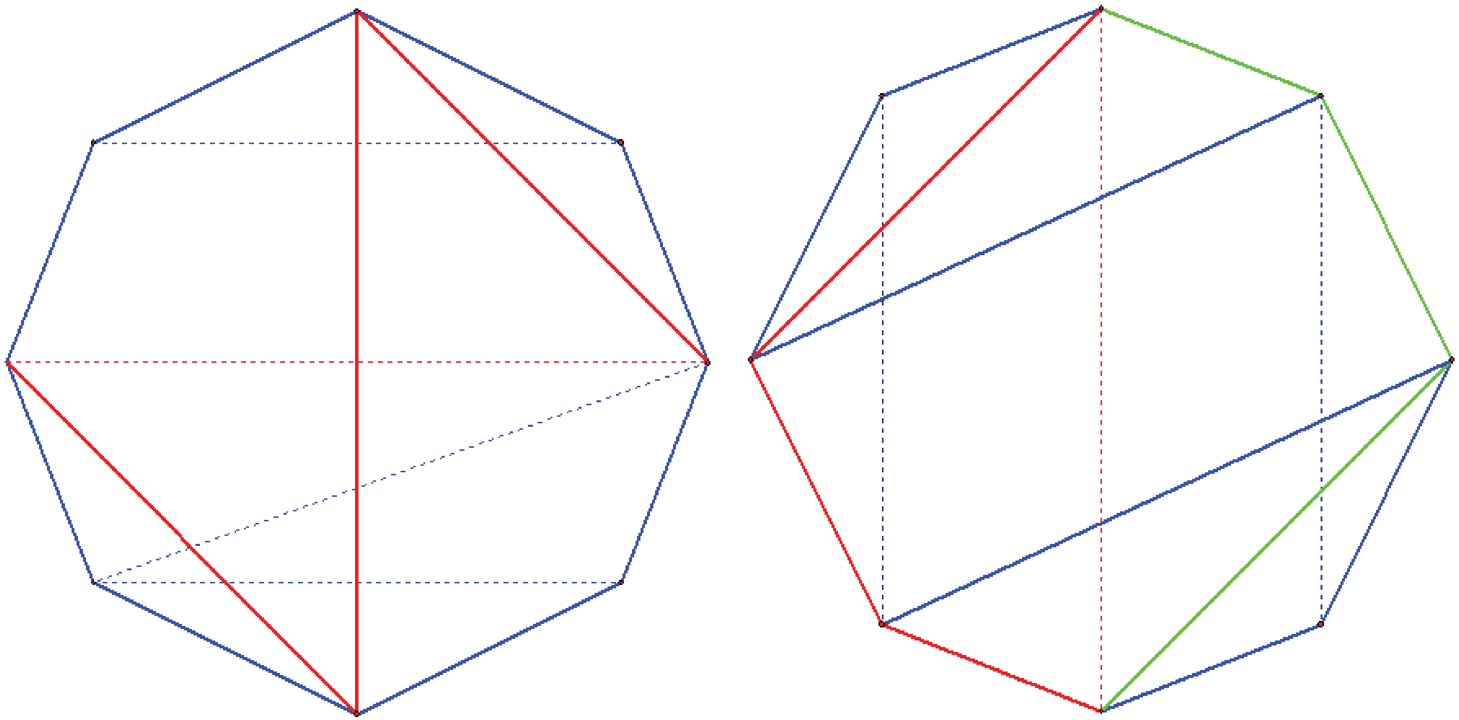

Figure 4: BD192 and BD199

Thank you for your attention