# Beyond Behavioural Economics

## A Process Realist Perspective

## Mathematics of Behavioural Economics and Finance

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Ultimate scientific explanation is attractive and important, but financial engineering cannot wait for full explanation. That is, it is legitimate to strive towards a second best: a "descriptive phenomenology" that is organized tightly enough to bring a degree of order and understanding.

Benoit B. Mandelbrot (1997)

# Overview

- Keynes on DM under uncertainty & role of conventions

- Ontological considerations
  - Objective not just subjective

- Applications:
  - Bounded sub-additivity (Tversky & Wakker)
  - Equivalent approach—fuzzy measure theory (approximation algebras)
  - Concept lattices and "framing"
  - Tsallis arithmetic: $q$-generalized binomial approximation

[…] civilization is a thin and precarious crust erected by the personality and will of a very few and only maintained by rules and conventions skillfully put across and guilefully preserved" (Keynes, X: 447)

The "love of money as a possession" is a "somewhat disgusting morbidity, one of those semi-criminal, semi-pathological propensities which one hands over with a shudder to the specialists in mental disease (IX: 329)
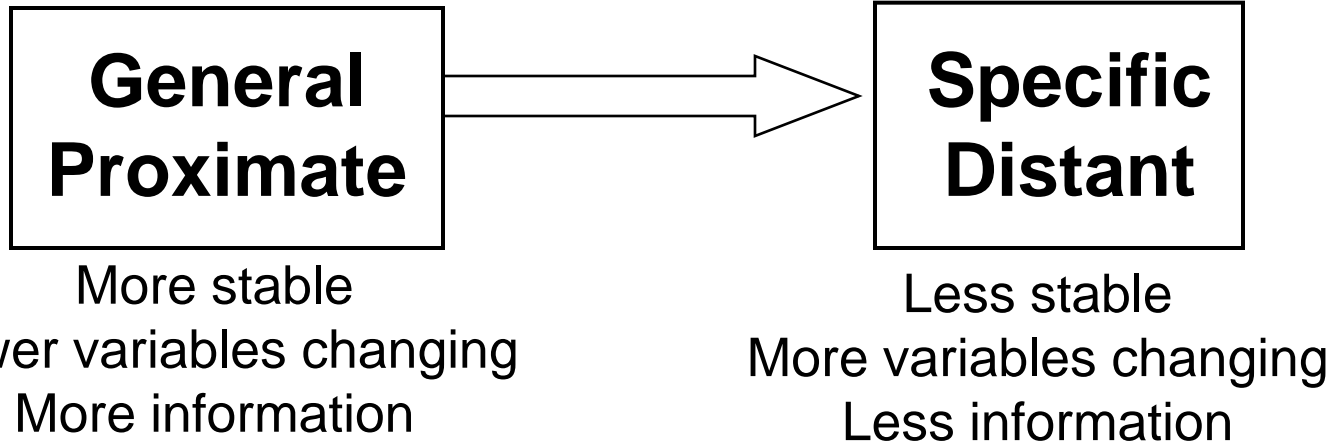
For we shall enquire more curiously than is safe today into the true character of this 'purposiveness' with which in varying degrees Nature has endowed almost all of us. For purposiveness means that we are more concerned with the remote future results of our actions than with their own quality or their immediate effects on the environment. The 'purposive' man is always trying to secure a spurious and delusive immortality for his acts by pushing his interest in them forward into time. He does not love his cat, but his cat's kittens; nor, in truth, the kittens, but only the kittens' kittens, and so on forward to the end of catdom. For him jam is not jam unless it is a case of jam tomorrow and never jam today. Thus, by pushing his jam forward into the future, he strives to secure for his act of boiling it an immortality. (IX 329-30)

Why should anyone outside a lunatic asylum wish to use money as a store of wealth? Because, partly on reasonable and partly instinctive grounds, our desire to hold money as a store of wealth is a barometer of the degree of distrust of our own calculations and conventions concerning the future. Even though this feeling about money is itself conventional or instinctive, it operates, so to speak, at a deeper level of our motivation. It takes charge at the moments when the higher, more precarious conventions have weakened. The possession of actual money lulls our disquietude; and the premium which we require to make us part with money is the measure of the degree of our disquietude (XIV: 116)

Winslow's (2005) argument: via Bloomsbury, Keynes was heavily influenced by, though was still wary of the more speculative aspects of, Freudian psychoanalysis with respect to his analysis of the "money love" exhibited by the *anal sadistic* character type (cruelty, authoritarianism, and obsessive attempts at control!)

Winslow's (1989) argument: Keynes was persuaded by Whitehead's organicist (Varzi, 2014) analysis of the extensive continuum to replace an *epistemological* (i.e. degree of rational belief + weight) with an *ontological* approach to probability, which further distinguished b/n SR & LR expectations as to their grounding

# Whitehead's Nested Ontology

| General Proximate | → | Specific Distant |
|---|---|---|

More stable
Fewer variables changing
More information

Less stable
More variables changing
Less information

e.g.   Living Organisms → Human beings → Entrepreneurs
e.g.      Short-run expectations → Long run expectations
### *Organic Interdependence*
Relevant factors limited to portion of 'extensive continuum'
- Relations between parts of complex as internal, hence necessary
- Future derived from past in way that preserves existence of this complex
- The resulting interdependence supports rational judgments based on partial knowledge
- Limited set of possibilities created for individual called 'real' potentiality
- Provides local foundation for frequency theory of probability

# Ontological considerations

- **Extreme scepticism** (Speculative Realism's *Principle of Unreason*): while we can have direct rational access to truths about the nature of being, this makes it clear that being itself can always become other than what it seems to us and as it is in itself!

- For Whitehead (and Keynes?), *fundamental* uncertainty (FU) is accounted for by ontological & *objective* structure/properties of the '**extensive continuum**' (strains) and not merely by epistemic & *subjective* informational constraints over what could otherwise become knowable by the agent through interactive learning or market selection/evolution

- Often justified by notion that society is an *open system*: as our ideas about the world & correlative institutions change, so does the social world

- Simonian view: conventions of decision-making are 'rational heuristics', not 'biases' or 'anomalies' (i.e. conceived as departures from a "true" neoclassical benchmark)

- Formally, they can be modelled by a topos within which monetary value measures (sheaves) come into play (Adachi; Baez, Fritz & Leinster, 2011, Vickers, 1989)

- Processes of **arbitrage** in financial markets ( i.e. Kotlikoff's law) must be modelled by *dagger compact* categories (which arise under the constraints imposed by relevant quadratic form, when arbitrage is represented by closed, but arbitrarily permutable, network diagrams) rather than *symmetric monoidal* categories applying to signal-flow-graphs! (Baez & Fong, 2014)

- A formal mathematical **equivalence** holds between 3 approaches to DM under uncertainty: multiple-priors, sub-additivity (Choquet integration) & fuzzy measure theoretic (via ortho-normal lattice structure & logics) (Schmeidler, Mirofushi & Sugeno, Gilboa & Schmeidler, Cattaneo & Ciucci, 2004)

# Ontological Considerations

In accounting for interactions between:

- Non-financial (real) and financial (nominal) parts of the economy
    - Must reject "**block recursivity**" of mainstream macroeconomic models which implies that former determines the latter (Sargent, 1986) e.g. via shocks to real forces of productivity and thrift (Brock, 1989;)
    - But we are also obliged to model **feedbacks** from the real to the financial via reduced aggregate demand

- Other developments
    - Catastrophe-theoretic inter-play between fundamentalist and chartists in **adaptive rational expectation equilibria** in asset-markets, alone (Brock and Hommes)
    - Alternatively, via **coupled Phillips Curve** mechanism (applying to both goods & labour markets) + by debt-deflation & asset price effects (Chiarella et al)
    - Ecological models featuring increasing numbers of (destabilizing) **predator-prey** interactions (Haldane and May)
    - Minskyian models: for which financial **fragility**, in turn, can lead to financial **instability**, which then, has real effects (compounded by real impact of policies: e.g. fiscal conservatism and real wage repression) (Taleb's coffee cups!)
    - Thermodynamic analysis of '**anomalous diffusion**' processes (Tsallis) and "far-from-equilibrium" systems characterised by emerging chaos and self-organization (Prigogine), yet little interaction!
    - Benoit Mandelbrot's scaling random fractals

- Uncertainties applying to multiple-priors, robust control, & sub-additivity approaches can always be interpreted *objectively* rather than *subjectively* (although "ambiguity" implies incomplete though *potentially* complete knowledge)

# Cumulative Prospect Theory

A prospect $(x_1, p_1;\ldots;x_n, p_n)$, yielding outcome $x_j$ with probability $p_j$, and in which outcomes are ranked in order of magnitude ,

$$x_1 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots x_n$$

the value of the prospect is given by the following function (Tversky and Wakker, 1995, p 1259):

$$\sum_{j=1}^{k} \pi_j^- v(x_j) + \sum_{j=k+1}^{n} \pi_j^+ v(x_j)$$

where the decision weights are defined by:

$$\pi_j^- = w^-(p_1 + \cdots p_j) - w^-(p_1 + \cdots p_{j-1}) \text{ and}$$

$$\pi_j^+ = w^+(p_j + \cdots p_n) - w^+(p_{j+1} + \cdots p_n)$$

Individuals are risk-seeking for gains and risk averse for losses of *low* probability, while on the other hand, they are risk averse for gains and risk seeking for losses of *high* probability

- Choquet Integral: [Reesor & McLeish]
    - distortion function, *g*, is any non-decreasing function on [0,1] such that $g(0) = 0$ and $g(1) = 1$.
    - If a random variable *X* under the probability measure *P* has a cumulative distribution function (cdf) *F* defined by $F(x) = P[X \leq x]$, and a decumulative distribution function (dff) *S* defined by $S(x) = P[X \geq x] = 1 - F(x^-)$, and if $g(u)$ is a left-continuous distortion function then $S^*(x) = g[S(x)]$ is a ddf corresponding to a distorted probability distribution
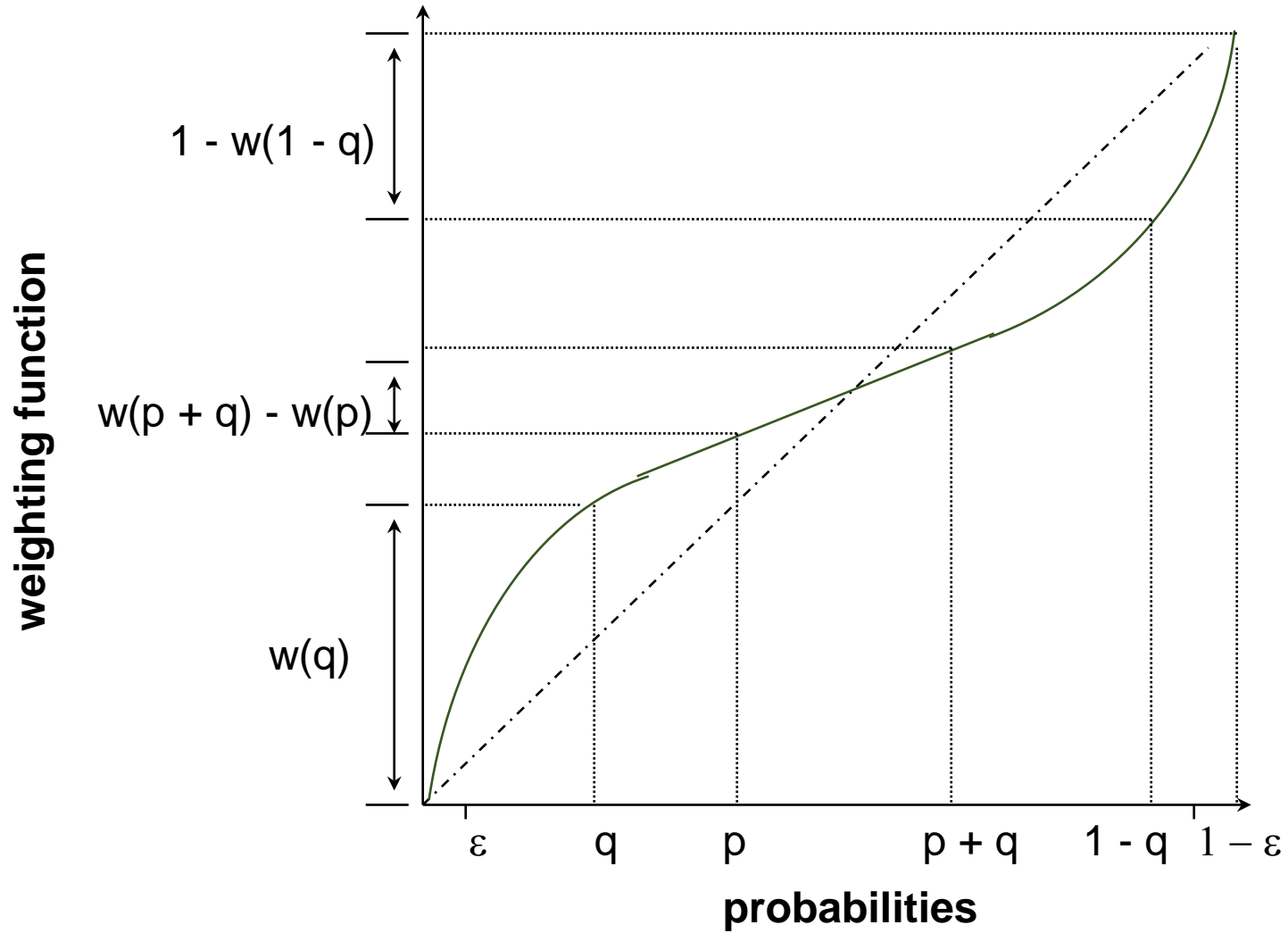
$$E_{P*}[X] = E_{P*}[X^+] - E_{P*}[X^-] = \int_0^\infty S^*(x)dx - \int_0^\infty F^*(-x)dx$$

$$= \int_0^\infty g(S(x))dx - \int_0^\infty \overline{g}(F(-x))dx = \int_0^\infty g(S(x))dx - \int_0^\infty [1 - g(S(-x))]dx$$

$$= \int_0^\infty g(S(x))dx + \int_{-\infty}^0 [g(S(x)) - 1]dx$$

- Choquet Integral with respect to a distortion function *g* is accordingly given by $H_g(X)$ in:
- Dual dist. fn.

$$\overline{g}(u) = 1 - g(1 - u)$$

$$H_g(X) = \int_0^\infty g[S(x)]dx + \int_{-\infty}^0 [g[S(x)] - 1]dx$$

$$= \int_0^\infty [1 - \overline{g}[F(x)]]dx - \int_{-\infty}^0 \overline{g}[F(x)]dx$$

**Subadditivity in Preferences (Taversky & Wakker)**

weighting function

probabilities

$1 - w(1 - q)$

$w(p + q) - w(p)$

$w(q)$

$\varepsilon$  $q$  $p$  $p + q$  $1 - q$  $1 - \varepsilon$

- In the case of decision-making under *risk, w* satisfies the condition of bounded sub-additivity: (Tversky & Wakker, 1995: 1260)

  if there exist boundary constants $\varepsilon \geq 0, \varepsilon' \geq 0$ such that

  $$0 + w(q) \geq w(p+q) - w(p) \text{ whenever } p+q \leq 1-\varepsilon$$

  $$\text{and } 1 - w(1-q) \geq w(p+q) - w(p) \text{ whenever } p \geq \varepsilon'$$

- In the case of decision-making under *uncertainty* the lower and upper sub-additivity conditions, respectively, are satisfied if there are events *E* and *E'* such that:

  $$\emptyset + W(B) \geq W(A \cap B) - W(A) \text{ whenever } W(A \cap B) \leq W(S-E)$$
  $$1 - W(S-B) \geq W(A \cup B) - W(A) \text{ whenever } W(A) \geq W(E')$$

# Fuzzy Measure Approach

- Work within partially ordered set (Cattaneo & Ciucci)
- Supplement existing connectives:
  - Negation: $\sim a := a \Rightarrow 0$
  - Meet: $a \leq b$ iff $a \wedge b = a$
  - Implication: $a \Rightarrow b := \sup\{c \mid a \wedge c \leq b\}$
- With two additional primitive implication predicatives:
  - Łukasiewicz: $a \rightarrow_L b := \min\{1, 1 - a + b\}$
  - Gödel: $a \rightarrow_G b := (1$ if $a \leq b$ or $b$ if $a > b)$
- And their associated negation connectives:
  $\neg a := a \rightarrow_L 0 = 1 - a$
  $\sim a := a \rightarrow_G 0 = (1$ if $a = 0$ or $0$ otherwise$)$
- Plus a third:
  $\flat\, a := \neg \sim \neg a = (0$ if $a = 1$ or $1$ otherwise$) = $ ***contingency***

1. $a \lor b := (a \rightarrow_L b) \rightarrow_L b = \max\{a, b\}$
2. $a \land b := \neg((\neg a \rightarrow_L \neg b) \rightarrow_L \neg b) = \min\{a, b\}$
3. $a \oplus b := \neg a \rightarrow_L b = \min\{1, a + b\}$
4. $a \odot b := \neg(a \rightarrow_L \neg b) = \max\{0, a + b - 1\}$
5. $\nu(a) := (a \rightarrow_L 0) \rightarrow_G 0 = (1$ if $a = 1$ or $0$ otherwise$)$
6. $\mu(a) := (a \rightarrow_G 0) \rightarrow_L 0 = (0$ if $a = 0$ or $1$ otherwise$)$
   - $\nu$ and $\mu$ are realizations of the modal–like connectives of **necessity** and **possibility**
   - They satisfy the properties of a S5 system
   - And the distributive properties of a Kleene lattice:
     - $\nu(a \lor b) = \nu(a) \lor \nu(b)$
     - $\mu(a \land b) = \mu(a) \land \mu(b)$
   - **And these in turn define the boundaries of a fuzzy set**

- S5 Axioms (principles of modal logic $\approx$ closure axiom in topology):
  - $\nu(1) = 1$: if a sentence is true, then also its necessity its true (N)
  - $a \leq b$ implies $\nu(a) \leq \nu(b)$: if a conditional and its antecedent are both necessary, then so is the consequent (K)
  - $\nu(a) \leq a \leq \mu(a)$: necessity implies actuality and actuality implies possibility (T)
  - $\nu(a) = \nu(\nu(a))$ and $\mu(a) = \mu(\mu(a))$: whatever is necessary (resp., possible) is necessarily necessary (resp., possibly possible)
  - $\mu(a) = \neg(\nu(\neg a))$: what is possible is just what is not–necessarily–not (DF)
  - $a \leq \nu(\mu(a))$: actuality implies necessity of possibility (B)
  - $\mu(a) = \nu(\mu(a)), \nu(a) = \mu(\nu(a))$: possibility is equal to the necessity of possibility; whereas necessity is equal to the possibility of necessity (S5)

# Concept Lattices [Wille, 1992]

- **Context** = triple ($G$, $M$, $I$) where $G$ and $M$ are sets while $I$ is a binary relation between $G$ and $M$, i.e.,
  - $I \subseteq G$ x $M$
  - the elements of $G$ and $M$ are called *objects* (in German: *Gegenstānde*) and *attributes* (in German: *Merkmale*)
  - *gIm,* i.e., ($g$, $m$) $\in I$, is read: the object $g$ has the attribute $m$
- **Derivation operators** (represented by "prime"):
  - $X \mapsto X' = \{m \in M | gIm \text{ for all } g \in X\}$
  - $Y \mapsto Y' = \{g \in G | gIm \text{ for all } m \in Y\}$
- Which form a Galois-connection b/n power-sets of $G$ & $M$

- Duality b/n objects and attributes:
  1) $X_1 \subseteq X_2$ implies $X'_2 \subseteq X'_1$ for $X_1, X_2 \subseteq G$
  1') $Y_1 \subseteq Y_2$ implies $Y'_2 \subseteq Y'_1$ for $Y_1, Y_2 \subseteq M$
  2) $X \subseteq X''$ and $X' = X'''$ for $X \subseteq G$
  2') $Y \subseteq Y''$ and $Y' = Y'''$ for $Y \subseteq M$
  3) $(\bigcup_{t \in T} X_t)' = \bigcap_{t \in T} X'_t$ for $X_t \subseteq G (t \in T)$
  3') $(\bigcup_{t \in T} Y_t)' = \bigcap_{t \in T} Y'_t$ for $Y_t \subseteq M (t \in T)$
- Concept as unit constituted by extension/intension:
  - Pair $(A, B)$ is formal *concept* of context $(G, M, I)$ if $A \subseteq G, B \subseteq M$, $A = B', B = A'$, with *A* = *extent*, *B* = *intent*
- Intuitively,
  - every object in *A* has every attribute in *B*,
  - for every object in *G* that is not in *A*, there is an attribute in *B* that the object does not have,
  - for every attribute in *M* that is not in *B*, there is an object in *A* that does not have that attribute
- For a set of objects *A:*
  - the set of their common attributes *A'* describes the similarity of objects of the set *A*
  - while the closed set *A"* is a cluster of similar objects with the set of common attributes *A'*

- Subconcept-superconcept-relation on set of all concepts, $\mathfrak{B}(G, M, I)$:
  - $(A_1, B_1)$ is a *subconcept* of the concept $(A_2, B_2)$ if $A_1 \subseteq A_2$ which is equivalent to by $B_1 \subseteq B_2$ by (1) and (1')
  - $(A_2, B_2)$ is then a *superconcept* of $(A_1, B_1)$
  - Since this definition yields an order relation, the subconcept-superconcept-relation is denoted by $\leq$

- Basic Theorem for Concept Lattices:
  - $\bigvee_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t (\bigcup_{t \in T} B_t)'')$
  - $\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t \in T} A_t)'', \bigcap_{t \in T} B_t)$

- A *concept lattice* approach to DM
  - DM within categorical set: [Davey & Priestly, 1998: Ch. 7]
    - that can be classified and ordered by 'ease of accessibility'
    - i.e. automatic recall on the basis of variety of factors such as 'prominence' & 'similarity'
    - Which, in turn, encompass within a unified frame Kahneman and Tversky's notions of 'representativeness', 'anchoring', and 'cognitive availability'
    - each of which are, themselves, influenced by *mood* and *affective valency*
    - (i.e. Louis Bunuel's "obscure object of desire" meets Whitehead's "lure of the proposition"!)
  - And all this can all be correlated with Keynes's arguments about:
    - Animal spirits, evidential weight, states of confidence, and
    - those specifically social influences over inferences that can be made from the past to the future on the basis of convention, average opinion, and the current plausibility or implausibility of the usual assumption that the existing state of affairs is likely to continue for an indefinite period into the future
    - but once again the formal link to uncertainty must be articulated within this formalism, which can be achieved by combining a pertinent concept lattice with another derived from an *approximation algebra* so that concepts applying to a specific context of DM under uncertainty can be correlated with the fuzzy measures determined with the lattice structure of this algebra

# Tsallis Entropy

- Justification: Let $\alpha \in (0, \infty)$. Suppose $F$ is any map sending morphisms in **FinMeas** to numbers in $[0, \infty)$, and obeying these four properties: (Baez et al., 2011)
  - Functoriality: $F(f \circ g) = F(f) \circ F(g)$
  - Additivity: $F(f \oplus g) = F(f) + F(g)$ for al $f$, g
  - Homogeneity of degree $\alpha$: $F(\lambda f) = \lambda^\alpha F(f)$   [for **Finprob**, $F(\lambda f \oplus (1 - \lambda)g) = \lambda^\alpha F(f) + (1 - \lambda)^\alpha F(g)$]
  - Continuity
- Then there exists  constant $c \geq 0$ such that for any morphism $f: p \to q$ in **FinMeas**, $F(f) = c(H_\alpha(p))$-$H_\alpha(q)$ where $H_\alpha$ is Tsallis entropy of order $\alpha$
- New limit theorems, burgeoning applications, growing empirical support
  - cold atoms in dissipative optical lattices, dusty plasmas, trapped ions, spin-glasses, turbulence, self-organized criticality, high-energy experiments at LHC/CMS/CERN and RHIC/PHENIX/Brookhaven, low-dimensional dissipative maps
- Already a large number of finance publications (see Tsallis biblio)
- Tsallis himself has remarked on congruence with weighting functions deployed in CPT (next overhead)
- But finance applications (e.g. Borland, L.) mostly based on hard-core stochastic calculus (Fokker-Planck equations)
- Are there simpler (numerical methods-based) alternatives? ANS. *Yes*!

# CVP, Prelec, & Tsallis Entropy

## Prelec's *S*-shaped Probability Weighting Functions

$$\upsilon(p) = \gamma \exp\left[-\beta(-\ln p)^\alpha\right]$$

Under *SA,* an event *B* has a greater impact when it turns impossibility into possibility or possibility into certainty, than when it merely makes a possibility more likely (Tversky & Wakker, 1995: 1264).

## Tsallis Entropy Equivalents to Weighting functions of CPT

$$\Pi_1(p) = \frac{p^q}{p^q + (1-p)^q}; \quad \Pi_2(p) = \frac{p^q}{\left[p^q + (1-p)^q\right]^{1/q}}, \quad \Pi_3(p) = \frac{p^q}{\left[p^q + A(1-p)^q\right]}, \quad A > 0.$$

Applicable to dynamic "far-from-equilibrium" systems characterised by emerging chaos and self-organization: e.g. triplet of *q*-parameters ($q_{stat}$, $q_{sen}$, $q_{rel}$) which characterise, respectively, the properties of the resulting meta- or quasi-stationary distribution, generalized exponential sensitivity to initial conditions, and generalized exponential relaxation of macroscopic quantities to thermal equilibrium (e.g. empirically confirmed for phenomena such as solar wind in the distant heliosphere) (Berlaga and Viñas, *Physica A*, 2005) NB. Suyari and Wada (2008) provide an explanation for the *q*-triplet, which draws on the asymptotic and duality-related properties of a discrete two-parameter general. of multinomial distribution] i.e. an **ontological not epistemological explanation!**

# A Case-study: deploying Tsallis's *q*-arithmetic Generalized Option Pricing

- Definition of *q*-product: (Suyari)

$$x \otimes_q y := \begin{cases} \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} + y^{1-q} - 1 > 0 \\ 0, & \text{otherwise} \end{cases}$$

- *q*-log & *q*-exponential

$$\ln_q(x) := \frac{x^{1-q} - 1}{1 - q} \quad x > 0, q \in \Re^+$$

$$\exp_q(x) := \begin{cases} \left[1 + (1-q)x\right]^{\frac{1}{1-q}} & \text{if } 1 + (1-q)x > 0, \ x > 0, q \in \Re^+ \\ 0 & \text{otherwise} \end{cases}$$

- *q*-sum:

$$x \oplus_q y \equiv x + y + (1-q)xy$$

- *q*-sum inverse:

$$\oplus_q^{-1} x \equiv \frac{-x}{1+(1-q)x}, \quad x \neq 1/(q-1)$$

- *q*-difference:

$$x \oplus_q^{-1} y \equiv x \oplus_q \left(\oplus_q^{-1} y\right) = \frac{x-y}{1+(1-q)y}, \quad y \neq 1/(q-1)$$

- *q*-product inverse:

$$1 \otimes_q^{-1} x \equiv \left[2 - x^{1-q}\right]_+^{1/(1-q)}, \quad x > 0$$

- *q*-product satisifies:

$$\ln_q\left(x \otimes_q y\right) = \ln_q x + \ln_q y$$

$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y)$$

- Inverse of *q*-product

$$x \otimes_q^{-1} y := \begin{cases} \left[x^{1-q} - y^{1-q} + 1\right]^{\frac{1}{1-q}} & \text{if } x > 0, y > 0, x^{1-q} - y^{1-q} - 1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Tsallis distribution
  – derived from *q*-product of likelihood function:

$$f(x) = \frac{\exp_q\left(-\beta_q x^2\right)}{\int \exp_q\left(-\beta_q x^2\right) dx}, \quad \beta_q > 0$$

# Derivation of Student T

$$\max S_q \ \ s.t. \ \ \left\langle x^2 \right\rangle_q \equiv \int dx \, x^2 \left[p(x)\right]^q = 1, \quad \int dx \, p(x) = 1$$

$$\phi_m^{(t)}(y) = \sqrt{\frac{1+m}{2\beta m}} \, p\!\left(y\sqrt{\frac{1+m}{2\beta m}}\right) = \frac{1}{\sqrt{\pi m}} \frac{\Gamma\!\left(\dfrac{1+m}{2}\right)}{\Gamma\!\left(\dfrac{m}{2}\right)} \frac{1}{\left(1+\dfrac{y^2}{m}\right)^{\frac{1+m}{2}}}$$

Student t-distribution with $m$ degrees of freedom

# Suyari's q-generalisation of BDist

- *q*-factorial for $n \in N$, $q > 0$

$$n!_q := 1 \otimes_q \cdots \otimes_q n = \left[ \sum_{k=1}^{n} k^{1-q} - (n-1) \right]^{\frac{1}{1-q}}$$

- *q*-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := (n!_q) \otimes^{-1}_q \left[ (k!_q) \otimes_q (n-k)!_q \right] \quad n,k(\leq n) \in \mathbf{N}$$

- Yields Tsallis distribution!

$$\ln_q \begin{bmatrix} n \\ k \end{bmatrix}_q \cong \begin{cases} \dfrac{n^{2-q}}{2-q} \cdot S_{2-q}\left( \dfrac{k}{n}, \dfrac{n-k}{n} \right) & \text{if } q > 0, \, q \neq 2 \\ -S_1(n) + S_1(k) + S_1(n-k) & \text{if } q = 2 \end{cases}$$

- Recursion Formula:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \left( n \otimes_q^{-1} k \right) \otimes_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

- Matlab Routines:
  - http://www.volopta.com/Matlab.html

# Exact Pricing Formula

**The Assumptions:**

**A1** Frictionless markets (no restrictions on short sales, all securities infinitely divisible, unrestricted borrowing and lending at risk-free interest rate)

**A2** Risk-free interest rate constant over life of option

**A3** Underlying asset pays no dividends over life of option

**A4** Stock prices follow multiplicative binomial process described by:

$$S \quad \overset{q}{\nearrow} \quad uS$$
$$\quad \underset{1-q}{\searrow} \quad dS \text{ where } d = 1/u$$
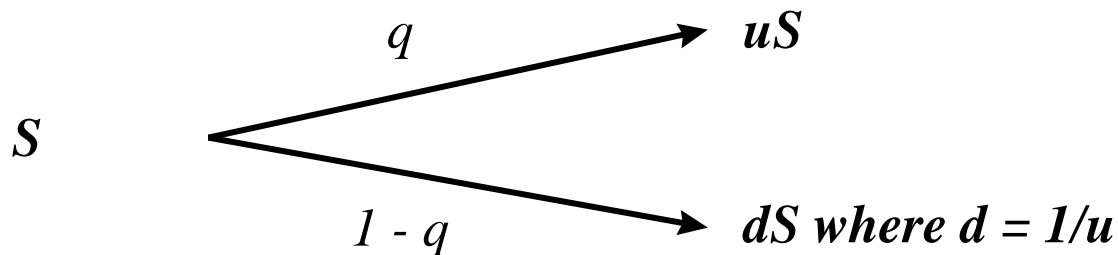
The **no-arbitrage** condition implies that [$u > (1 + r) > d$]

Again, let $S = 100, u = 1.8, d = 0.6$

$$u \equiv \frac{S^+}{S} = 1 + R^+$$

$$d \equiv \frac{S^-}{S} = 1 + R^-$$

$$N = \frac{C^+ - C^-}{(u-d)S}$$

$$B = \frac{dC^+ - uC^-}{(u-d)(1+r)}$$

$$C = \frac{pC^+ + (1-p)C^-}{1+r}$$

$$p = \frac{(1+r)-d}{u-d} =$$

e.g. $r = 8\%, u = 1.8, d = 0.6 \backslash$

implies $p = \frac{1.08 - 0.6}{1.8 - 0.6} = 0.4$

Let the time to expiration of the option, t, be sub-divided into *n* equal sub-intervals of length . Starting at the expiration date and working backwards, the multiplicative binomial option pricing formula for *n*-periods becomes:

$$C = \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^{j}(1-p)^{n-j} \max(u^{j}d^{n-j}S - E,\, 0)}{(1+\text{r})^{n}}$$

If *m* is the minimum number of upward moves *j* over *n* periods necessary for the option to be exercised or finish in the money (i.e., ) then the formula can be written as:

$$C = S\Phi[m; n, p'] - \frac{E}{(1+r)^n} \Phi[m; n, p]$$

where $\Phi$ is the complementary binomial distribution function giving the probability of at least $m$ ups out of $n$ steps

$$\equiv \Phi[m; n, p] \equiv \sum_{j=m}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

and $p' \equiv \left(\frac{u}{1+r}\right) p$

By choosing the parameters ($u$, $d$, and $p$) so that the mean and variance of the continuously compounded rate of return of the discrete binomial process are consistent with their continuous counterparts, the stock-price will become normally distributed and the F[.] function will converge to the standard normal distribution function, $N(.)$. Specifically, by setting

$$u = e^{\sigma\sqrt{h}}$$

$$d = 1/u$$

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{h}$$

where $\mu \equiv \ln r - \dfrac{1}{2}\sigma^2$ and $h \equiv \tau/n$ is the trading interval

As $n \to \infty$, $\Phi[m; n, p'] \to N(x)$

so that the binomial formula converges to the continuous time Black - Scholes formula

$$C = SN(x) - E(1 + r)^{-\tau} N\left(x - \sigma\sqrt{\tau}\right)$$

where $x \equiv \dfrac{\ln\left(S/E(1+r)^{-\tau}\right)}{\sigma\sqrt{\tau}} + \dfrac{1}{2}\sigma\sqrt{\tau}$

Other factors being equal, the value of a call option is higher (1) the higher the value of the underlying asset, $S$; (2) the longer the time to expiration, $t$; (3) the lower the exercise price, $E$; (4) the higher the variance of asset returns, $s^2$; and (5) the higher the riskless interest rate, $r$.h

$$C = S\Phi[m; n, p'] - \frac{E}{(1+r)^n} \Phi[m; n, p]$$

where $\Phi$ is the complementary binomial distribution function giving the probability of at least $m$ ups out of $n$ steps

$$\equiv \Phi[m; n, p] \equiv \sum_{j=m}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

and $p' \equiv \left(\frac{u}{1+r}\right) p$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \left(n \otimes_q^{-1} k\right) \otimes_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

# Conclusion

- In the light of the above ontological conception of DM under uncertainty, scope exists for further developing certain approaches to modelling

- I have considered the following:
  - one variant of fuzzy measure theory based on approximation algebras (with current applications limited to modalities in computation)
  - Concept lattices & framing (viz psychometric techniques)
    - Both of which can be brought together under the umbrella of "fuzzy concept lattices"—a growing, but fairly tecky literature
  - Tsallis version of Cox, Ross, & Rubinstein as a feasible vehicle, which could be accomplished in Excel using Visual Basic!

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