



Figure: Organic Math Workshop, Simon Fraser University, December 12-14, 1995

“Sometimes it is easier to see than to say”

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Setting: You are introducing definite integrals to your calculus students





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2. **The Definite Integral.** Suppose  $f$  is a continuous function defined on the closed interval  $[a, b]$ . We divide  $[a, b]$  into  $n$  subintervals of equal width  $\Delta x = (b - a)/n$ . Let

$$x_0 = a, x_1, x_2, \dots, x_n = b$$

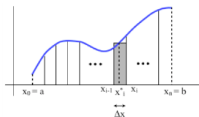
be the end points of these subintervals. Let

$$x_1^*, x_2^*, \dots, x_n^*$$

be any **sample points** in these subintervals, so  $x_i^*$  lies in the  $i$ th subinterval  $[x_{i-1}, x_i]$ .

Then the **definite integral of  $f$  from  $a$  to  $b$**  is written as  $\int_a^b f(x) dx$ , and is defined as follows:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$



# Setting: Example 1



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The image shows a classroom scene with a lecturer and students. The lecturer is standing in front of a large projection screen that displays a series of mathematical equations. The equations are as follows:

$$\begin{aligned}\sum_{k=1}^n f\left(a+k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) &= \sum_{k=1}^n \left(-\left(1+\frac{2k}{n}\right)^2+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-1-\frac{4k}{n}-\frac{4k^2}{n^2}+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(\frac{4}{n}-\frac{8k}{n^2}-\frac{8k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{4}{n} - \sum_{k=1}^n \frac{8k}{n^2} - \sum_{k=1}^n \frac{8k^2}{n^3} \\ &= \frac{4}{n} \sum_{k=1}^n 1 - \frac{8}{n^2} \sum_{k=1}^n k - \frac{8}{n^3} \sum_{k=1}^n k^2\end{aligned}$$

# Setting: Example 1

The image shows a lecture hall with a professor presenting to a class. A large projection screen displays the following mathematical derivation:

$$\begin{aligned}\sum_{k=1}^n f\left(a+k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) &= \sum_{k=1}^n \left(-\left(1+\frac{2k}{n}\right)^2+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-1-\frac{4k}{n}-\frac{4k^2}{n^2}+3\right)\left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(\frac{4}{n}-\frac{8k}{n^2}-\frac{8k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{4}{n}-\sum_{k=1}^n \frac{8k}{n^2}-\sum_{k=1}^n \frac{8k^2}{n^3} \\ &= \frac{4}{n}\sum_{k=1}^n 1-\frac{8}{n^2}\sum_{k=1}^n k-\frac{8}{n^3}\sum_{k=1}^n k^2\end{aligned}

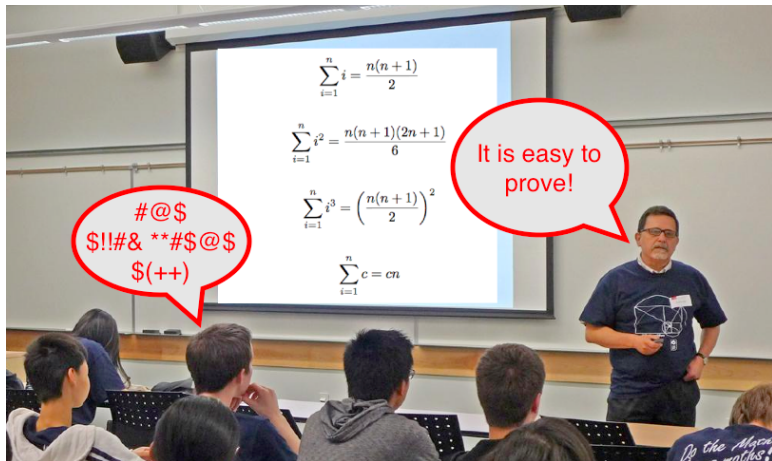
The final equation is highlighted with a red box. The professor is standing to the right of the screen, and several students are visible in the foreground, looking towards the screen.$$

# Setting: Help!



$$\begin{aligned}\sum_{k=1}^n f\left(a + k\frac{(b-a)}{n}\right) \left(\frac{b-a}{n}\right) &= \sum_{k=1}^n \left(-\left(1 + \frac{2k}{n}\right)^2 + 3\right) \left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-\left(1 + \frac{4k}{n} + \frac{4k^2}{n^2}\right) + 3\right) \left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(-1 - \frac{4k}{n} - \frac{4k^2}{n^2} + 3\right) \left(\frac{2}{n}\right) \\ &= \sum_{k=1}^n \left(\frac{4}{n} - \frac{8k}{n^2} - \frac{8k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{4}{n} - \sum_{k=1}^n \frac{8k}{n^2} - \sum_{k=1}^n \frac{8k^2}{n^3} \\ &= \frac{4}{n} \sum_{k=1}^n 1 - \frac{8}{n^2} \sum_{k=1}^n k - \frac{8}{n^3} \sum_{k=1}^n k^2\end{aligned}$$

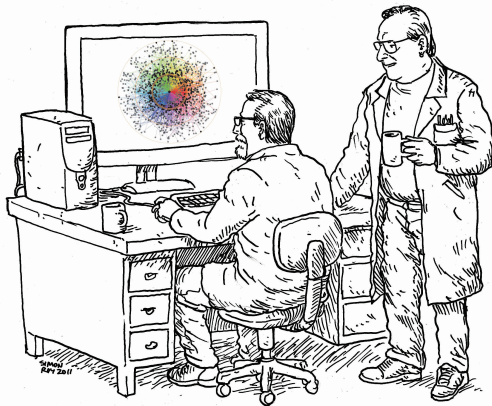
# Setting: It worked!



## Problem:

How can you quickly convince your students that those identities are true?

## Reminder:



*"Sometimes it is easier to see than to say."*



## Dynamical Visual Models - One:

$$1 + 3 + \dots + (2n - 1) = n^2$$

Figure: The sum of the first  $n$  positive odd integers

## Facts:

- ▶ Known to Pythagoras, c. 570 – 500 BCE

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- ▶ The first inductive proof has been attributed to Francesco Maurolico, 1494 – 1575,

## Dynamical Visual Models - Two:

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Figure: The sum of the first  $n$  positive integers

Fact:

We follow Pythagoras' proof.

## Dynamical Visual Models - Three:

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Figure: The sum of the squares of the first  $n$  positive integers

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- ▶ Sometimes it is called *the Greek rectangle method*.

## Dynamical Visual Models - Four:

$$1^3 + 2^3 + \dots + n^3 = \left( \frac{n \cdot (n + 1)}{2} \right)^2$$

Figure: The sum of the cubes of the first  $n$  positive integers

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- ▶ We follow the idea that is attributed to Abu Bakr al-Karaji.

## Proof:

- ▶ Let  $n \in \mathbb{N}$  and let  $A = \left[0, \frac{n(n+1)}{2}\right] \times \left[0, \frac{n(n+1)}{2}\right]$ .

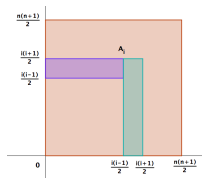
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For  $i \in [1, n]$ , let  $A_i = \left[0, \frac{i(i-1)}{2}\right] \times \left[\frac{i(i-1)}{2}, \frac{i(i+1)}{2}\right] \cup \left[\frac{i(i-1)}{2}, \frac{i(i+1)}{2}\right] \times \left[0, \frac{i(i+1)}{2}\right]$ .





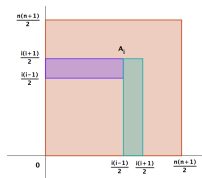
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▶

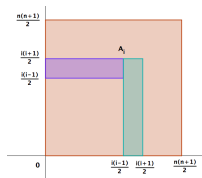
- ▶  $\bigcup_{i=1}^n A_i = A$ ,  $i \neq j \Rightarrow \mu(A_i \cap A_j) = 0$ , and  $\mu(A_i) = i^3$



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- ▶  $\sum_{i=1}^n \mu(A_i) = \mu(A) \Rightarrow \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Disclaimer:

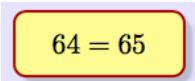

$$64 = 65$$

Figure: Fibonacci Jigsaw Puzzle

# Why Dynamical Visual Models in a Math Classroom?

*Gaining insight and intuition or just knowledge.*

- the first of “Eight Rules for Computation” by David Bailey and Jonathan Borwein.

## Acknowledgments:

Visual models created by Damir Jungić.

Thank you!

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