Experimental computations in random walks on groups

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Outline

1. Introduction
   1. Random walks on groups.
   2. The Poisson boundary.
   3. Applications in group theory.
2. How experimental computations can aid in describing the Poisson boundary.

Random walks on groups

- A *random walk* is a l.c. group G with a probability measure μ on G.
- Start at the identity and randomly multiply on the right by group elements distributed by μ.
- If a group is finitely generated and we choose uniformly over a symmetric generating set, then the walk is called *simple*.
- Example: Simple random walk on the $(\mathbb{Z}, +)$.
  - Start a counter at 0
  - Repeatedly increment or decrement the counter by 1 with equal probability.

0   -1   -2   -3   -2   -1   -2   -1   -2   ...

Random walks on groups

- A group with measure can be identified with a discrete time homogeneous Markov chain with state space G and transition probabilities given by

$$P(g, A) = \mu(g^{-1}A).$$

and initial distribution δ_e.
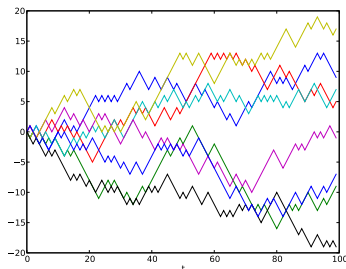
- The transition probabilities are left invariant

$$p(hg, hA) = p(g, A) \quad \forall g, h \in G, A \in \mathscr{B}(G).$$

The path space

- Let $G^\infty = \prod_{i \in \mathbb{N}} G$ with the *product sigma algebra*.
- An element $y = (y_0, y_1, y_2 \ldots) \in G^\infty$ is called a *path*.
- G has a measurable action on G^∞.

$$g \cdot y := (gy_0, gy_1, gy_2 \ldots)$$

- Let $\mathbb{P}^\mu$ be the product measure $\delta_e \times \mu \times \mu^2 \times \mu^3 \times \cdots$ on G^∞ where μ^n is the nth convolution power of μ.
- The pair $(G^\infty, \mathbb{P}^\mu)$ is called the *path space*.

The Poisson boundary

- Let $\sim$ be the equivalence relation on G^∞ where $x \sim y$ if $\exists N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that:

$$x_i = y_{i+k} \ \forall i > N.$$

- Let $\Gamma = X/\sim$ and let $p : G^\infty \to \Gamma$ be the projection map.
- Let Σ_Γ be the largest sigma algebra so that p is measurable.
- Let $\nu = \mu \circ p^{-1}$ be the push-forward measure with respect to p. Call ν the *hitting measure*.
- G has a measurable action on Γ using the G^∞ action on the elements of each coset.
- The pair (Γ, ν) is called the Poisson boundary of (G, μ)

The Poisson boundary

- G can also act measurably on $G \sqcup \Gamma$ with the appropriate action.
- A *measure topology* on $G \sqcup \Gamma$ may be defined so that for $\mathbb{P}^\mu$-almost every path y the sequence $\{y_n\}$ converges to a point in Γ, and if y_n converges to $y^\infty \in \Gamma$, then gy_n converges to gy^∞.
- If G is embedded in a topological space B so that $\mathbb{P}^\mu$-a.e path converges in B, and G acts on B measurably in this way, then it is possible that B with the resulting hitting measure could be identified with the Poisson boundary.

Group properties and the Poisson boundary

- The Poisson boundary is always trivial for nilpotent groups. (Chu and Hilberdink 1996)
- It can be non-trivial for solvable groups.
- Every countable amenable group has a non-degenerate symmetric probability measure such that the boundary is trivial (Rosenblatt 1981, Kaimanovich and Vershik 1983).
- If supp μ generates a non-amenable group, then the Poisson boundary is non-trivial (Rosenblatt 1981, and later Kaimanovich and Vershik 1983).
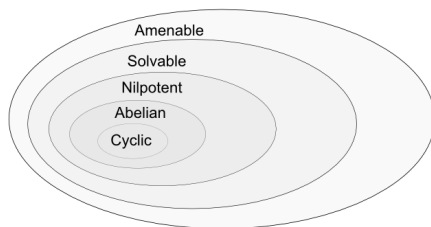
Some applications in group theory

- If G is a finitely generated solvable group then it admits a symmetric measure with non-trivial Poisson boundary if and only if the group is not virtually nilpotent (Erschler 2004).
- With respect to the word length metric, the growth of a finitely-generated group is a measure of how quickly the balls of radius r grow.
- Gromov's theorem states that G has polynomial growth if and only if G is virtually nilpotent.
- These facts can be combined to give a lower bound on the growth of finitely generated solvable groups.

A family of matrix groups

- Let G'_n be the group of the upper triangular matrices with integer powers of two on the diagonal, and dyadic rationals in entries above the diagonal. For example

$$G'_3 = \left\{ \begin{pmatrix} 2^x & f & h \\ 0 & 2^y & g \\ 0 & 0 & 2^z \end{pmatrix} : x, y, z \in \mathbb{Z}, f, g, h \in \mathbb{Z}[1/2] \right\}$$

- Kaimanovich suggested that it would be interesting to describe the Poisson boundary of G'_n for certain measures. I'm currently trying to do this.

As a semi-direct product

- G_n' is isomorphic to a semi-direct product $G_n = H_n \ltimes N_n$.
  - H_n is the group of $n \times n$ diagonal matrices whose non-zero entries are integer powers of 2. e.g.

$$H_3 = \left\{ \begin{pmatrix} 2^x & 0 & 0 \\ 0 & 2^y & 0 \\ 0 & 0 & 2^z \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

  - N_n is the group of $n \times n$ upper unitriangular matrices whose entries are dyadic rationals. e.g.

$$N_3 = \left\{ \begin{pmatrix} 1 & f & h \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix} : f, g, h \in \mathbb{Z}[1/2] \right\}$$

  - G_n is just the set $H_n \times N_n$ with multiplication
    $(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 h_1 n_2 h_1^{-1})$ for $h_1, h_2 \in H_n$ and $n_1, n_2 \in N_n$.

- If $((y^{(1)}, \varphi^{(1)}), (y^{(2)}, \varphi^{(2)}), \cdots)$ is a path in G_n then $(y^{(1)}, y^{(2)}, \cdots)$ is a path in $\mathbb{Z}_n$. So a walk in G_n contains n random walks on $(\mathbb{Z}, +)$ with measures μ_{x_p} given by the push-forward of μ under the maps

$$(x, f) \mapsto \log_2[x]_{pp}, \ p \in \{1, \cdots, n\}$$

for each $p \in \{1, \cdots, n\}$.

- For each p, let $\bar{\mu}_{x_p} = \sum_{z \in \mathbb{Z}} z \mu_{x_p}(z)$ be the mean of each measure μ_{x_p}.

The drift matrix

- Let D_{ij} be the *drift matrix* given by

$$D_{ij} = \begin{cases} \bar{\mu}_{x_i} - \bar{\mu}_{x_j} & : i \leq j \\ 0 & : i \geq j. \end{cases}$$

  I will refer to the entries of D_{ij} as *drifts*.

- The boundary behaviour depends on the entries of the drift matrix.

- We restrict ourselves to considering measures on G_n which have *finite first moment*. That is,

$$\int_G |x|_K d\mu(x) < +\infty$$

  for any generating set K. This forces the drift matrix to have finite, real-valued entries.

- G_1 is isomorphic to the abelian group $\mathbb{Z}$ - the boundary is trivial.
- For G_2, it is possible to show that

$$[\varphi^{(m)}]_{1,2} = \sum_{l=0}^{m-1} \left([f^{(l+1)}]_{1,2} 2^{y_1^{(l)} - y_2^{(l)}} \right)$$

  so, if $D_{12} < 0$ ($D_{12} > 0$), then $[\varphi^{(m)}]_{1,2}$ is a.s. convergent $\mathbb{R}$ ($\mathbb{Q}_2$). Furthermore, the boundary is non-trivial, and isomorphic to $\mathbb{R}$ ($\mathbb{Q}_2$) with the hitting measure. The boundary is trivial if $D_{12} = 0$.
- In general, the entries of $\varphi^{(m)}$ are given by a recurrence relation.

Searching for a candidate boundary space

- Computer simulations are helpful in searching for a candidate boundary space.
- The process is simple: perform random walks and look for interesting asymptotic behaviour, e.g. convergence in some space.
- Assuming that the drifts are all non-zero, there are at least as many drift cases to consider as there are orderings of the set $\{\bar{\mu}_{x_p} : p \in \{1, \cdots, n\}\}$.
- I have been able to spot entry-wise convergence in $\mathbb{R}$ or $\mathbb{Q}_2$ in many drift cases.

The Poisson boundary of G_n

- Currently, I can show the following for any n:
    - If the drifts are all negative (positive), then $\varphi^{(m)}$ will $\mathbb{P}^{\mu}$-a.s. converge to a unitriangular real (2-adic) matrix, and the boundary is isomorphic to this space with the action of left multiplication and the hitting measure.
    - If the drifts are all zero then the boundary is trivial.
    - If the drifts on the first super diagonal are non-zero, then the first super diagonal entries of $\varphi^{(m)}$ will $\mathbb{P}^{\mu}$-a.s. converge in either $\mathbb{R}$ or $\mathbb{Q}_2$.
    - There are conditions on the drifts which can $\mathbb{P}^{\mu}$-a.s. make any particular entry $\varphi^{(m)}{}_{ij}$ converge in $\mathbb{Q}_2$ or $\mathbb{R}$.
- I suspect that there mixed drift cases which have more complicated behaviour, but I'm not sure yet.

Is the candidate space possibly the Poisson boundary?

- After a candidate boundary is found, perhaps with the aid of computation, the next step is proving that it is indeed the Poisson boundary. The following result given by Kaimanovich is very helpful:

- If G is a finitely generated group, μ is a probability measure on G with finite first moment and there is a sequence of measurable maps π_n from a candidate boundary B to the group so that

$$\frac{1}{n} d(\pi_n(y_\infty), y_n) \to 0$$

where d is the word length metric then (B, λ) is the Poisson boundary of the pair (G, μ) where λ is the hitting measure.

Is the candidate space possibly the Poisson boundary?

- The word length in G_n is not easy to calculate efficiently but it has the following bound

$$|(x, f)|_K \leq C \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \|f_{ij}\|$$

for some real constant C where $|z|$ is the ordinary absolute value, K is a particular generating set and

$$\|f\| = \begin{cases} 1 + \max\{|d_-(f)|, |d_+(f)|\} & : f \neq 0 \\ 0 & : f = 0 \end{cases}$$

where $d_-(f)$ $(d_+(f))$ is the place value of the smallest (largest) non-zero digit in the unique binary decimal expansion of $f \neq 0$.

- A candidate boundary B and a sequence of approximations π_n for G_n can be easily tested on a computer using this bound.
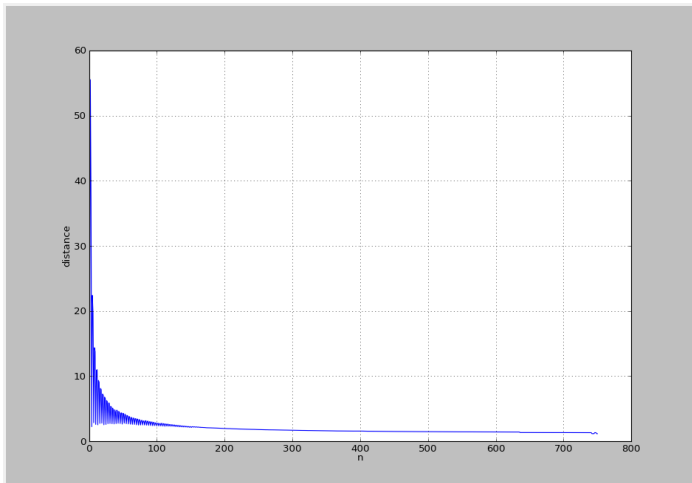
Figure: A 'bad' guess for π_n and all negative drifts in G_3.
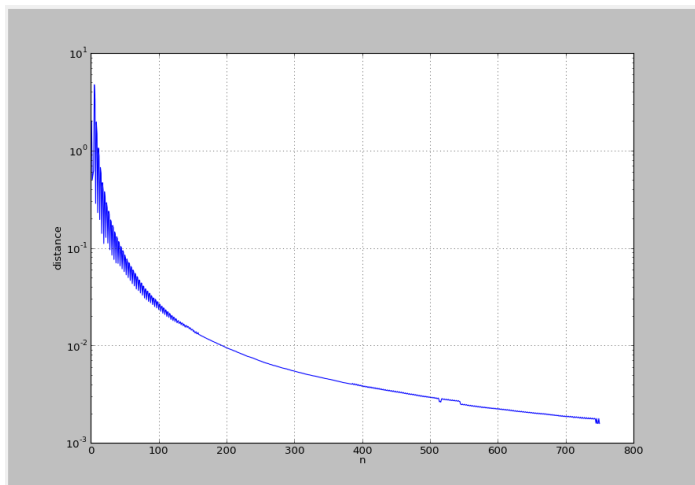
Figure: A 'good' guess for π_n and all negative drifts in G_3 (note that the y axis is now logarithmic).

Hitting distributions

- It is also interesting to describe the hitting measure ν.
- At least for negative drifts on G_n, the hitting measure on the boundary can be visualised by performing many walks and plotting a histogram of the final values $[\varphi_\infty]_{ij}$.
- Unsurprisingly, the resulting distribution depends on the measure.
- The next slide has an example for G_2.

Example hitting distribution for G_2
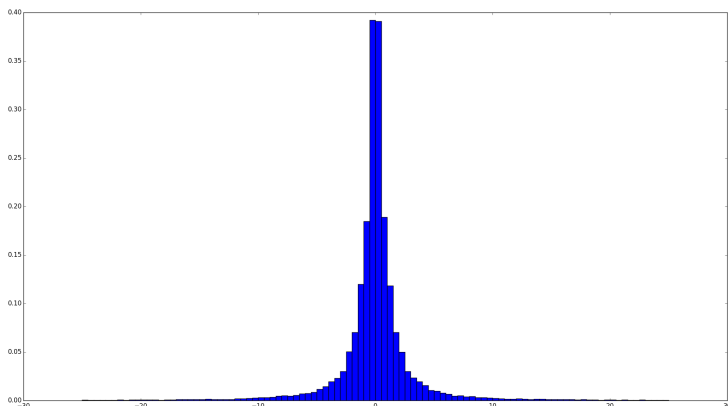
Figure: Histogram of real hitting points for a random walk on G_2. Negative drift. Equal weight on all off-diagonal generators and their inverses.

Boundary triviality from entropy

- Let G be a countable discrete group. The (Shannon) *entropy* of a probability μ on G will be denoted by $H(\mu)$

$$H(\mu) = - \sum_{g \in \text{supp}\,\mu} \mu(g) \log \mu(g)$$
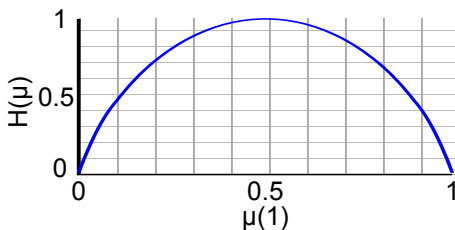
Figure: Entropy vs p for the measure $\mu(x) = p\delta_1(x) + (1-p)\delta_{-1}(x)$ on $\mathbb{Z}$.

Boundary triviality from entropy

- Suppose that $H(\mu)$ is finite. Then the limit

$$h(G, \mu) = \lim_n \frac{H(\mu^n)}{n},$$

where μ^n is the nth convolution power of μ, is called the entropy of the pair (G, μ).

- This may be thought of as the mean specific entropy contained in one factor of the product $y_n = x_1 \cdot \ldots \cdot x_n$ in the random walk.

- Suppose that $H(\mu)$ is finite. Then the equality

$$\lim_n (1/n) \log \mu^n(y_n) = -h(G, \mu)$$

holds for almost all paths $y \in G^\infty$ (Kaimanovich and Vershik 1983).

Boundary triviality from entropy

- Let G be a discrete countable group, μ be a probability measure on G with finite entropy. Then the Poisson boundary of the pair (G, μ) is trivial if and only if $h(G, \mu) = 0$. (Kaimanovich and Vershik 1983)
- If $\mu^n(y_n)$ can be computed efficiently then we can test for boundary triviality by performing long walks calculating $\mu_n(y_n)$
- Rounding error can be a problem if $\mu_n(y_n)$ gets too small.
- Depending on the group and the measure, computing $\mu_n(y_n)$ can be difficult. The support of μ_n can grow exponentially, and the word problem can be an issue.

Figure: Output of a sample program for the simple random walk on $\mathbb{F}_2$. $\mu_n(y_n)$ can be computed quickly as it can be given in terms of word length. As expected, the output suggests the boundary is non-trivial.

Summary

- Experimental computations can aid in describing the Poisson boundary of a group with measure in the following ways:
    1. Finding candidate boundary spaces.
    2. Testing if the candidate space is likely to be the boundary (using e.g. Kaimanovich's ray criterion).
    3. Visualising the hitting measure.
    4. Testing for boundary triviality.
- Rounding error can be an issue for walks in matrix groups.
- Can work in multiple ways. E.g. matrices or words in a generating set.

Thank you.