On lexicographic tangents

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Based on joint work with **Levent Tunçel** (University of Waterloo).

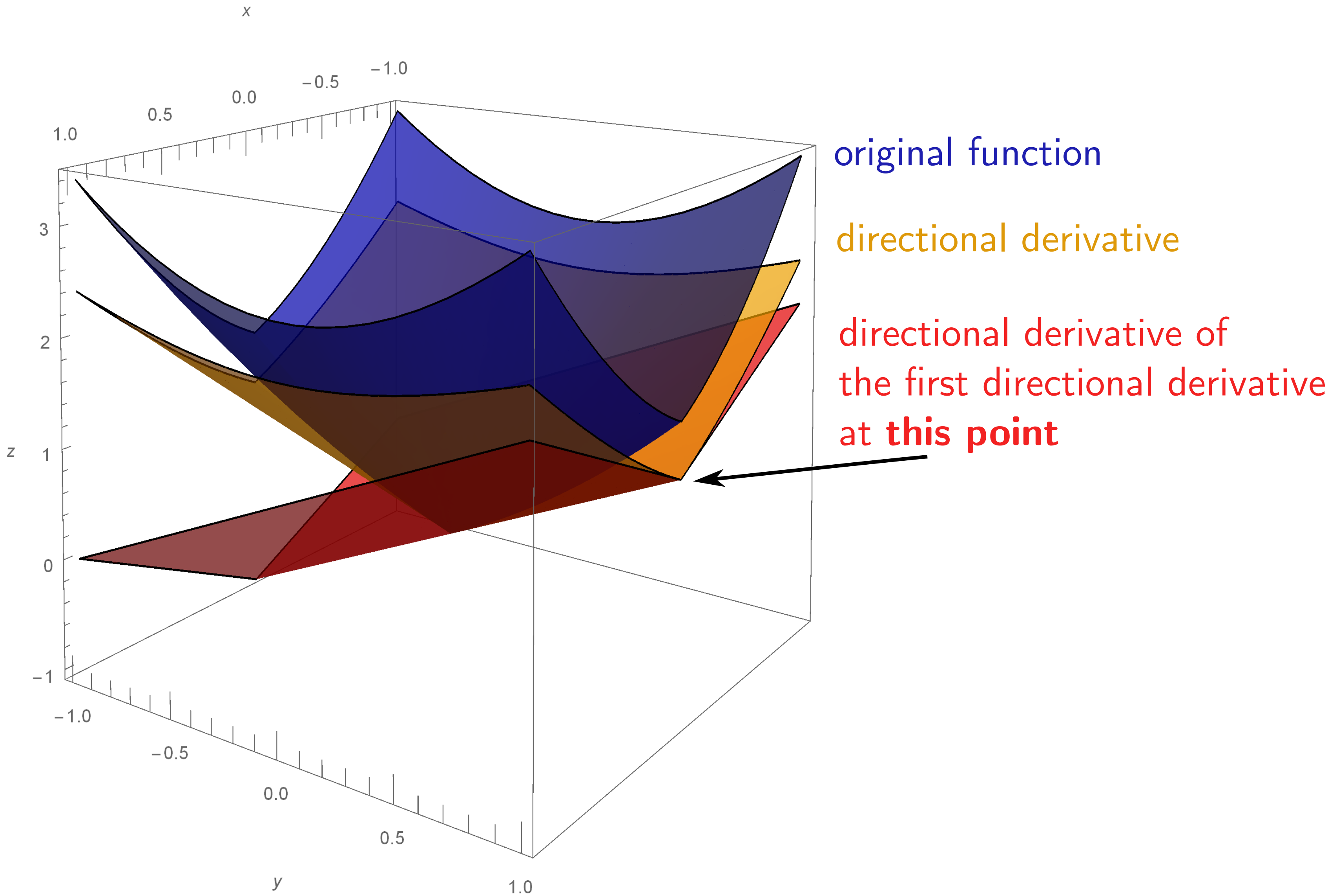
The directional derivative

Let $f : \mathbb{R}^n \to \mathbb{R}^p$, the (Dini) directional derivative:

$$f'(x;h) := \lim_{\alpha \downarrow 0} \frac{f(x+\alpha h) - f(x)}{\alpha}.$$

Why not do this several times?

Directional derivative of directional derivative



Lexicographic smoothness

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be locally Lipschitz and dir. diff. at $x \in \mathbb{R}^n$. The *homogenisation operator* H_x with respect to $x \in \mathbb{R}^n$:

$$\varphi = H_x[f] \quad \Leftrightarrow \quad \varphi(h) = f'(x; h).$$

Definition 1 (Nesterov)**.** Let $U = \{u_k\}_{k=1}^m$ be a set of directions in $\mathbb{R}^n$. The sequence of recursively defined functions

$$\begin{aligned} f_{x,U}^{(0)} &= H_x[f], \\ f_{x,U}^{(k)} &= H_{u_k}[f_{x,U}^{(k-1)}] \end{aligned}$$

is the *homogenisation sequence* of f generated by (x, U).

Definition 2 (Nesterov)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is called lexicographically smooth on $Q \subseteq \mathbb{R}^n$ if its homogenisation sequence is well defined for any $x \in Q$ and any set of directions $U \subset \mathbb{R}^n$.

[Yu. Nesterov, Lexicographic differentiation of nonsmooth functions.]

Lexicographic derivative

Denote by $L_k(U)$ the linear span of the first k vectors of the sequence $U = \{u_1, \ldots, u_m\}$, and let $L_0(U) := \{0\}$.

Theorem 3 (Yu. Nesterov)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be lexicographically smooth at $x \in \mathbb{R}^n$. If the sequence $U = \{u_1, \ldots, u_m\}$ is full-dimensional ($L_m(U) = \mathbb{R}^n$), then there exists a number k_0, $0 \leq k_0 \leq m$, such that for all $k \geq k_0$ the functions $f^{(k)}_{x,U}(h)$ are linear.*

The smallest such k_0 is called the degree of non-differentiability of f at x along U and is denoted by $d(f, U, x)$. The relevant Jacobians (gradients) are called the lexicographic derivatves of $f(x)$ along the sequence U.

It is enough to consider sequences of orthonormal vectors.

Differentiable and convex functions

For differentiable functions the degree of non-differentiability is one, and the lexicographic derivatives coincide with the Jacobian (gradient) of the function.

Theorem 4 (Nesterov). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. For a basis $S = \{s_1, \ldots, s_n\}$ of $\mathbb{R}^n$, define recursively*

$$\begin{aligned} B_0 &= \partial f(x), \\ B_k &= \operatorname*{Arg\,max}_{y \in B_{k-1}} \langle s_k, y \rangle, \qquad k = 1, \ldots, n. \end{aligned}$$

Then the lexicographic gradient $g^T(f, S, x)$ is the unique point in the singleton B_n.

Classes of L-smooth functions

- Differentiable functions
- Component-wise convex functions
- Component-wise quasidifferentiable functions
- Tame functions

Calculus

Compositions of L-smooth functions (defined on appropriate open sets) are L-smooth, and their lexicographic derivatives can be computed explicitly.

One can define the lexicographic subdifferential $\partial_L f(x)$ as the collection of all Jacobians (gradients) for all orthonormal bases. The lexicographic subdifferential is a subset of the Clarke subdifferential.

Lexicographic derivatives can be integrated along smooth curves,

$$f(\gamma(T)) - f(\gamma(0)) = \int_0^T g(f, S, \gamma(t))\dot{\gamma}(t)dt,$$

where $g(f, S, x)$ is the lexicographic derivative of f at x along S.

Definable functions are L-smooth

Theorem 5 (Khan). *Given an open set $X \subset \mathbb{R}^n$, a locally Lipschitz continuous function $f : X \to \mathbb{R}$ is L-smooth if and only if f is directed subdifferentiable at x.*

Theorem 6 (Baier, Farkhi, R. with help from A. Daniilidis and D. Drusvyatskiy). *Locally Lipschitz tame functions are directed subdifferentiable.*

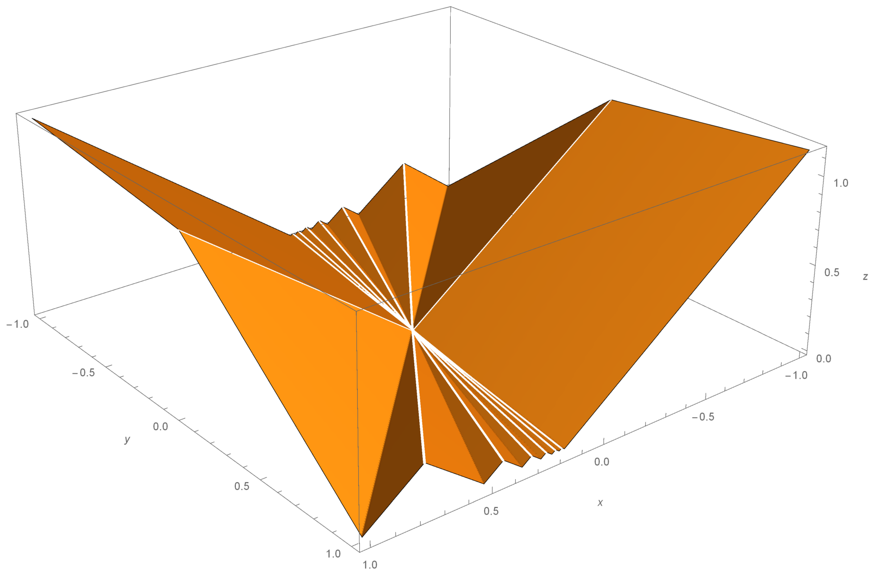
It follows that locally Lipschitz tame functions are directed subdifferentiable and hence L-smooth. In particular, semialgebraic locally Lipschitz functions are L-smooth.

[K. Khan, Relating Lexicographic Smoothness and Directed Subdifferentiability]

[R. Baier, E. Farkhi, V. Roshchina, Directed Subdifferentiable Functions and the Directed Subdifferential Without Delta-Convex Structure]

An example of a function which is L-smooth

The idea of this example was suggested by Jeffrey Pang.



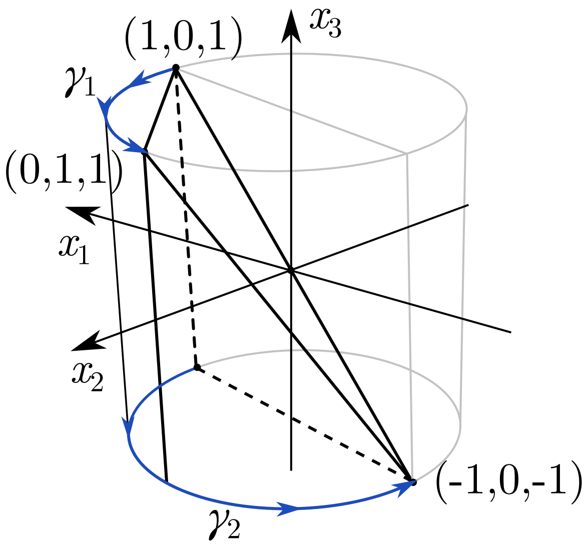
Let $f : \mathbb{R}^2 \to \mathbb{R}$,

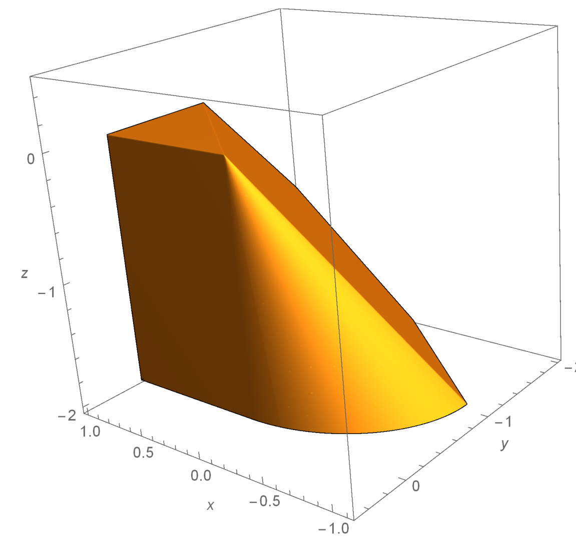
$$f(x, y) := \inf_{k \in \mathbb{N}} \left| x - \frac{y}{k} \right|.$$

This function is neither quasidifferentiable nor tame, but is nevertheless L-smooth.

Lexicographic tangents

Some geometric irregularities only show in tangents.





$\gamma_1(t) = (\cos t, \sin t, 1), t \in [0, \pi/2], \quad \gamma_2(t) = (\cos t, \sin t, -1), t \in [0, \pi].$

The set $C := \text{conv}\{\gamma_1, \gamma_2\}$ is facially exposed, but its tangent is not.

[V. Roshchina, L. Tunçel, Facially Dual Complete (Nice) cones and lexicographic tangents]

Facially dual complete (FDC) cones

When feasible solutions belong to a face F of a constraint cone K in a conic optimisation problem, we may replace K by $(K \cap \text{span}(F))$. The dual cone constraint becomes

$$s \in (K \cap \text{span}(F))^* = \text{cl}\left(K^* + F^\perp\right),$$

where $F^\perp := \{s \in \mathbb{R}^n : \langle s, x\rangle = 0 \;\; \forall x \in F\}$. If $\left(K^* + F^\perp\right)$ happens to be closed, then we can remove the closure operation. *Facially Dual Complete* (nice) cones are closed convex cones K with

$$\left(K^* + F^\perp\right) \text{ is closed for every proper face } F \text{ of } K.$$

FDC is important in duality theory (artificial strong duals and facial reduction), in determining whether the image of a convex set is closed under a linear map and also for lifted representations.

[V. Roshchina, L. Tunçel, Facially Dual Complete (Nice) cones and lexicographic tangents]

Characterisations of FDC cones

Let $C \subseteq \mathbb{R}^n$ and $x \in C$. Then, the tangent cone for C at x is

$$T(x; C) := \operatorname*{Lim\,sup}_{t \to +\infty} t(C - x).$$

We say that a closed convex set $C \subset \mathbb{R}^n$ has *tangential exposure* property if

$$T(x; K) \cap \operatorname{span} F = T(x; F) \quad \forall F \text{ faces of } K,\ \forall x \in F. \tag{1}$$

Theorem 7. *If a closed convex cone K is facially dual complete, then for every face F of K and every $x \in F$, we have*

$$T(x; K) \cap \operatorname{span} F = T(x; F). \tag{2}$$

Characterisations of FDC cones

A second-order tangent cone for C at $x \in C$ and $v \in T(x;C)$ is:

$$\begin{aligned} T\left[v; T(x;C)\right] &= \operatorname*{Lim\,sup}_{t_2 \to +\infty} t_2 \left[T(x;C) - v\right] \\ &= \operatorname*{Lim\,sup}_{t_2 \to +\infty} t_2 \left\{ \left[\operatorname*{Lim\,sup}_{t_1 \to +\infty} t_1 (C - x) \right] - v \right\}. \end{aligned}$$

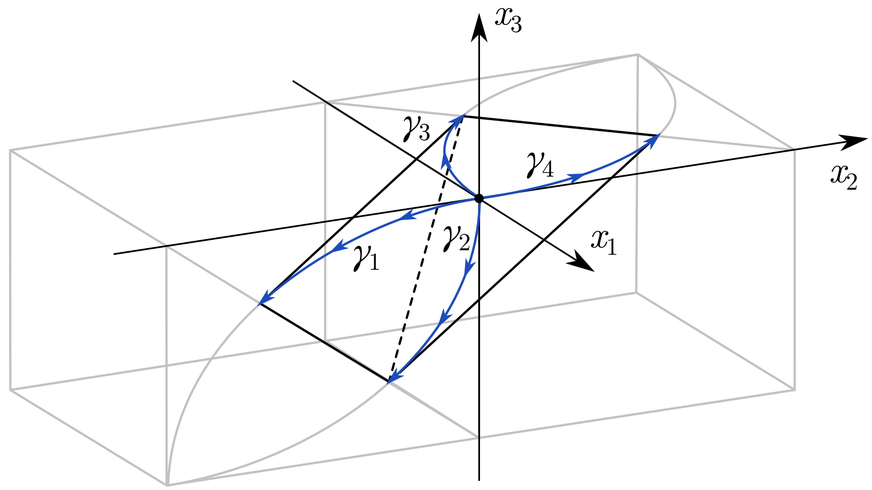
Recursive application of this construction generates k^{th} order lexicographic tangent cones for every nonnegative integer k.

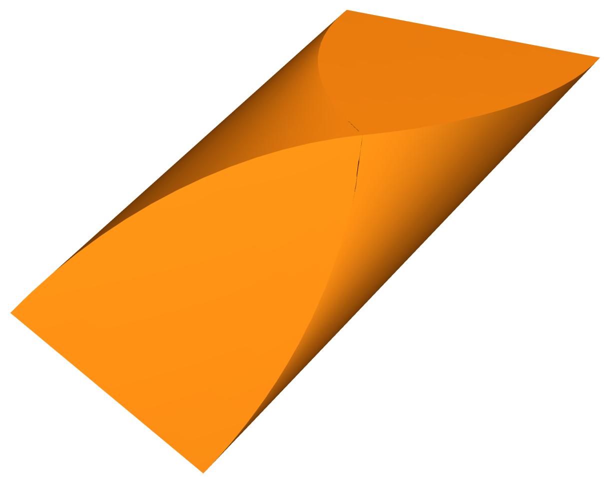
We say that a closed convex set is *strongly tangentially exposed* if it is tangentially exposed along with all of its lexicographic tangent cones.

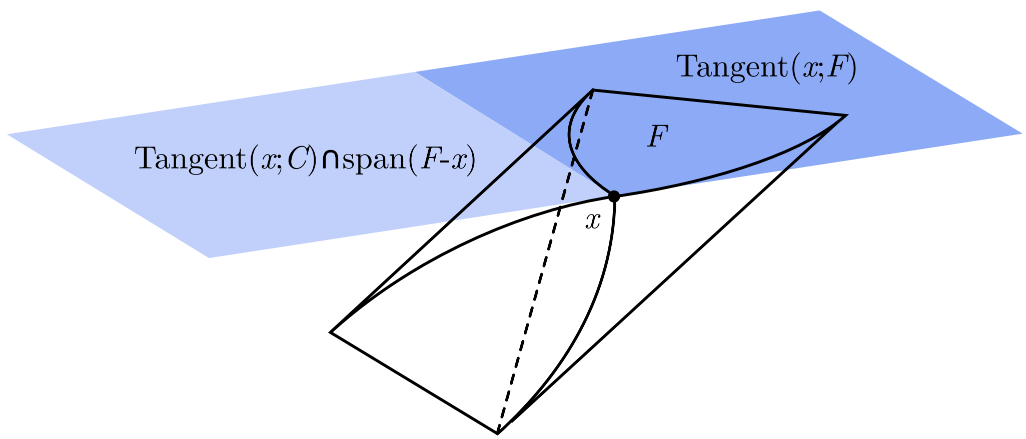
Theorem 8. *If a closed convex cone $K \subseteq \mathbb{R}^n$ is strongly tangentially exposed, then it is facially dual complete.*

Breakdown of strong tangential exposure

First order

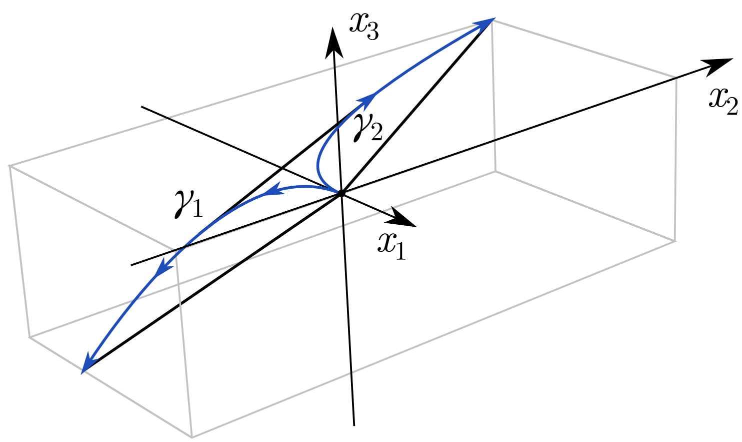


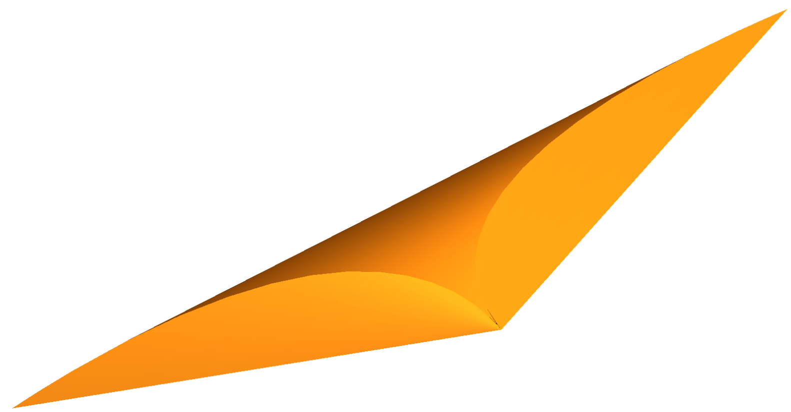


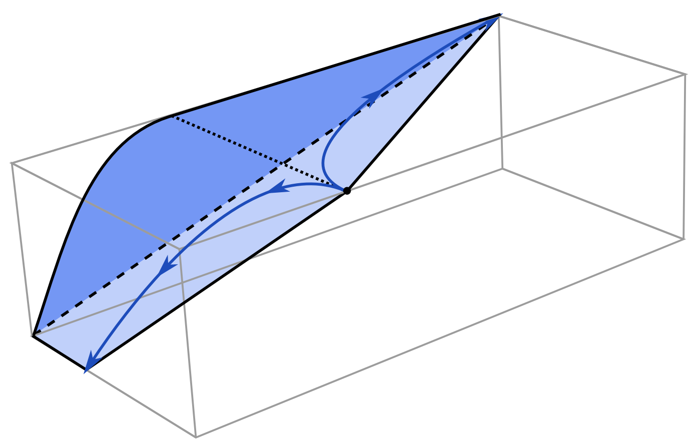


Breakdown of strong tangential exposure

Second order







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