Figure: Organic Math Workshop, Simon Fraser University, December 12-14, 1995

“Sometimes it is easier to see than to say”

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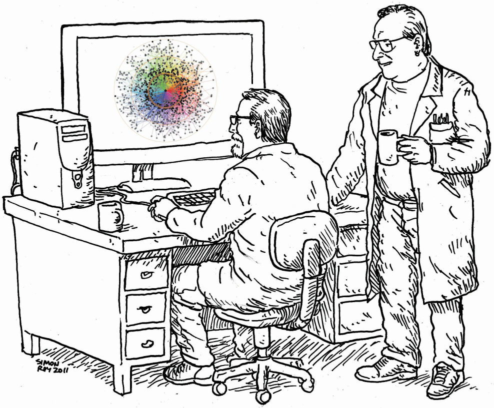
September 27, 2017



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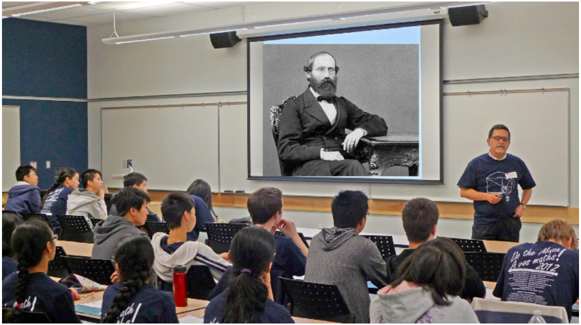
Setting: You are introducing definite integrals to your calculus students



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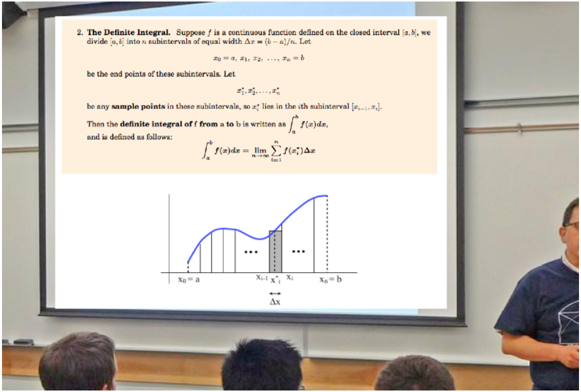

Setting: Example 1

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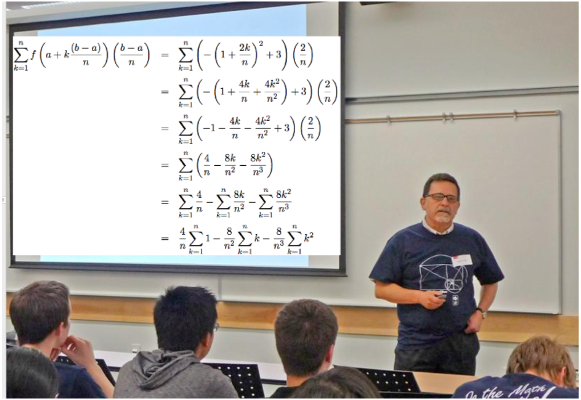

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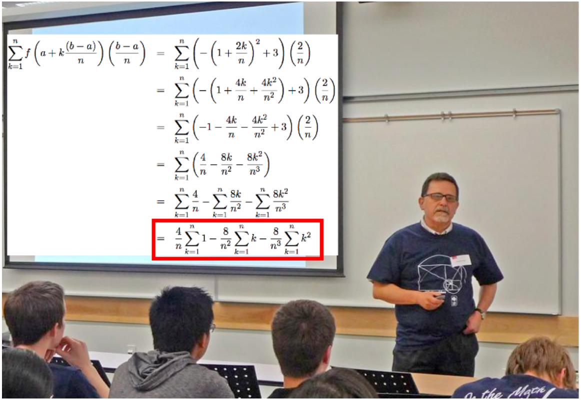
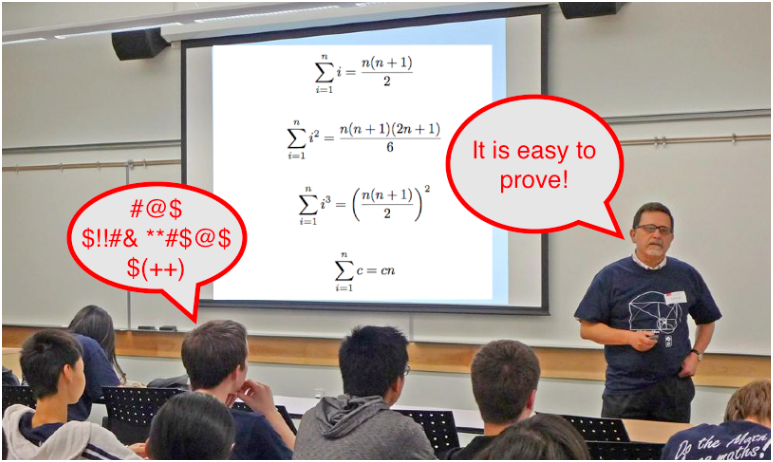


Setting: Help!

$$\begin{aligned}
\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) &= \sum_{k=1}^{n}\left(-\left(1+\frac{2k}{n}\right)^2+3\right)\left(\frac{2}{n}\right)\\
&= \sum_{k=1}^{n}\left(-\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)+3\right)\left(\frac{2}{n}\right)\\
&= \sum_{k=1}^{n}\left(-1-\frac{4k}{n}-\frac{4k^2}{n^2}+3\right)\left(\frac{2}{n}\right)\\
&= \sum_{k=1}^{n}\left(\frac{4}{n}-\frac{8k}{n^2}-\frac{8k^2}{n^3}\right)\\
&= \sum_{k=1}^{n}\frac{4}{n}-\sum_{k=1}^{n}\frac{8k}{n^2}-\sum_{k=1}^{n}\frac{8k^2}{n^3}\\
&= \boxed{\frac{4}{n}\sum_{k=1}^{n}1-\frac{8}{n^2}\sum_{k=1}^{n}k-\frac{8}{n^3}\sum_{k=1}^{n}k^2}
\end{aligned}$$

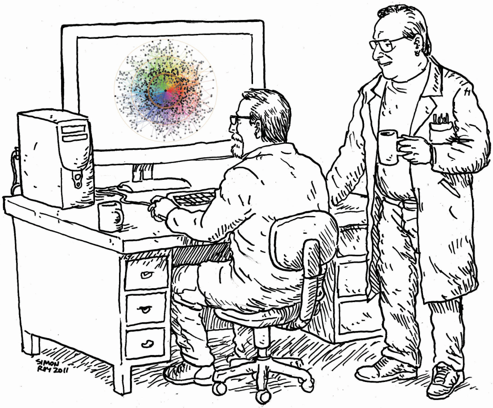
Setting: It worked!



Problem:

How can you quickly convince your students that those identities are true?

Reminder:



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Dynamical Visual Models - One:

$$1 + 3 + \ldots + (2n - 1) = n^2$$

Figure: The sum of the first *n* positive odd integers

Facts:

- Known to Pythagoras, c. 570 – 500 BCE

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- The first inductive proof has been attributed to Francesco Maurolico, 1494 – 1575,

Dynamical Visual Models - Two:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Figure: The sum of the first n positive integers

Fact:

We follow Pythagoras' proof.

Dynamical Visual Models - Three:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Figure: The sum of the squares of the first n positive integers

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- Sometimes it is called *the Greek rectangle method*.

Dynamical Visual Models - Four:

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n \cdot (n+1)}{2}\right)^2$$

Figure: The sum of the cubes of the first *n* positive integers

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- We follow the idea that is attributed to Abu Bakr al-Karaji.

Proof:

- Let $n \in \mathbb{N}$ and let $A = \left[0, \frac{n(n+1)}{2}\right] \times \left[0, \frac{n(n+1)}{2}\right]$.

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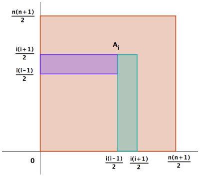
For $i \in [1, n]$, let $A_i = \left[0, \frac{i(i-1)}{2}\right] \times \left[\frac{i(i-1)}{2}, \frac{i(i+1)}{2}\right] \cup \left[\frac{i(i-1)}{2}, \frac{i(i+1)}{2}\right] \times \left[0, \frac{i(i+1)}{2}\right]$.



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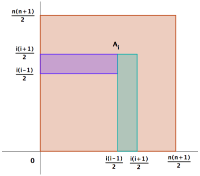

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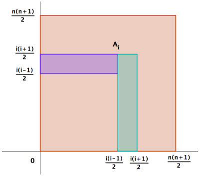

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- $\cup_{i=1}^{n} A_i = A$, $i \neq j \Rightarrow \mu(A_i \cap A_j) = 0$, and $\mu(A_i) = i^3$
- $\sum_{i=1}^{n} \mu(A_i) = \mu(A) \Rightarrow \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Disclamer:

$$64 = 65$$

Figure: Fibonacci Jigsaw Puzzle

Why Dynamical Visual Models in a Math Classroom?

Gaining insight and intuition or just knowledge.

– the first of "Eight Rules for Computation" by David Bailey and Jonathan Borwein.

Acknowledgments:

Visual models created by Damir Jungić.

Thank you!

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