An Abstract Variational Theorem

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$$\text{argmax}(f) := \{x \in X : f(y) \leq f(x) \text{ for all } y \in X\}.$$

Definition

We shall say that a function $f : X \to [-\infty, \infty)$ defined on a normed linear space $(X, \|\cdot\|)$ attains a (or has a) strong maximum at $x_0 \in X$ if, $f(x_0) = \sup\{f(x) : x \in X\}$ and $\lim_{n\to\infty} x_n = x_0$ whenever $(x_n : n \in \mathbb{N})$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \sup\{f(x) : x \in X\} = f(x_0)$.

Definition

Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to (-\infty, \infty]$ be a proper function. Then the Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, \infty]$ defined by, (here, and elsewhere, X^* denotes the dual space of X)

$$f^*(x^*) := \sup\{x^*(x) - f(x) : x \in X\}.$$

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The final notion that we need before we can precisely state our theorem is that of a G_δ set

Definition

A subset Y of a topological space (X, τ) is called a G_δ set if there exists a countable family $\{O_n : n \in \mathbb{N}\}$ of open subsets of X such that $Y = \bigcap_{n \in \mathbb{N}} O_n$.

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Theorem (Abstract Variational Theorem)

Let $f : X \to (\infty, \infty]$ be a proper function on a Banach space $(X, \|\cdot\|)$. If there exists a nonempty open subset A of $\mathrm{Dom}(f^*)$ such that $\mathrm{argmax}(x^* - f) \neq \varnothing$ for each $x^* \in A$, then there exists a dense and G_δ subset R' of A such that

$$(x^* - f) : X \to [-\infty, \infty)$$

has a strong maximum for each $x^* \in R'$. In addition, if $0 \in A$ and $\varepsilon > 0$ then there exists an $x_0^* \in X^*$ with $\|x_0^*\| < \varepsilon$ such that $(x_0^* - f) : X \to [-\infty, \infty)$ has a strong maximum.

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The second one is a result from topology concerning norm continuity of minimal usco mappings.

The third one is the “Brøndsted-Rockafellar Theorem”.

Definition

Suppose that $f : X \to (-\infty, \infty]$ is a convex function defined on a normed linear space $(X, \|\cdot\|)$ and $x \in \mathrm{Dom}(f)$. Then we define the subdifferential $\partial f(x)$ by,
$\partial f(x) := \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x) \text{ for all } y \in \mathrm{Dom}(f)\}$.

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We now state a convex analysts' version of James' theorem.

Theorem (GJT - Generalised James' Theorem)

Let $(X, \|\cdot\|)$ be a Banach space and let A be a nonempty, open, convex subset of X^*. If $\varphi : A \to \mathbb{R}$ is a continuous, convex function and $\partial\varphi(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in A$, then $\partial\varphi(x^*) \subseteq \widehat{X}$ for all $x^* \in A$.

Here, and elsewhere, $\widehat{X}$ denotes the natural embedding of X into X^{**}.

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Suppose that C is a nonempty closed and bounded convex subset of a Banach space $(X, \|\cdot\|)$ with $0 \in C$.

Define $p : X^* \to \mathbb{R}$ by,

$$p(x^*) := \sup_{c \in C} x^*(c) = \sup_{c \in C} \widehat{c}(x^*) \quad \text{for all } x^* \in X^*.$$

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Then $\widehat{C} \subseteq \partial p(0)$, since $p(0) = 0$, and so

$$\partial p(0) = \{F \in X^{**} : F(x^*) \leq p(x^*) \text{ for all } x^* \in X^*\}.$$

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$$\partial p(0) = \{F \in X^{**} : F(x^*) \leq p(x^*) \text{ for all } x^* \in X^*\}.$$

If every $x^* \in X^*$ attains its supremum over C then

$$\partial p(x^*) \cap \widehat{X} \neq \varnothing \quad \text{for every } x^* \in X^*.$$

This last fact follows because, if $x^* \in X^* \setminus \{0\}$, $c \in C$ and $p(x^*) = x^*(c)$, then $\widehat{c} \in \partial p(x^*)$, since for any $y^* \in X^*$

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$$\begin{aligned} \widehat{c}(y^*) - \widehat{c}(x^*) &= y^*(c) - x^*(c) \\ &= y^*(c) - p(x^*) \\ &\leq p(y^*) - p(x^*), \text{ by the definition of } p(y^*). \end{aligned}$$

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Thus, by the GJT,

$$\overline{\widehat{C}}^{w^*} \subseteq \partial p(0) \subseteq \widehat{X} \quad \text{since, } \partial p(0) \text{ is weak}^*\text{-closed.}$$

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Hence, C is weakly compact since the relative weak and weak* topologies agree on $\widehat{X}$.

Definition

A set-valued mapping φ from a topological space A into subsets of a topological space (X, τ) is τ-upper semicontinuous at a point $x_0 \in A$ if for each τ-open set W in X, containing $\varphi(x_0)$, there exists an open neighbourhood U of x_0 such that $\varphi(U) \subseteq W$.

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A cusco from a topological space A into subsets of a linear topological space X is said to be a minimal cusco if its graph does not contain, as a proper subset, the graph of any other cusco on A.

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Our interest in cusco mappings is revealed in the next result.

Theorem

If $\varphi : U \to \mathbb{R}$ is a continuous convex function defined on a nonempty open convex subset U of a normed linear space $(X, \|\cdot\|)$, then the subdifferential mapping, $x \mapsto \partial\varphi(x)$, is a minimal weak*-cusco on U.

We shall say that a set-valued mapping $\Phi : A \to 2^X$ from a topological space (A, τ) into subsets of a normed linear space $(X, \|\cdot\|)$ is single-valued and norm upper semicontinuous at a point $x_0 \in A$ if: (i) $\Phi(x_0)$ is a singleton and (ii) for every $\varepsilon > 0$ there exists an open neighbourhood U of x_0 such that $\Phi(U) \subseteq B[\Phi(x_0), \varepsilon]$.

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Theorem (CT - J. P. R. Christensen's Theorem, 1982)

Let $\Phi : A \to 2^{X^{**}}$ be a minimal weak* cusco from a complete metric space A into subsets of the second dual of a Banach space $(X, \|\cdot\|)$. If $\Phi(x) \subseteq \widehat{X}$ for all $x \in A$, then Φ is single-valued and norm upper semicontinuous at the points of a dense and G_δ subset of A.

The key notion here is the "ε-subgradient".

Definition

Suppose that $f : X \to (-\infty, \infty]$ is a convex proper lower semicontinuous function on a normed linear space $(X, \|\cdot\|)$ and $x \in \mathrm{Dom}(f)$. Then, for any $\varepsilon > 0$, we define the *ε-subdifferential* $\partial_\varepsilon f(x)$ by,

$$\partial_\varepsilon f(x) := \{x^* \in X^* : x^*(y-x) \leq f(y) - f(x) + \varepsilon \text{ for all } y \in \mathrm{Dom}(f)\}.$$

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Theorem (BRT - Brøndsted-Rockafellar Theorem)

Suppose that $f : X \to (-\infty, \infty]$ is a convex proper lower semicontinuous function on a Banach space $(X, \|\cdot\|)$. Then, given any point $x_0 \in \mathrm{Dom}(f)$, $\varepsilon > 0$ and any $x_0^* \in \partial_\varepsilon f(x_0)$, there exists $x \in \mathrm{Dom}(f)$ and $x^* \in X^*$ such that $x^* \in \partial f(x)$, $\|x - x_0\| \leq \sqrt{\varepsilon}$ and $\|x^* - x_0^*\| \leq \sqrt{\varepsilon}$.

Proposition

Suppose that $f : X \to (-\infty, \infty]$ *is a proper function on a Banach space* $(X, \|\cdot\|)$ *Then,*

(i) f^* *is a convex and weak** *lower semicontinuous function on* $\mathrm{Dom}(f^*)$*;*

(ii) f^* *is continuous on* $\mathrm{int}(\mathrm{Dom}(f^*))$*;*

(iii) *if* $x^* \in \mathrm{Dom}(f^*)$ *and* $x \in \mathrm{argmax}(x^* - f)$ *then* $\widehat{x} \in \partial f^*(x^*)$*;*

(iv) *if* $\varepsilon > 0$, $x^* \in \mathrm{Dom}(f^*)$, $x \in X$ *and* $f^*(x^*) - \varepsilon < x^*(x) - f(x)$ *then* $\widehat{x} \in \partial_\varepsilon f^*(x^*)$*;*

(v) *if* $x_0^* \in \mathrm{int}(\mathrm{Dom}(f^*))$, $x \in \mathrm{argmax}(x_0^* - f)$ *and* $x^* \mapsto \partial f^*(x^*)$ *is single-valued and norm upper semicontinuous at* x_0^* *then* $x_0^* - f$ *has a strong maximum at* x*.*

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(i) For each $x \in \mathrm{Dom}(f)$ define $g_x : X^* \to \mathbb{R}$ by, $g_x(x^*) := \widehat{x}(x^*) - f(x)$. Then each function g_x is weak* continuous and affine. Now for each $x^* \in X^*$,

$$f^*(x^*) = \sup_{x \in \mathrm{Dom}(f)} g_x(x^*).$$

Thus, as the pointwise supremum of a family of weak* continuous affine mappings, the Fenchel conjugate of f, is itself convex and weak* lower semicontinuous. [Recall the general fact that the pointwise supremum of a family of convex functions is convex and the pointwise supremum of a family of lower semicontinuous mappings is again lower semicontinuous].

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(iii) Let y^* be any element of $\mathrm{Dom}(f^*)$. Then,

$$\begin{aligned}
\widehat{x}(y^*) - \widehat{x}(x^*) &= y^*(x) - x^*(x) \\
&= [y^*(x) - f(x)] - [x^*(x) - f(x)] \\
&= [y^*(x) - f(x)] - f^*(x^*) \\
&\leq f^*(y^*) - f^*(x^*).
\end{aligned}$$

Therefore, $\widehat{x} \in \partial f^*(x^*)$.

(iv) Let y^* be any element of $\mathrm{Dom}(f^*)$. Then,

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&\leq [y^*(x) - f(x)] - [f^*(x^*) - \varepsilon] \\
&\leq f^*(y^*) - f^*(x^*) + \varepsilon.
\end{aligned}$$

Therefore, $\widehat{x} \in \partial_\varepsilon f^*(x^*)$.

(v) Let $(x_n : n \in \mathbb{N})$ be a sequence in X such that

$$\lim_{n\to\infty}(x_0^* - f)(x_n) = \sup_{x' \in X}(x_0^* - f)(x') = f^*(x_0^*).$$

We will show that $(x_n : n \in \mathbb{N})$ converges to x. Let $\varepsilon > 0$. By (iii) and the assumption that $\partial f^*(x_0^*)$ is a singleton we have that $\partial f^*(x_0^*) = \{\widehat{x}\}$. Since, $x^* \mapsto \partial f^*(x^*)$, is norm upper semicontinuous at x_0^* there exists a $0 < \delta < \varepsilon$ such that if $\|x^* - x_0^*\| \leq \delta$ then $\|F - \widehat{x}\| < \varepsilon$ for all $F \in \partial f^*(x^*)$. Choose $N \in \mathbb{N}$ such that $(x_0^* - f)(x_n) > f^*(x_0) - \delta^2$ for all $n > N$. Then, by (iv), $\widehat{x_n} \in \partial_{\delta^2} f^*(x_0^*)$ for all $n > N$. Let $n > N$. Then, by the Brøndsted-Rockafellar Theorem, there exist $x_n^* \in \mathrm{Dom}(f^*)$ and $F_n \in X^{**}$ such that $F_n \in \partial f^*(x_n^*)$, $\|x_n^* - x_0^*\| \leq \delta$ and $\|F_n - \widehat{x_n}\| \leq \delta < \varepsilon$. Therefore,

$$\|x_n - x\| = \|\widehat{x_n} - \widehat{x}\| \leq \|\widehat{x_n} - F_n\| + \|F_n - \widehat{x}\| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Theorem (Abstract Variational Theorem)

Let $f : X \to (-\infty, \infty]$ be a proper function on a Banach space $(X, \|\cdot\|)$. If there exists a nonempty open subset A of $\mathrm{Dom}(f^*)$ such that $\mathrm{argmax}(x^* - f) \neq \varnothing$ for each $x^* \in A$, then there exists a dense and G_δ subset R' of A such that

$$(x^* - f) : X \to [-\infty, \infty)$$

has a strong maximum for each $x^* \in R'$. In addition, if $0 \in A$ and $\varepsilon > 0$ then there exists an $x_0^* \in X^*$ with $\|x_0^*\| < \varepsilon$ such that $(x_0^* - f) : X \to \mathbb{R} \cup \{-\infty\}$ has a strong maximum.

Proof.

Consider $\partial f^* : A \to 2^{X^{**}}$. Then, by the Proposition part (iii), $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in A$. Thus, by GJT, $\partial f^*(x^*) \subseteq \widehat{X}$ for all $x^* \in A$. Therefore, by CT, there exists a dense and G_δ subset R' of A such that ∂f^* is single-valued and norm upper semicontinuous at each point of R'. So, by the Proposition part (v), $(x^* - f)$ has a strong maximum for each $x^* \in R'$. $\square$

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The paper

"A Gentle Introduction to James' Weak Compactness Theorem and Beyond"

contains all the results presented in this talk.

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Definition

Let C be a nonempty closed and bounded convex subset of a normed linear space $(X, \|\cdot\|)$. We shall say that a point $x_0 \in C$ is a strongly exposed point of C if there exists an $x^* \in X^*$ such that $x^*|_C$ has a strong maximum at x_0.

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If $x^* \in X^*$ and $x^* - f$ has a strong maximum at $x_0 \in X$ then $x_0 \in C$ and, x_0 is in fact a strongly exposed point of C.

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Definition

Let C be a nonempty closed and bounded convex subset of a normed linear space $(X, \|\cdot\|)$. We shall denote by $\mathrm{Exp}(C)$ the set of all strongly exposed points of C.

Theorem

If C is a weakly compact convex subset of a Banach space $(X, \|\cdot\|)$ then $C = \overline{co}(Exp(C))$.

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Proof: Let $f : X \to (-\infty, \infty]$ be defined by, $f(x) := 0$ if $x \in C$ and by $f(x) := \infty$ otherwise. Then, since C is weakly compact, $\text{argmax}(x^* - f) = \text{argmax}(x^*|_C) \neq \varnothing$.

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Proof: Let $f : X \to (-\infty, \infty]$ be defined by, $f(x) := 0$ if $x \in C$ and by $f(x) := \infty$ otherwise. Then, since C is weakly compact, $\text{argmax}(x^* - f) = \text{argmax}(x^*|_C) \neq \varnothing$. Therefore, by the Abstract Variational Theorem, there exists a dense and G_δ subset R of X^* such that $(x^* - f)$ has a strong maximum for each $x^* \in R$.

Theorem

If C is a weakly compact convex subset of a Banach space $(X, \|\cdot\|)$ then $C = \overline{co}(Exp(C))$.

Proof: Let $f : X \to (-\infty, \infty]$ be defined by, $f(x) := 0$ if $x \in C$ and by $f(x) := \infty$ otherwise. Then, since C is weakly compact, $\text{argmax}(x^* - f) = \text{argmax}(x^*|_C) \neq \varnothing$. Therefore, by the Abstract Variational Theorem, there exists a dense and G_δ subset R of X^* such that $(x^* - f)$ has a strong maximum for each $x^* \in R$. Now suppose, in order to obtain a contradiction, that $C \neq \overline{\text{co}}(\text{Exp}(C))$.

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$$\sup\{x^*(c) : c \in \overline{\text{co}}(\text{Exp}(C))\} < x^*(x_0).$$

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Since C is bounded and R is dense in X^* we can assume, without loss of generality, that $x^* \in R$.

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Thank you for your attention and for the opportunity to present my work.