Concentration near a hyperplane in quasi-normed spaces

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Outline

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- Quasi-norms
- Small-ball estimates and structure of vectors
- Esseen inequality
- Euclidean vs non-euclidean result
- The real problem

Quasi-norms

Definition (Star-shaped domain)

A body $K \subseteq \mathbb{R}^d$ is star-shaped if $\mathrm{conv}(\{x, 0\}) \subseteq K$ $\forall x \in K$.

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Definition (Quasi-norm in $\mathbb{R}^d$)

$\|\cdot\|_K$ as defined above is a quasi-norm: same as a norm but instead of the triangle inequality,

$$\|x + y\|_K \leq C_K \left(\|x\|_K + \|y\|_K \right), \quad C_K \geq 1.$$

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Example: ℓ_p^d

- Take $\mathbb{R}^d$ with $\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$, $p > 0$.
- This is a quasi-norm with $C_p = \max\{2^{1/p-1}, 1\}$ ($\Longrightarrow$ if $p \geq 1$ this is a norm).
- Let B_p^d be the unit ball of this (quasi-)norm.

Small-Ball Probability

- $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ a family of n fixed vectors.
- $\varepsilon_1, \ldots, \varepsilon_n$ independent symmetric Bernoulli random variables.

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Definition (Small-Ball Probability)

Let $r > 0$, $K \subseteq \mathbb{R}^d$ symmetric star-shaped, V as above. Define

$$\rho_r^K(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\Big(\sum_{j=1}^n \varepsilon_j v_j \in x + rK \Big).$$

Small-Ball and structure of V

Small-Ball Probability

$$\rho_r^K(V) = \sup_{x\in\mathbb{R}^d} \mathbb{P}\Big(\sum_{j=1}^{n} \varepsilon_j v_j \in x + rK\Big).$$

$\rho_r^K(V)$ is large

$\Updownarrow$

$\sum \varepsilon_j v_j$ is highly concentrated

$\Updownarrow$

Much cancellation between members of V

$\Updownarrow$

V is 'well-structured'

Examples in $\mathbb{R}^1$

Theorem (Erdős '45)

$v_1, \ldots, v_n$ integers, then

$$\rho_0^{B_2^1}(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\Big(\sum_{j=1}^{n} \varepsilon_j v_j = x \Big) = O(n^{-1/2}).$$

Theorem (Sárközy-Szemerédi '65)

$v_1, \ldots, v_n$ different integers, then

$$\rho_0^{B_2^1}(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\Big(\sum_{j=1}^{n} \varepsilon_j v_j = x \Big) = O(n^{-3/2}).$$

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In gereral: several ways of defining 'well-structured'.

Esseen Inequality

Let X_V be the random vector $\sum_{j=1}^{n} \varepsilon_j v_j$.

Theorem (Esseen inequality '66)

$$\rho_r^{B_2^d}(V) \leq \left(\frac{r}{\sqrt{d}} + \sqrt{d} \right)^d \int_{B_2^d} \left| \mathbb{E}(i\langle X_V, \xi \rangle) \right| d\xi.$$

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Theorem (Esseen inequality for quasi-norms, FGG '14)

$$\rho_r^K(V) \leq C_K^d r^d \int_{\mathbb{R}^d} \left|\mathbb{E}(i\langle X_V, \xi\rangle)\right| e^{-\frac{r^2\|\xi\|_2^2}{2}} d\xi.$$

Using the Esseen for quasi-norms, can obtain more general versions of euclidean results.

Concentration near a hyperplabe for quasi-norms

Definition

Let ω_K be the smallest number such that $B_2^d \subseteq \omega_K K$.
For example: $\omega_{B_2^d} = \omega_{B_\infty^d} = 1$, $\omega_{B_1^d} = \sqrt{d}$.

Theorem (Concentration near a hyperplane in quasi-normed space, FGG '15)

Let $\|\cdot\|_K$ be a quasi-norm on $\mathbb{R}^d$. Assume that $\ell \leq n$ is such that $\rho_r^K(V) \geq \left(\frac{C_K}{\sqrt{\ell}}\right)^d$. Then there exists a hyperplane H and at least $n-\ell$ vectors from V that satisfy

$$\mathrm{dist}_K(v_j, H) = \inf_{h\in H} \|v_j - h\|_K \leq \omega_K r.$$

This result was proved for the euclidean norm by Tao-Vu '12.

A question from combinatorics

$$\mathcal{P}_r^K(d,n) = \sup_V \rho_r^K(V).$$

Sup over all sets of size n of vectors of length ≥ 1.
Question: Estimate $\mathcal{P}_r^K(n,d)$.

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Theorem (Erdős '65)

$$\mathcal{P}_r^{B_2^1}(n,1) = 2^{-n} S(n, \lfloor r \rfloor + 1).$$

$S(n,m)$ is sum of largest m binomial coefficients.

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Estimating $\mathcal{P}_r^K(d,n)$, $K \neq B_2^d$ is still open...

The End