On a family of polynomials related to $\zeta(2,1) = \zeta(3)$

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Riemann's (single) zeta values

In the region $\operatorname{Re} s > 1$, the *Riemann zeta function* may be defined by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,.$$

One of interesting (still unsolved!) problems is determining all polynomial relations over $\mathbb{Q}$ for the numbers $\zeta(s)$, $s = 2, 3, 4, \dots$. They are known as *zeta values*.
The first breakthrough in this direction is due to Euler, who showed that $\zeta(2k)$ is always a rational multiple of π^{2k}.
Much less is known on the arithmetic nature of the values of the zeta function at odd integers $s = 3, 5, 7, \dots$: in 1978, Apéry has proved the irrationality of the number $\zeta(3)$ and there are more recent but partial linear independence results about the other odd zeta values.

Multiple zeta values (MZVs)

The series for $\zeta(s)$ enables the following multidimensional generalization. For positive integers $s_1, s_2, \ldots, s_l$ with $s_1 > 1$, consider the values of the l-tuple zeta function

$$\zeta(\boldsymbol{s}) = \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}};$$

the corresponding multi-index $\boldsymbol{s} = (s_1, s_2, \ldots, s_l)$ is regarded as *admissible*. The quantities so defined are called the *multiple zeta values* (and abbreviated MZVs), or the *multiple harmonic series*, or the *Euler sums*.

The double sums (corresponding to $l = 2$) trace back to Euler, who derived a family of identities connecting them and single zeta values. In particular, Euler proved the identity $\zeta(2,1) = \zeta(3)$, which was several times rediscovered after.

Thirty-two plus one variations

In a nice expository paper

> *"Thirty-two Goldbach variations," Intern. J. Number Theory* **2** *(2006), no. 1, 65–103,*

J. Borwein and D. Bradley list (approximately) 32 proofs of the Euler(–Goldbach) identity $\zeta(2,1) = \zeta(3)$.
One of the goals of this talk is to produce variation no. 33.
This immediately brings us to Mozart's Symphony no. 33 in B flat major (which is approximately 21 minutes duration):

`http://www.youtube.com/watch?v=RCwB3LPeNIY`

Multiple polylogarithms

The MZVs can be thought of as the special $z = 1$ values of the multiple polylogarithms (where $s_1 = 1$ is now allowed!)

$$\mathrm{Li}_{\boldsymbol{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}}.$$

It is elementary to see that

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Li}_{s_1, s_2, \ldots, s_l}(z) = \begin{cases} \dfrac{1}{z} \mathrm{Li}_{s_1 - 1, s_2, \ldots, s_l}(z) & \text{if } s_1 > 1, \\ \dfrac{1}{1-z} \mathrm{Li}_{s_2, \ldots, s_l}(z) & \text{if } s_1 = 1. \end{cases}$$

This allows us to write

$$\mathrm{Li}_{\boldsymbol{s}}(z) = \int \cdots \int_{z > z_1 > \cdots > z_k > 0} \omega_{\varepsilon_1}(z_1) \cdots \omega_{\varepsilon_k}(z)$$

while decoding $\boldsymbol{s} \mapsto \omega_0^{s_0 - 1} \omega_1 \cdots \omega_0^{s_l - 1} \omega_1 = \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_k}$ over the alphabet in just two letters $\omega_0(z) = \mathrm{d}z/z$ and $\omega_1(z) = \mathrm{d}z/(1-z)$.

Euler's identity from the title

For example,

$$\mathrm{Li}_{2,1}(z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1^2 n_2} = \iiint_{z > z_1 > z_2 > z_3 > 0} \frac{\mathrm{d}z_1}{z_1} \frac{\mathrm{d}z_2}{1 - z_2} \frac{\mathrm{d}z_3}{1 - z_3}$$

and

$$\mathrm{Li}_3(z) = \sum_{n_1 \geq 1} \frac{z^{n_1}}{n_1^3} = \iiint_{z > z_1 > z_2 > z_3 > 0} \frac{\mathrm{d}z_1}{z_1} \frac{\mathrm{d}z_2}{z_2} \frac{\mathrm{d}z_3}{1 - z_3},$$

so that

$$\zeta(2,1) = \iiint_{1 > z_1 > z_2 > z_3 > 0} \frac{\mathrm{d}z_1}{z_1} \frac{\mathrm{d}z_2}{1 - z_2} \frac{\mathrm{d}z_3}{1 - z_3}, \quad \zeta(3) = \iiint_{1 > z_1 > z_2 > z_3 > 0} \frac{\mathrm{d}z_1}{z_1} \frac{\mathrm{d}z_2}{z_2} \frac{\mathrm{d}z_3}{1 - z_3}.$$

The two quantities are equal as can be read off from the change of variable $(z_1, z_2, z_3) \mapsto (1 - z_3, 1 - z_2, 1 - z_1)$.

Generalized Euler's identities

More generally, the argument implies

$$\zeta(\underbrace{2,1,2,1,\dots,2,1}_{2l \text{ entries}}) = \zeta(\underbrace{3,3,\dots,3}_{l \text{ entries}}) \qquad \text{for} \quad l = 1,2,\dots,$$

and even

$$\zeta\big(\{m+1,\underbrace{1,1,\dots,1}_{k-1 \text{ entries}}\}^l\big) = \zeta\big(\{k+1,\underbrace{1,1,\dots,1}_{m-1 \text{ entries}}\}^l\big) \qquad \text{for} \quad l = 1,2,\dots,$$

where $\{\,\cdot\,\}^l$ denotes the l-repetition of the corresponding multi-index.

However it does not imply the (similar-looking) identity

$$\zeta(\{3,1\}^l) = 4^l \zeta(\{4\}^l) = \frac{2\pi^{4l}}{(4l+2)!} \qquad \text{for} \quad l = 1,2,\dots .$$

Borwein–Bradley–Broadhurst identity

To prove

$$\zeta(\{3,1\}^l) = \frac{2\pi^{4l}}{(4l+2)!} \qquad \text{for} \quad l = 1, 2, \dots,$$

one uses the fact that the generating function

$$L(z;t) = 1 + \sum_{l=1}^{\infty} \mathrm{Li}_{\{3,1\}^l}(z) t^{4l}$$

is annihilated by the differential operator

$$\left((1-z)\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 - t^4.$$

The product $\Pi(z;t)$ of two Gauss hypergeometric functions

$${}_2F_1\left(\left.\begin{matrix} \frac{1}{2}(1+i)t, -\frac{1}{2}(1+i)t \\ 1 \end{matrix}\,\right|\, z\right) \quad \text{and} \quad {}_2F_1\left(\left.\begin{matrix} \frac{1}{2}(1-i)t, -\frac{1}{2}(1-i)t \\ 1 \end{matrix}\,\right|\, z\right)$$

satisfies exactly the same linear differential equation. By inspecting the initial part of the z-expansions we see that $L(z;t) = \Pi(z;t)$. It remains to substitute $z = 1$ and apply the Gauss summation theorem to the two ${}_2F_1$-series.

Generalized Euler's identity revisited

Our principal result is the existence of a similar proof of the identity

$$\zeta(\{2,1\}^l) = \zeta(\{3\}^l) \qquad \text{for} \quad l = 1, 2, \ldots,$$

and slightly more (though partly experimental).

Theorem

In notation

$$B(z;t) = \sum_{l=0}^{\infty} \mathrm{Li}_{\{2,1\}^l}(z) t^{3l}, \quad \textit{and} \quad C(z;t) = \sum_{l=0}^{\infty} \mathrm{Li}_{\{3\}^l}(z) t^{3l},$$

we have

$$B(1;t) = C(1;t) = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{j^3}\right).$$

Note that the evaluation of $C(1;t)$ is straightforward.

Polynomial coefficients

In order to prove the theorem we re-expand $B(z;t)$ as the Taylor series in z,

$$\begin{aligned} B(z;t) &= \sum_{l=0}^{\infty} \mathrm{Li}_{\{2,1\}^l}(z)t^{3l} = \sum_{n=0}^{\infty} B_n(t)z^n \\ &= 1 + \tfrac{1}{4}t^3 z^2 + \tfrac{1}{6}t^3 z^3 + \left(\tfrac{1}{192}t^6 + \tfrac{11}{96}t^3\right)z^4 + \left(\tfrac{3}{400}t^6 + \tfrac{1}{12}t^3\right)z^5 \\ &\quad + \left(\tfrac{1}{34560}t^9 + \tfrac{47}{5760}t^6 + \tfrac{137}{2160}t^3\right)z^6 + \cdots . \end{aligned}$$

The linear z-differential equation for $B(z;t)$ can be translated into the 3-term recursion

$$n^3 B_n - (n+1)^2(2n+1)B_{n+1} + (n+2)^2(n+1)B_{n+2} = t^3 B_n$$

for the coefficients $B_n = B_n(t) \in \mathbb{Q}[t^3]$; the initial values are $B_0 = 1$ and $B_1 = 0$.
After some experimentation we arrive at

An explicit hypergeometric formula

Theorem

If

$$B(z;t)=\sum_{l=0}^{\infty}\mathrm{Li}_{\{2,1\}^l}(z)t^{3l}=\sum_{n=0}^{\infty}B_n(t)z^n$$

then the following explicit formula is valid:

$$\begin{aligned} B_n(t) &= \frac{1}{n!}\sum_{k=0}^{n}\frac{(\omega t)_k(\omega^2 t)_k(t)_{n-k}(-t+k)_{n-k}}{k!\,(n-k)!} \\ &= \frac{(t)_n(-t)_n}{n!^2}\,{}_3F_2\left(\begin{matrix}-n,\,\omega t,\,\omega^2 t\\ -t,\,1-n-t\end{matrix}\,\middle|\,1\right), \end{aligned}$$

where $\omega=e^{2\pi i/3}$ *and* ${}_3F_2$ *denotes the generalized hypergeometric function.*

Note that the fact $B_n(t)\in\mathbb{Q}[t^3]$ (that is, the invariance of $B_n(t)$ under the change $t\mapsto\omega t$) is hardly seen from the formula.
The explicit formula and some hypergeometric identities demonstrate the required equality $B(1;t)=A(1;t)$.

Special polynomials

Curiously enough, the above polynomials generalise to the one-parameter family of polynomials

$$B_n^\alpha(t) = \frac{1}{n!}\sum_{k=0}^{n} \frac{(\omega t)_k(\omega^2 t)_k(\alpha+t)_{n-k}(\alpha-t+k)_{n-k}}{k!\,(n-k)!}$$
$$= \frac{1}{n!}\sum_{k=0}^{n} \frac{(\alpha+\omega t)_k(\alpha+\omega^2 t)_k(t)_{n-k}(\alpha-t+k)_{n-k}}{k!\,(n-k)!}.$$

Lemma

For each $\alpha \in \mathbb{C}$, the polynomials $B_n^\alpha(t)$ satisfy the 3-term recurrence relation

$$((n+\alpha)^3 - t^3)B_n^\alpha - (n+1)(2n^2+3n(\alpha+1)+\alpha^2+3\alpha+1)B_{n+1}^\alpha + (n+2)^2(n+1)B_{n+2}^\alpha = 0$$

and the initial conditions $B_0^\alpha = 1$, $B_1^\alpha = \alpha^2$.
In particular, $B_n^\alpha(t) \in \mathbb{C}[t^3]$ for $n = 0, 1, 2, \dots$.

Furthermore, we have other interesting properties of the polynomials.

Some properties of the polynomials

Lemma

We have

$$B_n^{1-n-\alpha}(t) = B_n^{\alpha}(t).$$

Lemma

We have

$$\sum_{n=0}^{\infty} B_n^{\alpha}(t) z^n = (1-z)^{1-2\alpha} \sum_{n=0}^{\infty} B_n^{1-\alpha}(t) z^n.$$

Numerical verification suggests that for any real α the zeroes of B_n^{α} viewed as polynomials in $x = t^3$ lie on the real half-line $(-\infty, 0]$.

Borwein–Bradley–Broadhurst (ex-)conjecture

A modified version of the MZV identity,

$$\sum_{n_1>m_1>n_2>m_2>\cdots>n_l>m_l\ge 1} \frac{(-1)^{n_1+n_2+\cdots+n_l}}{n_1^2 m_1 n_2^2 m_2 \cdots n_l^2 m_l} = 8^l \sum_{n_1>m_1>n_2>m_2>\cdots>n_l>m_l\ge 1} \frac{1}{n_1^2 m_1 n_2^2 m_2 \cdots n_l^2 m_l}$$

for $l = 1, 2, \dots,$ was a conjecture for twelve years.
(Note that the right-hand side is the familiar $\zeta(\{2,1\}^l)$. The left-hand side is its "alternating" version. Recall a similar looking identity $\zeta(\{3,1\}^l) = 4^l \zeta(\{4\}^l)$ for $l = 1, 2, \dots$.)
The general identity was demonstrated by J. Zhao in a tour-de-force of manipulations with various algebraic (shuffle) structures of (alternating) MZVs.
Is there a civilized proof of the elegant identity?

An equivalent form

The identity is equivalent to proving that the polynomials $A_n(t) \in \mathbb{Q}[t^3]$ (of degree $[n/2]$ in $x = t^3$) produced by the recursion

$$(n^3 - (-1)^n t^3) A_n(t) + (n+1)^2 (2n+1) A_{n+1}(t) + (n+2)^2 (n+1) A_{n+2}(t) = 0$$

and the initial conditions $A_0 = 1$, $A_1 = 0$ satisfy

$$\sum_{k=0}^{\infty} A_k(t) = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{8j^3}\right).$$

Another equivalent form

Equivalently, the polynomials $\widetilde{A}_n(t) = \sum_{k=0}^{n} A_k(t)$ that come from the recursion

$$(n^3 - (-1)^n t^3)\widetilde{A}_{n-1} + (2n+1)n\widetilde{A}_n - (n+1)^2 n\widetilde{A}_{n+1} = 0$$

satisfy

$$\lim_{n\to\infty} \widetilde{A}_n(t) = \prod_{j=1}^{\infty}\left(1 + \frac{t^3}{8j^3}\right).$$

Experimentally, the zeroes of the polynomials A_n and $\widetilde{A}_n$ as polynomials in $x = t^3$ lie on the negative half-line $(-\infty, 0)$. Unfortunately, no explicit formulas for A_n and $\widetilde{A}_n$ are known making the proof of any equivalent form of the Borwein–Bradley–Broadhurst (ex-)conjecture really hard.

Asymptotics of solutions of recursions

There is a general, and quite powerfull, technique known as the Birkhoff–Trjitzinsky method to determine asymptotic behaviour of any solution of a difference equation with polynomial in n coefficients:

$$\widetilde{A}_n \sim \frac{C \cdot A^n}{n^\alpha} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots \right) \quad \text{as } n \to \infty.$$

Let me cite D. Zeilberger here:

> *"The Birkhoff–Trjitzinsky method suffers from one drawback. It only does the asymptotics up to a multiplicative constant C. But nowadays this is hardly a problem. Just crank-out the first ten thousand terms of the sequence using the very recurrence whose asymptotics you are trying to find, not forgetting to furnish the few necessary initial conditions, and then estimate the constant empirically. If you are lucky, then Maple can recognize it in terms of 'famous' constants like e and π, by typing `identify(C);`."*

Asymptotics of the particular polynomials

Well, finding the constant C is not a problem from the computational point of view: in our case we easily get $A = 1$, $\alpha = 0$ and even have the prediction

$$C = C(t) = \prod_{j=1}^{\infty} \left(1 + \frac{t^3}{8j^3}\right).$$

Proving the value of C rigorously is hard.

Any simple idea?

Be multiple.

Mozart's Symphony no. 33