Phase Portraits of Hyperbolic Geometry

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https://carma.newcastle.edu.au/scott/

Outline I

Outline II

Complex Phase Portraits

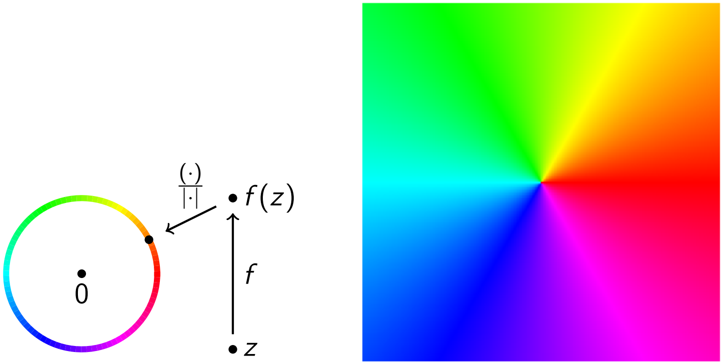
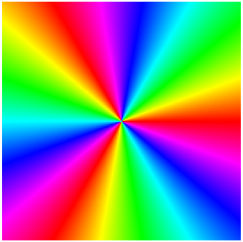


Figure: Construction of a complex phase portrait

The Basics

- Phase plotting is a way of visualizing complex functions $f : \mathbb{C} \to \mathbb{C}$.
- Where $f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2}$, we plot the domain space, coloring points according to argument of image θ_2
- Top right: $z \to z$. Bottom right: $z \to z^3$.



Some Examples

Figure: Left to right: $z \cdot e^z, W(z), \zeta(z)$.

Recapturing the Modulus

- We can also plot in 3d to recapture the modulus information.
- Let $f(r_1 e^{i\theta_1}) = r_2 e^{i\theta_2}$
- Again we plot over the domain space, coloring points according to argument of image θ_2
- We also give them vertical height corresponding to their modulus r_2.

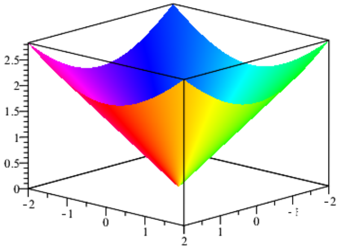


Figure: Phase portrait with modulus included: $z \to z$.

Recapturing the Modulus

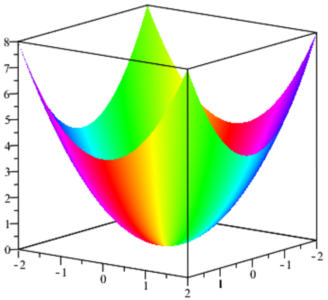


Figure: 3d phase portrait for $z \to z^2$.

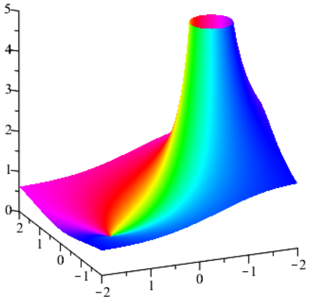


Figure: 3d phase portrait for inversion in circle radius $\sqrt{2}$ centered at $-i$

History

- Phase plotting is a relatively new tool.
- Recent attention
    - Elias Wegert's "Visual Complex Functions" published in 2013 [7]
    - "Complex Beauties" annual calendar (of which Jonathan Borwein was quite fond) [4]
- Wegert's Matlab code is available for download on his site.

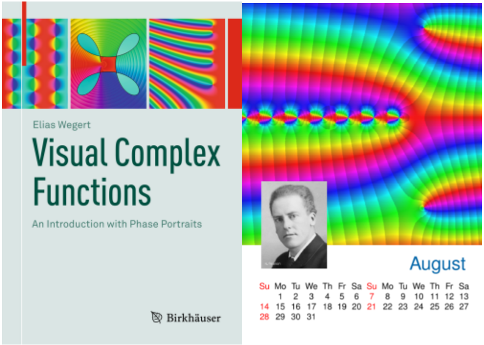


Figure: Left: Elias Wegert's "Visual Complex Functions." Right: Right: "Moment function of a 4-step Pplanar random walk" by Jonathan M. Borwein and Armin Straub from 2016 Complex Beauties calendar.

Differential Geometry

- Conformal Mappings are mappings which preserve the angles at which lines meet (and signs thereof)
- Direct Motions are mappings such that the distance between points is equal to the distance between their images.
- Parallel axiom: for a line L and point p there exists exactly one line through p which doesn't intersect L.
- Geometries which do not obey the parallel axiom:
    - Spherical Geometry (no lines through p)
    - Hyperbolic Geometry (more than one line through p)
    - Both have constant curvature (*intrinsic* property)
- The type of geometry determines how many types of direct motions there are.
- This is because conformal maps can be expressed as compositions of reflections across lines.

What Has Been Done

Phase plotting on the Riemann sphere has already been employed by Wegert.

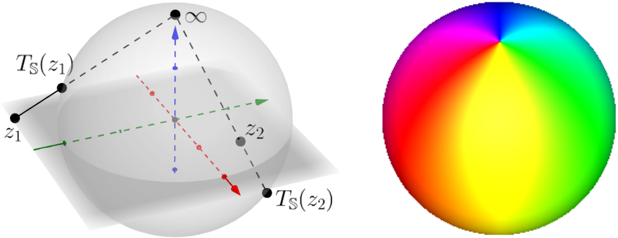


Figure: Left: The construction of the Riemann Sphere with stereographic projection. Right: phase plotting for a Möbius transformation (direct motion) on the Riemann Sphere.

Pseudosphere

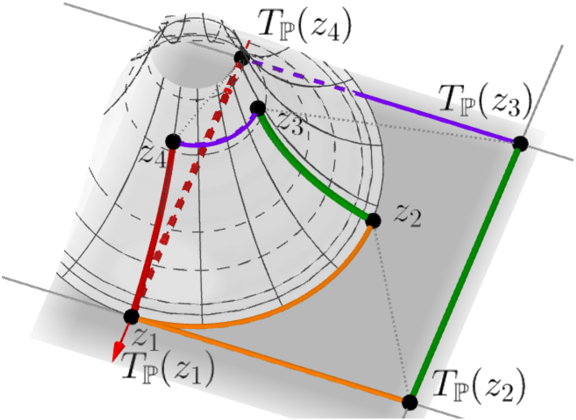


Figure: The conformal map from the pseudosphere to the hyperbolic upper half plane.

Pseudosphere

- The map is between the pseudosphere and a small area of the upper half plane
- If we colored according to the planar phase plotting rules, problems:
    - Fewer colors for visualization
    - Coloring would be tied to Euclidean geometry rather than Hyperbolic geometry, warping perspective.
    - Unable to tell if points mapped out of visible region.

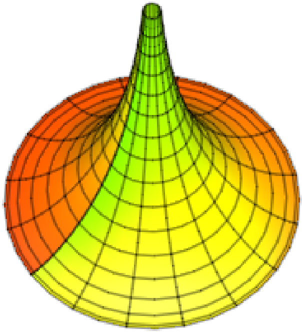


Figure: Colors change along tractrices rather than Euclidean subspaces.

What Has Been Done

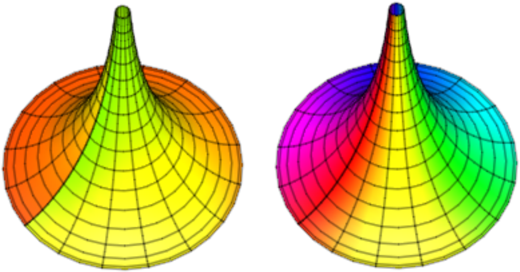


Figure: Left: naive plotting on pseudosphere. Right: assigning unique colors to tractrices.

Solution: defined a new coloring scheme unique to hyperbolic space.

- But what to do about points mapped out of viewable region?
- Solution: color them black

$$\mathcal{C}_{\mathbb{P}} : \mathbb{P} \to [0, 2\pi] \cup \{\text{black}\}$$
$$z \mapsto \begin{cases} \mathrm{Re} \circ f \circ T_{\mathbb{P}}(z) & \mathrm{Re} \circ f \circ T_{\mathbb{P}}(z) \in [0, 2\pi] \\ \text{black} & \text{otherwise} \end{cases}.$$

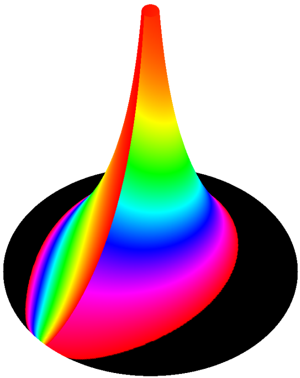
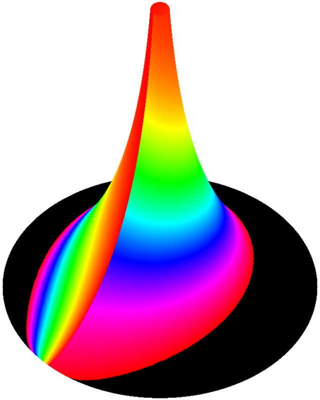


Figure: An h-rotation plotted on a pseudosphere.



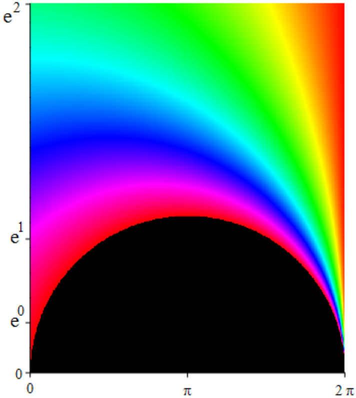


Figure: A rotation on hyperbolic space plotted in the Poincaré upper half plane and on the pseudosphere.

We can use the same plot in the upper half plane for comparison.

Poincaré Disc

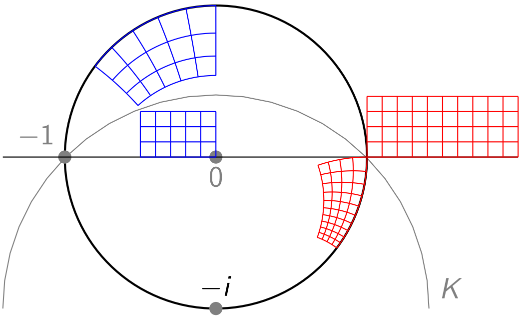


Figure: Anticonformal map I_K

Poincaré Disc Asymptotic

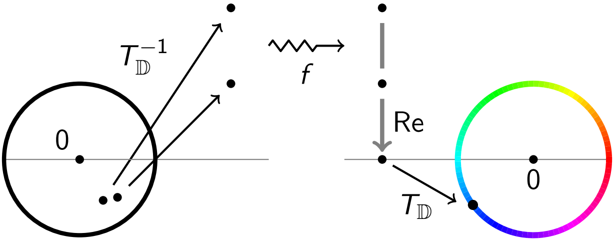
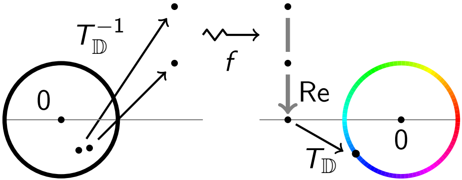


Figure: Construction of the coloring map $\mathcal{C}_{\mathbb{D},1}$ for uniquely coloring *asymptotic* parallel lines.

Poincaré Disc Asymptotic



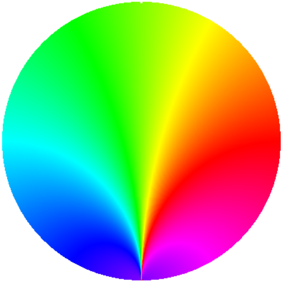
Theorem 1

Let $f : \mathbb{C}_+ \to \mathbb{C}_+$ represent a function on hyperbolic space. The function

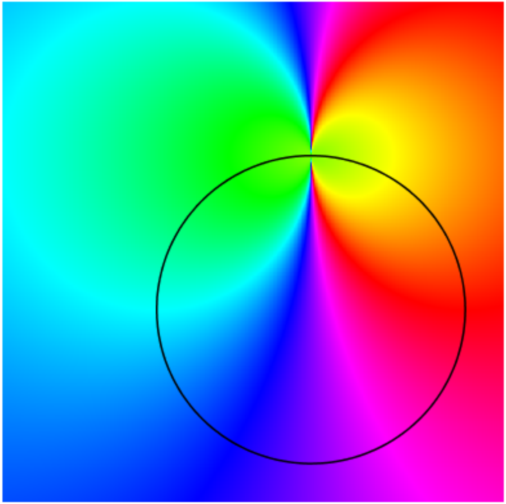
$$\mathcal{C}_{\mathbb{D},1} : \mathbb{D} \to [0, 2\pi)$$
$$z \mapsto (\arg \circ T_{\mathbb{D}} \circ \mathrm{Re} \circ f \circ T_{\mathbb{D}}^{-1})(z)$$

assigns a unique color to the preimage of each tractrix line.

Poincaré Asymptotic



A true phase portrait!



Poincaré Asymptotic

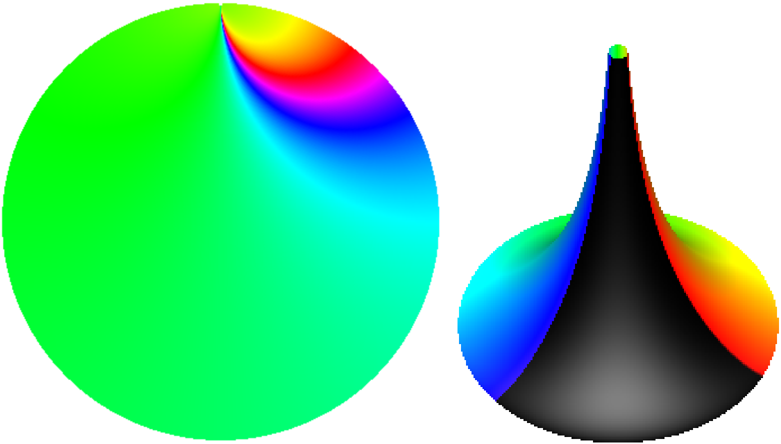


Figure: A limit rotation on hyperbolic space.

Poincaré Ultra Parallel

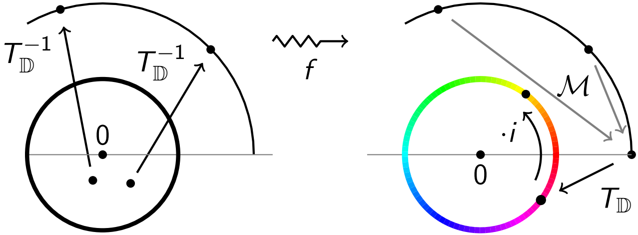
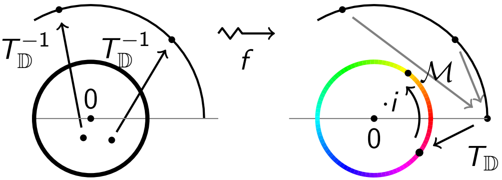


Figure: Construction of a phase plot $\mathcal{C}_{\mathbb{P},2}$ uniquely coloring *ultra*-parallel lines on the Poincaré Disc.

Poincaré Ultra Parallel



Theorem 2

Let $f : \mathbb{C}_+ \to \mathbb{C}_+$ represent a function on hyperbolic space. The function

$$\begin{aligned} \mathcal{C}_{\mathbb{D},2} &: \mathbb{D} \to [0, 2\pi) \\ & z \mapsto (2 \cdot \arg \circ\, i \cdot T_{\mathbb{D}} \circ \mathcal{M} \circ f \circ T_{\mathbb{D}}^{-1})(z) \end{aligned}$$

assigns a unique color to the preimage of each h-line corresponding to a circle in $\mathbb{C}_+$ centered at 0.

Poincaré Disc Ultra Parallel

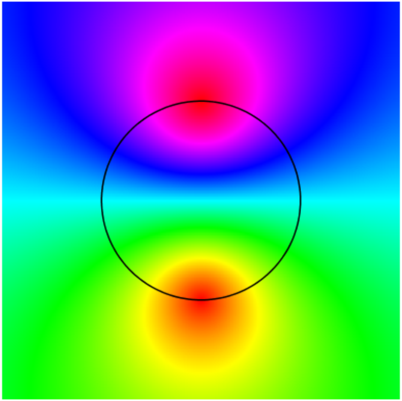


Figure: The phase plot $\mathcal{C}_{\mathbb{P},2}$ is also a true phase plot.

Multiple views: motivation

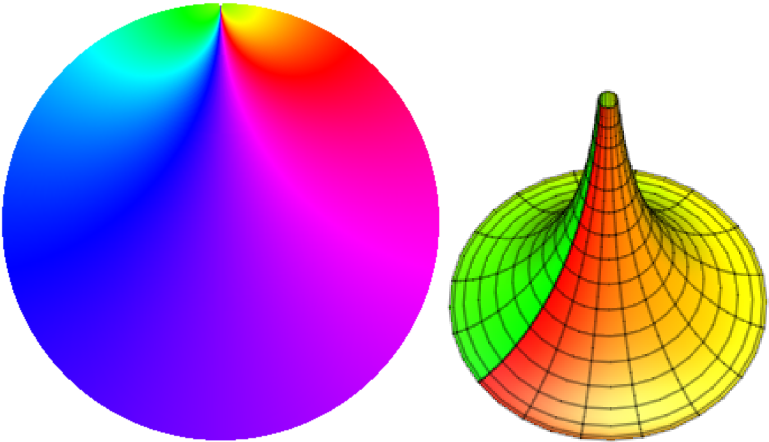


Figure: A translation on hyperbolic space.

Multiple views: motivation

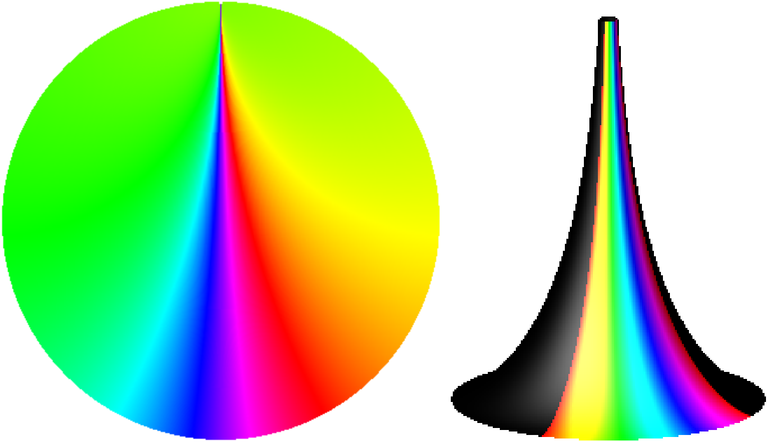


Figure: A translation on hyperbolic space.

Multiple views: motivation

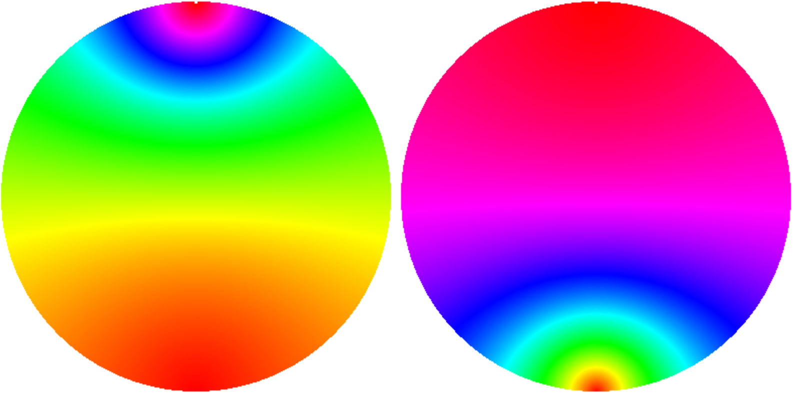


Figure: Translations on hyperbolic space.

Combining Views

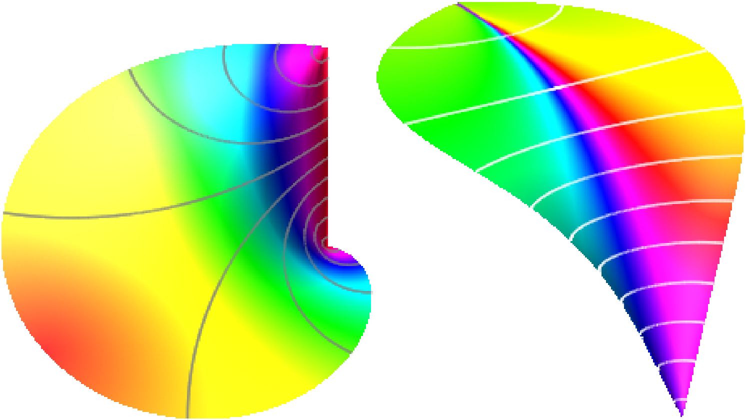


Figure: Colored landscapes on $\mathbb{D}$.

Combining Views

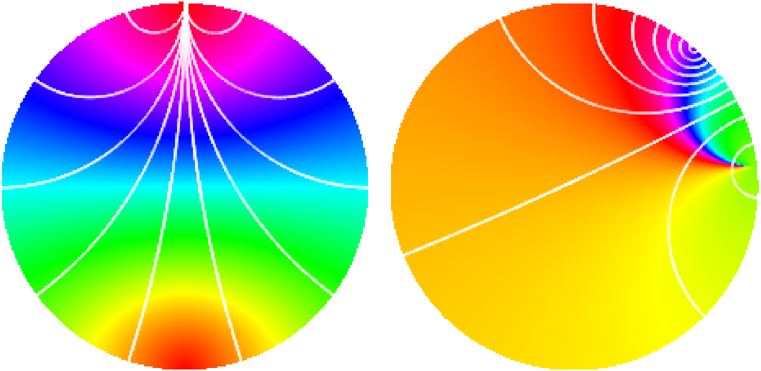
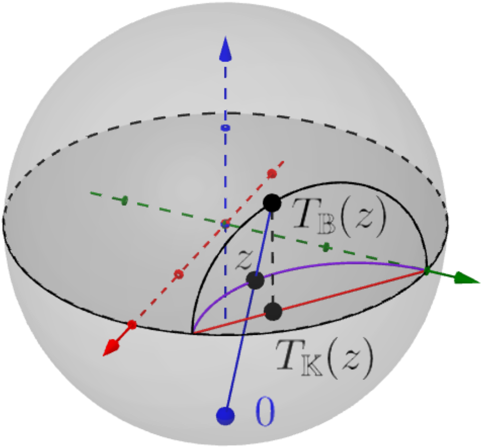
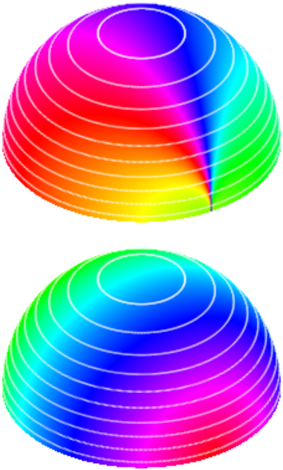


Figure: Colored contour plots on $\mathbb{D}$.

Beltrami Upper Half Sphere and Klein Disc





Klein Disc

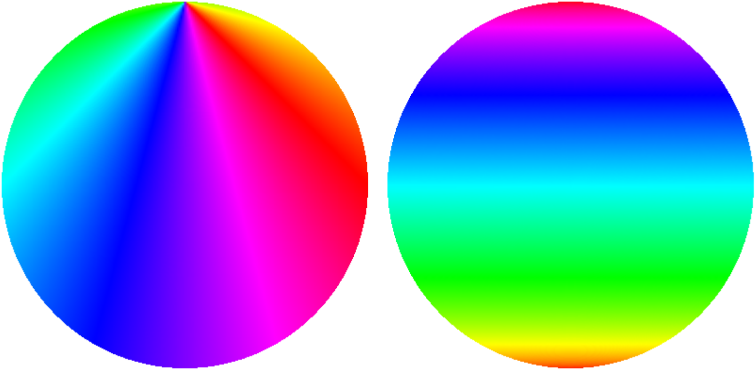
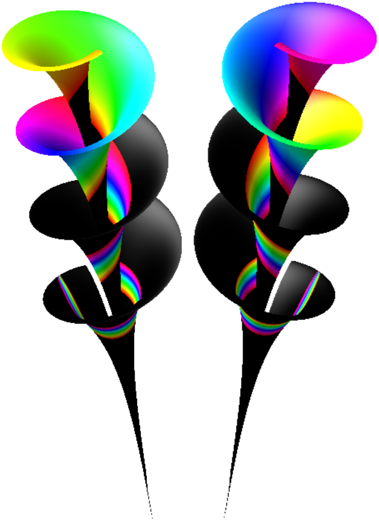


Figure: Phase plotting on $\mathbb{K}$

Dini's Surface

- We can also improve the pseudosphere view.
- Twisting the pseudosphere as we wrap, our view becomes less limited.
- We can have fun with symmetry! Reflect the lines about π and twist the pseudosphere in the other direction to obtain a mirror image with colors reversed!



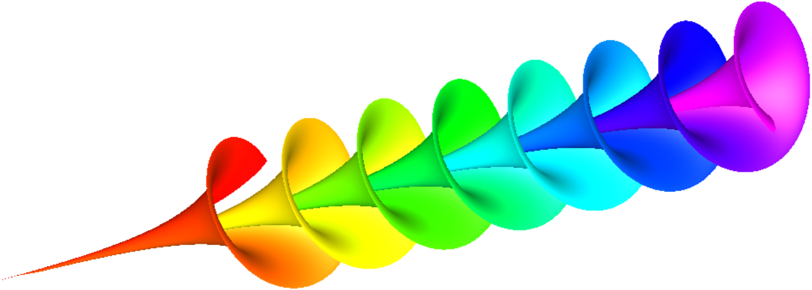
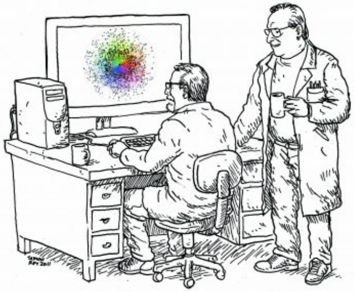


Figure: Translation on Dini's surface.

Our *Maple* package is available at
`http://hdl.handle.net/1959.13/1346401`

This work is dedicated to the memory of Jonathan Borwein: our advisor, mentor, and friend.



"Sometimes it is easier to see than to say."

Image drawn by Simon Roy at request of Jon and Veselin Jungic:

(http://jonborwein.org/2016/08/jon-borwein-a-friend-and-a-mentor/)

References I

[1] D. N. Arnold and J. Rogness, "Möbius transformations revealed." *Notices of the AMS*, 55 (2008).

[2] D. N. Arnold and J. Rogness, [jonathanrogness]. (2007, June 3). *Moebius Transformations Revealed* [Video file]. Retrieved from https://www.youtube.com/watch?v=JX3VmDgiFnY

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[6] T. Needham, *Visual Complex Analysis*. Oxford University Press, 1997.

References II

[7] E. Wegert, *Visual Complex Functions: An Introduction with Phase Portraits*. Springer, 2012.

[8] E. Wegert and G. Semmler, "Phase Plots of Complex Functions: A Journey in Illustration," *Notices of the American Mathematical Society* **58(6)**, 2011.