Entropy Maximization in Finance

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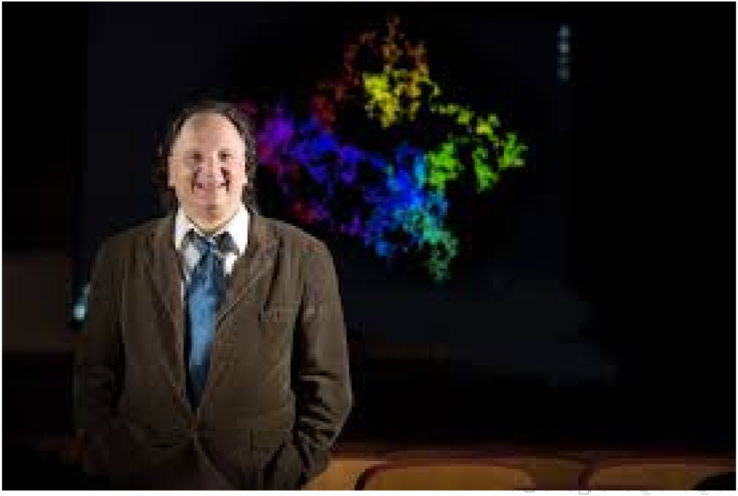
Meeting Jon for the first time

Inspiring leader at CECM and CARMA

CAMAR



Unregulated Kangaroo

What Jon had taught me

- Be curious, working on different projects.
- To explore, using computers when you can.
- Be engaged.
- "I think in blocks."

The joint work with Jon below is a nice illustration.

Entropy Maximization

- J. W. Gibbs proposed the principle of maximum entropy in statistical mechanics in 1902 based on earlier work of Ludwig Boltzmann (1870).
- E. T. Jaynes (1957) relates this principle to Shannon's Information Theory.
- The macroscopic state of a system is defined by a distribution on the microstates.
- Observations (finite) provide constraints on such a distribution but cannot completely determine it.
- The principle of maximum entropy chose one among all possible distributions that contains least information.

Mathematical formulation

Mathematical formulation

$$\min[f(x) : Ax = b]$$

where $-f$ is the entropy and $Ax = b$ models the observation.

Entropy maximization and its dual

If constraint qualification (CQ) condition

$$b \in \mathrm{ri}\ A^\top \mathrm{dom}\ f$$

holds then

$$\begin{aligned} p &:= \inf_x [f(x) : Ax = b] \\ &= \max_z [\langle z, b\rangle - f^*(A^\top z)] = (f^* \circ A^\top)^*(b) =: d. \end{aligned} \tag{1}$$

Let $\bar{x}, \bar{z}$ be solutions to p, d, respectively, Then

$$\bar{x} \in \partial f^*(A^\top \bar{z}) = (\partial f)^{-1}(A^\top \bar{z})$$

$$A^\top \bar{z} \in \partial f(\bar{x}) = (\partial f^*)^{-1}(\bar{x})$$

and $\bar{z}$ is the Lagrange multiplier of problem p.

Convex duality

Entropy maximization is a special case of the more general convex duality theory, which has wider application in finance, see:

Peter Carr and Qiji J. Zhu

Convex Duality and Financial Mathematics

However, problems that can be fit into the entropy maximizarion framework can take advantage of its special structure.

Economy

- State of an economy is represented as a probability space (Ω, P) with a finite sample space $\Omega = \{\omega_1, \dots, \omega_N\}$.
- We use $RV(\Omega, P)$ to denote the Hilbert space of all random variables endowed with the inner product

$$\langle x, y \rangle = \mathbb{E}(xy) = \sum_{\omega \in \Omega} x(\omega) y(\omega) P(\omega)$$

- Elements in $RV(\Omega, P)$ represent the payoffs of assets.
- We consider the one period model in which transaction can only take place at either the beginning $(t = 0)$ or the end of the period $(t = 1)$.

Financial Markets

- A financial market is modeled by random vectors

$$S_t = (S_t^0, S_t^1, \ldots, S_t^M), t = 0, 1 \ on \ \Omega.$$

- S_t^0 represent the price of a risk free bond so that $R = S_1^0/S_0^0$, is a constant.
- and $\hat{S}_t = (S_t^1, \ldots, S_t^M)$ represent the prices of risky assets at time t.
- We assume that S_0^i is a constant (current price is known) and S_1^i is a random variable (ending price depends on economic status).

Portfolio

Portfolio

A portfolio is a vector $x = [x_0, x_1, \ldots, x_M]^\top \in \mathbb{R}^{M+1}$ where x_i signifies the number of units of the ith asset. The value of a portfolio x at time t is $S_t \cdot x$, where notation "$\cdot$" signifies the usual dot product in $\mathbb{R}^{M+1}$.

Markowitz portfolio problem

Given a portfolio of risky assets and a fixed return, minimizing the variance as a measure for the risk.

Use $\hat{x} = (x_1, \ldots, x_M)$ to denote a portfolio corresponding to risk assets. For a given expected payoff μ and an unit initial wealth 1 one seeks to

$$\begin{aligned} \text{minimize} \quad & \mathrm{Var}(\hat{S}_1 \cdot \hat{x}) \\ \text{subject to} \quad & \mathbb{E}[\hat{S}_1 \cdot \hat{x}] = \mu \\ & \hat{S}_0 \cdot \hat{x} = 1. \end{aligned} \tag{2}$$

Markowitz portfolio problem

We can rewrite the primal as an entropy maximization problem

$$\begin{array}{rl} p := \min\limits_{x} & f(x) := \frac{1}{2} x^{\top} \Sigma x \\ \text{subject to} & Ax = b. \end{array} \tag{3}$$

Here $\mathbb{E}(\hat{S}_1) = [\mathbb{E}(\hat{S}_1^1), \ldots, \mathbb{E}(\hat{S}_1^M)]$,

$$A = \left[\begin{array}{c} \mathbb{E}(\hat{S}_1) \\ \hat{S}_0 \end{array} \right], \text{ and } b = \left[\begin{array}{c} \mu \\ 1 \end{array} \right],$$

and

$$\begin{aligned} \Sigma &= \mathbb{E}[(\hat{S}_1 - \mathbb{E}(\hat{S}_1))^{\top}(\hat{S}_1 - \mathbb{E}(\hat{S}_1))] \\ &= (\mathbb{E}[(S_1^i - \mathbb{E}(S_1^i))(S_1^j - \mathbb{E}(S_1^j))])_{i,j=1,\ldots,M}, \end{aligned} \tag{4}$$

is assumed to be a full rank covariance matrix.

Dual

The dual problem is

$$d := \max_{y} \; b^\top y - \frac{1}{2} y^\top A\Sigma^{-1}A^\top y$$

with a solution

$$\bar{y} = (A\Sigma^{-1}A^\top)^{-1}b, \; d = \frac{1}{2} b^\top (A\Sigma^{-1}A^\top)^{-1}b.$$

- The dual as a maximization problem without constraint is easier to solve than the primal.
- The dual is also simpler for having only 2 variables.

From dual to primal solution

CQ is $b \in \mathrm{range} A$. Assuming CQ is satisfied, strong duality implies that $p = d$. The relationship between primal and dual solution leads to

$$\bar{x} = (f^*)'(A^\top \bar{y}) = \Sigma^{-1} A^\top \bar{y}.$$

Thus the optimal portfolio is

$$\bar{x} = \Sigma^{-1} A^\top (A \Sigma^{-1} A^\top)^{-1} b.$$

Since $b = (\mu, 1)$ this is an affine function of the return μ.

Markowitz bullet

Let $\alpha = \mathbb{E}(\hat{S}_1)\Sigma^{-1}\mathbb{E}(\hat{S}_1)^\top$, $\beta = \mathbb{E}(\hat{S}_1)\Sigma^{-1}\hat{S}_0^\top$ and $\gamma = \hat{S}_0\Sigma^{-1}\hat{S}_0^\top$.
We have

Markowitz bullet

Given unit wealth 1 and expected payoff μ, the minimum standard deviation is determined by

$$\sigma(\mu) = \sqrt{\frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2}} \tag{5}$$

and

$$\hat{x}(\mu) = \frac{\mathbb{E}(\hat{S}_1)(\gamma\mu - \beta) + \hat{S}_0(\alpha - \beta\mu)}{\alpha\gamma - \beta^2}\Sigma^{-1} \tag{6}$$

Markowitz bullet: graphic illustration

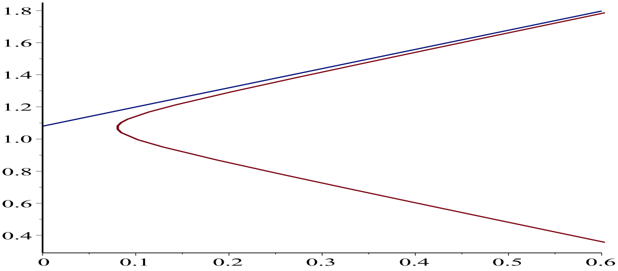


Figure: Markowitz Bullet

The upper half is called the Markowitz efficient frontier.

Two fund Theorem

The affine relationship

$$\hat{x}(\mu) = \frac{\mathbb{E}(\hat{S}_1)(\gamma\mu - \beta) + \hat{S}_0(\alpha - \beta\mu)}{\alpha\gamma - \beta^2}\Sigma^{-1}$$

leads to

Two fund Theorem

Any portfolio on the Markowitz frontier can be represented as a linear combination of two distinct portfolios on this frontier.

Math: two point determines a line.
Implications: For investing two good mutual funds suffice.

Capital asset pricing model

Capital asset pricing model (CAPM) deals with the problem of pricing risky asset. It is based on the principle that adding an appropriately priced asset should not change the structure of the existing market mix (in sense of Markowitz efficiency).

The core is a generalization of the Markowitz portfolio theory by allowing risk free asset in the portfolio (Treynor, Sharpe,Lintner,Mossin, 1961–1966).

Optimization problem

$$\begin{aligned} \text{minimize} \quad & \operatorname{Var}(S_1 \cdot x) \\ \text{subject to} \quad & \mathbb{E}[S_1 \cdot x] = \mu \\ & S_0 \cdot x = 1. \end{aligned} \tag{7}$$

Here again μ is the expected return.

As an entropy maximization

Define

$$f(x) = \frac{1}{2} Var(S_1 \cdot x) = \frac{1}{2} x^\top \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix} x, \tag{8}$$

and

$$A = \begin{bmatrix} \mathbb{E}(S_1) \\ S_0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{9}$$

We can rewrite the problem as an entropy maximization

$$\min[f(x) : Ax = b].$$

Linear structure

Direct computation yields

$$\sigma^2/2 := (f^* \circ A^\top)^*(b) = \frac{(\mu - R)^2}{2\Delta} \tag{10}$$

We see that in the (σ, μ) plane all the optimal portfolios are on the following straight line called the capital market line

$$\sigma = \frac{\mu - R}{\sqrt{\Delta}} \text{ or } \mu = R + \sigma\sqrt{\Delta}, \tag{11}$$

where $\Delta := \alpha - 2\beta R + \gamma R^2 > 0$ if $\text{rank} A = 2$.

Capital market line

Since two points determine a line we consider

- A portfolio of only bonds which must corresponds to $(0, R)$.
- A portfolio contains only risky assets which must be on the Markowitz efficient frontier denoted (σ_M, μ_M).
- The portfolio corresponding to (σ_M, μ_M) (denoted x_M) is called the capital market portfolio.
- Since the capital market line is optimal in terms of the tradeoff between risk and return, it must be tangent to the Markowitz bullet at (σ_M, μ_M).

One fund theorem

One fund theorem

The efficient portfolios in the CAPM (corresponding to μ) are of the form

$$\frac{\mu_M - \mu}{\mu_M - R}(1, \hat{0}) + \frac{\mu - R}{\mu_M - R} x_M.$$

To find x_M one may need two (efficient) funds.

Capital market line: picture

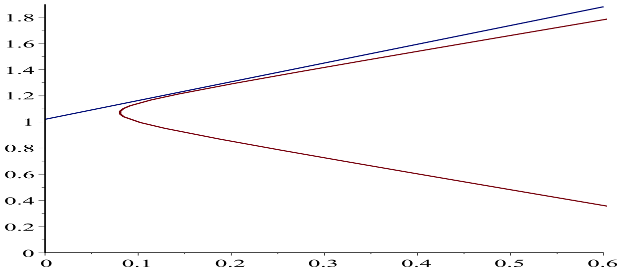


Figure: Capital market line

FTAP

Roughly speaking the fundamental theorem of asset pricing (FTAP) tells us that no arbitrage is equivalent to that there exists an equivalent martingale measure for pricing any assets in the market. It was first proposed by Ross and Cox (1976) and developed over the years by Harrison, Kreps, Pliska, Delbaen, Schachermayer and many others.

Arbitrage

Arbitrage

We say a portfolio θ is an arbitrage if

$$(S_1 - S_0) \cdot \theta \geq 0 \text{ and } (S_1 - S_0) \cdot \theta \neq 0.$$

Arbitrage is an opportunity to make money without any risk.

Equivalent martingale measure

Equivalent martingale measure

We say that Q is an equivalent martingale measure on economy (Ω, P) for financial market S provided that, for any $\omega \in \Omega$, $Q(\omega) \neq 0$ if and only if, $P(\omega) \neq 0$, and

$$\mathbb{E}^Q[S_1] = S_0.$$

Given a martingale measure Q for any contingent claim with payoff $\phi(S_1)$ we can price it as

$$\mathbb{E}^Q[\phi(S_1)].$$

Utility maximization

We approach by viewing FTAP as the result of duality of a utility maximization problem:

$$p = \sup_{\theta} \mathbb{E}[u(w_0 + (S_1 - S_0) \cdot \theta)],$$

where w_0 is the initial endowment. We can write it as an entropy maximization problem

$$p = -\min\{f(x) = \mathbb{E}[-u(y)] : y - (S_1 - S_0) \cdot \theta = w_0, \ x = (y, \theta)\}.$$

Duality approach

Outline

- No arbitrage if and only if p is finite.
- Then, dual solution exists (CQ is easy to check).
- A dual solution is the Lagrange multiplier of the primal problem.
- With the right class of utility functions, the scaling of a dual solution gives us an equivalent martingale measure also known as a risk neutral measure.

A closer look at the dual solution

The Lagrangian is

$$L((y,\theta),\lambda) = -\mathbb{E}[u(y)] + \mathbb{E}[\lambda y] - \mathbb{E}[\lambda(\theta \cdot (S_1 - S_0) - w_0)].$$

Thus, at optimal solution $(\bar{y}, \bar{\theta})$ we have

$$\lambda = u'(\bar{y}) > 0 \text{ and } \mathbb{E}[\lambda(S_1 - S_0)] = 0.$$

Now $Q = \lambda/\mathbb{E}(\lambda)$ is a P-equivalent martingale measure.

Fundamental Theorem of Asset Pricing

Refined Fundamental Theorem of Asset Pricing

In appropriate setting TFAE:

(i) There exists no arbitrage portfolio.

(ii) The portfolio utility optimization problem has a finite value and attains its solution.

(iii) There is an equivalent martingale measure proportional to the subdifferential of the negative of utility function at the optimal solution of the utility maximization problem.

Comments

The entropy maximization approach shows

- We can understand the martingale measure as a dual solution or Lagrange multiplier of an portfolio optimization problem.
- A martingale measure usually reflects the risk aversion of a market agent.
- The perception that one can find a universal “fair” price independent of agent’s behavior is, in general, incorrect.

Omitted technical details

- Different portfolio may yield same gain. We work in the quotient space of equivalent portfolios $port[S]$ in terms of the gains.
- One can define a norm (the minimum norm) on $port[S]$ to make it a Banach space. The norm can be seen as a gauge for the leverage level of the portfolio.
- We need to limit to a class of (quite general) utility functions.

Utility function

Consider a utility function u with the following key properties:

(u1) (Risk aversion) u is strictly concave,

(u2) (Profit seeking) u is strictly increasing and $\lim_{t\to+\infty} u(t) = +\infty$,

(u3) (Bankruptcy forbidden) For any $t < 0$, $u(t) = -\infty$,

Pricing using martingale measure

- FTAP tells us that no arbitrage is equivalent to the existence of an equivalent martingale measure Q.
- If an contingent claim has a payoff $\psi(S_1)$ at $t = 1$ then

$$\mathbb{E}^Q[\psi(S_1)]$$

  is a price of this contingent claim that will not create any arbitrage opportunity.
- However, usually there are many martingale measures in an incomplete market.
- Such we are facing the problem of how to choose a pricing martingale measure.

Selecting pricing martingale measure

One way is to selecting a martingale measure that maximizes an entropy (Borwein, Choksi, Marechal 2003). The math formulation is

$$\min[f(Q) : Q \in \mathcal{M}]$$

where $\mathcal{M}$ is the set of martingale measures on market S and f is the negative of an entropy.

Dual problem

We can rewrite the problem as

$$\min[f(Q) : \langle Q, S_1 - S_0 \rangle = 0]$$

where $\langle Q, y \rangle := \mathbb{E}^Q[y]$. Its dual problem is

$$\max[-f^*((S_1 - S_0) \cdot x)].$$

This can be viewed as seeking optimal portfolio x under utility function $-f^*$.

Comments

Selecting pricing martingale measure by maximizing entropy eventually is still related to utility optimization.

Super Hedging Bound

Let $\phi(S_1)$ be the payoff of the contingent claim at $t = 1$. Define

$$\begin{aligned} U &= \max\{\mathbb{E}^Q[\phi(S_1)] \mid Q \in \mathcal{M}\} \\ &= -\min\{\mathbb{E}^Q[\phi(S_1)] \mid \mathbb{E}^Q[S_1] = S_0\}. \end{aligned} \tag{12}$$

Then U is the upper bound of no arbitrage pricing, called the super hedging bound. The dual representation

$$U = \min_x \sup_{\omega\in\Omega}[[(S_1 - S_0)) \cdot x](\omega) + \phi(S_1)(\omega)]$$

provides a strategy of implementing the super-hedge.

Conclusion

- Many important results in financial math can be put in a framework of entropy maximization shows the heavy influence of methods in physical science in financial research.
- However, behavior of financial systems is the result of interaction among participants.
- It is well known that human psychology plays important role in financial markets which makes it quite different from a physical system.
- Thus, those fundamental results discussed above should be treated as a rough sketch of a road map that needs to be used with caution.
- More attention should be directed to recent progress in behavior finance and application of game theoretical methods in finance.

Jon, thanks for your mentoring, friendship and inspiration