Sandwich semigroups in locally small categories

WESTERN SYDNEY UNIVERSITY

James East

ALLY

Semigroups and Operator Algebras
Univ Newcastle
James O'Clock, 24 July 2017

Joint work with:

Igor Dolinka ...

Joint work with:

Ivana Đurđev ...

Joint work with:

Kritsada Sangkhanan, Jintana Sanwong, Preeyanuch Honyam ...

Joint work with:

and Worachead Sommanee

Based on:

- Variants of finite full transformation semigroups
  - Dolinka, East
  - IJAC (2015)
- Semigroups of matrices under a sandwich operation
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- Four pages per minute...

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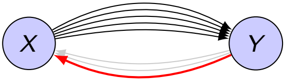

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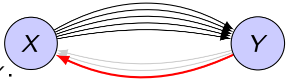

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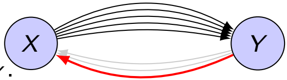

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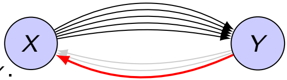

- Then $\mathcal{S}^a_{XY} = (S_{XY}, \star_a)$ is a sandwich semigroup.
- Special case: if $X = Y$ and $a = \mathrm{id}_X$, then $S^a_{ij} = \mathrm{End}(X)$.

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  - Hickey (1980s–2010s), Khan and Lawson (2001),
  - General theory for one-object (small) categories...

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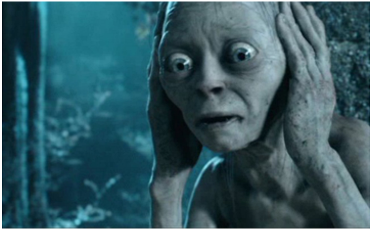

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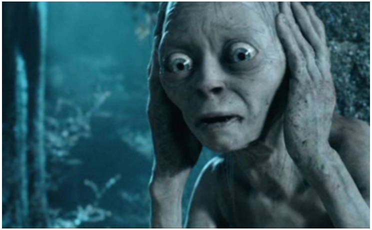

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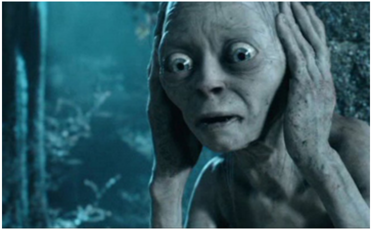

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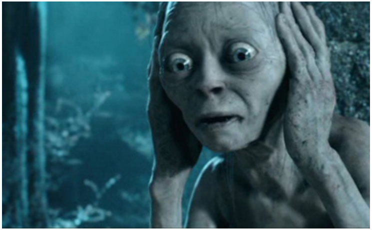

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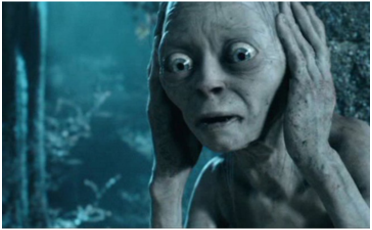

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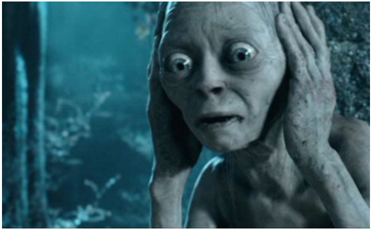

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- extended a lot of that to arbitrary categories,
- mostly concentrating on structural/combinatorial questions.

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- Exercise: Find a semigroup with pairwise-nonisomorphic variants.

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- How do [facts about S] relate to [facts about S^a]?
- If S belongs to an interesting family of semigroups, how does a variant S^a relate to other members of this family?

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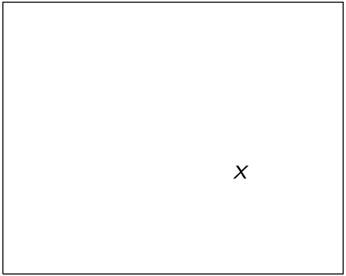

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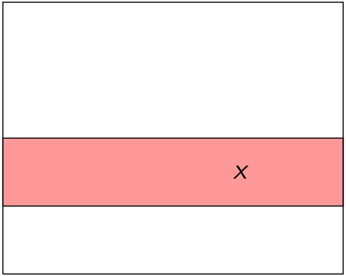

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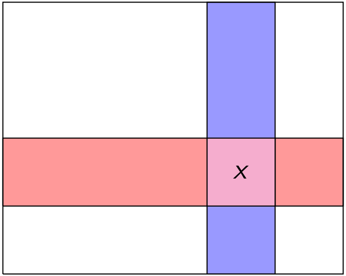

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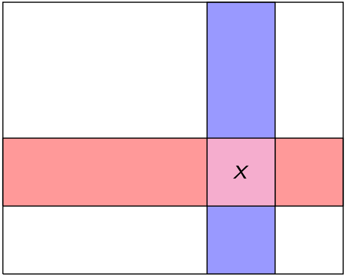

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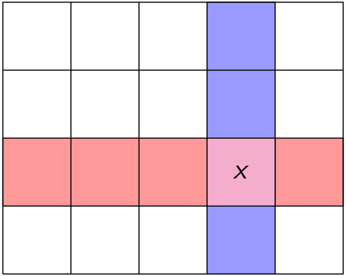

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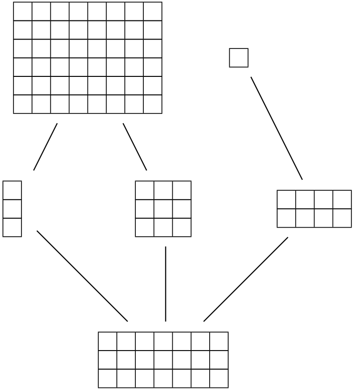

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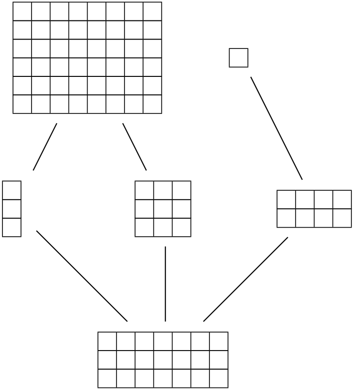

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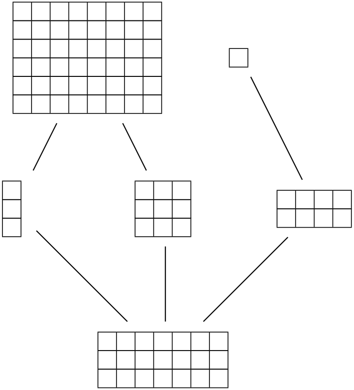
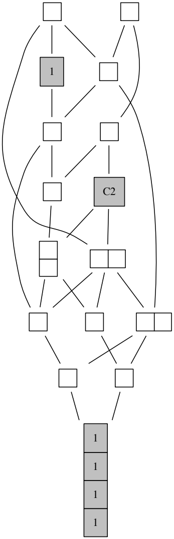


- Theorem: If S is finite, then $\mathscr{J} = \mathscr{D}$. So $\mathscr{J}$-classes are $\mathscr{D}$-classes.

Semigroups — egg-boxes (GAP) — thanks, JDM + AE-N



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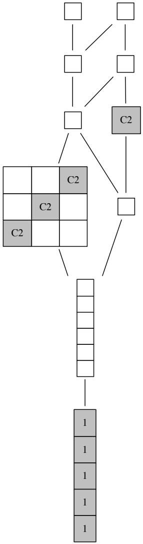

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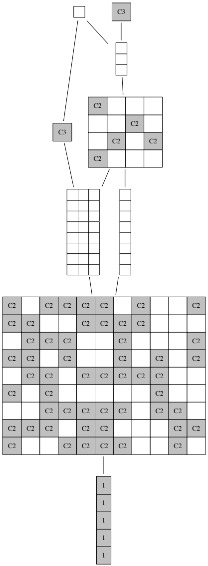

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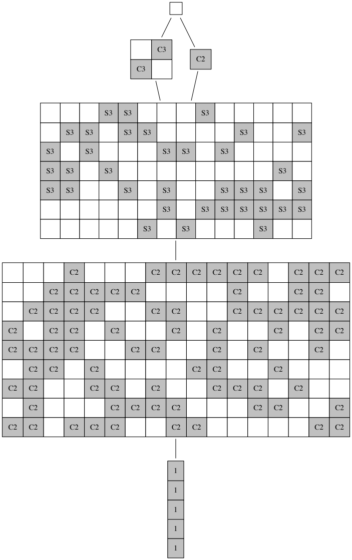

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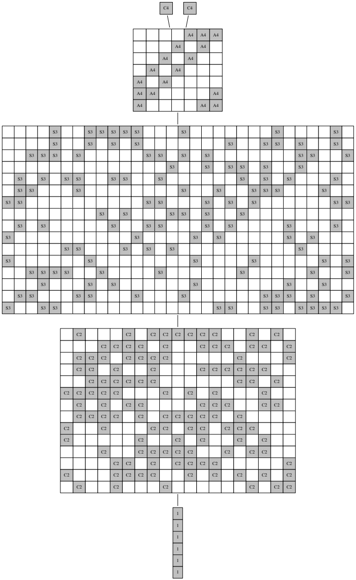

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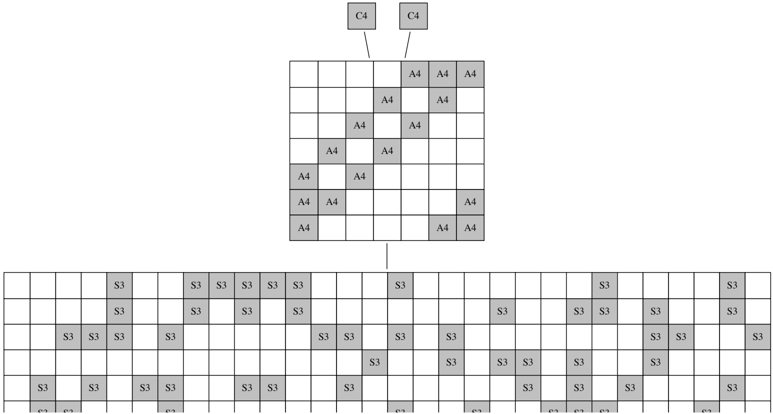

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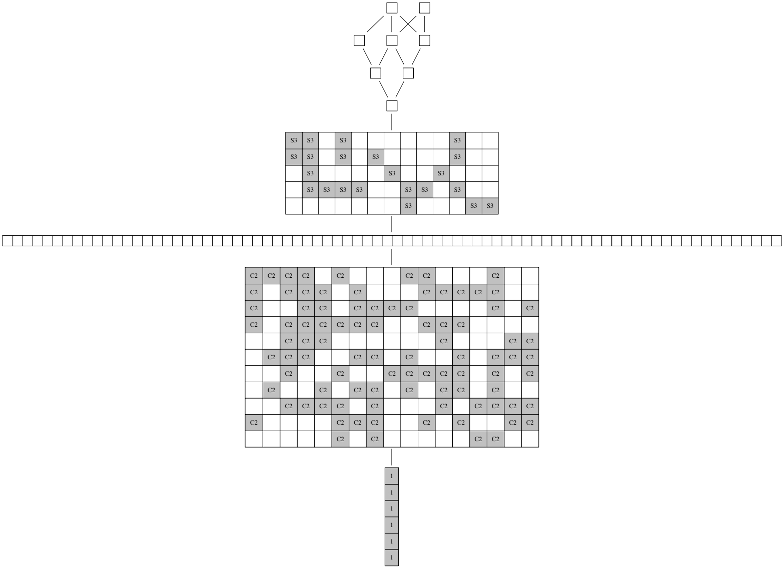

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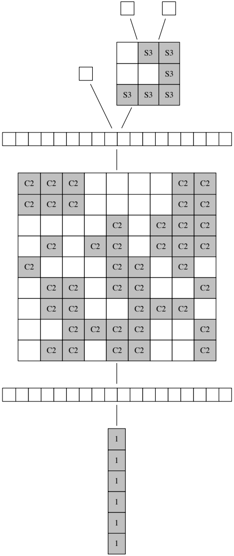

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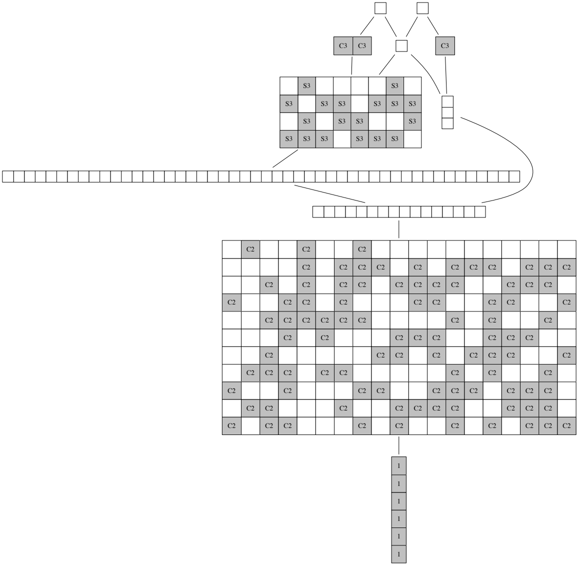

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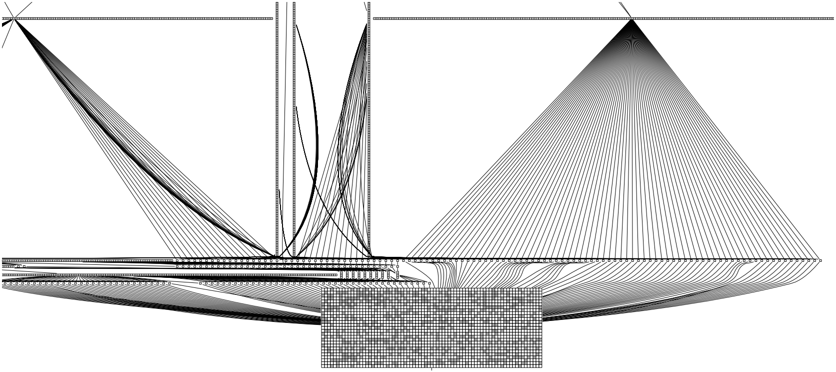

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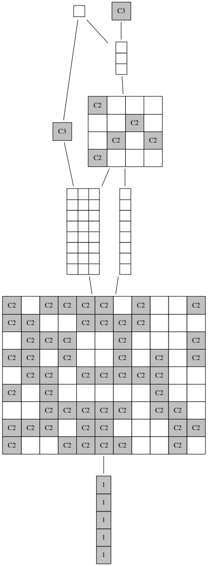

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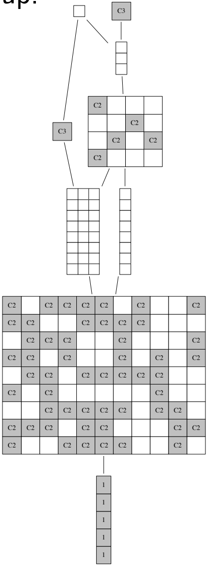

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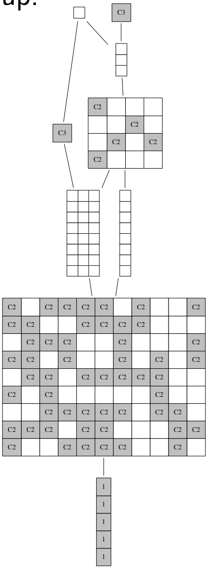

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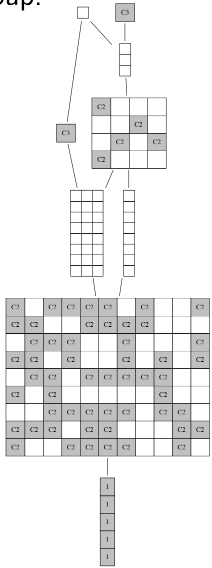

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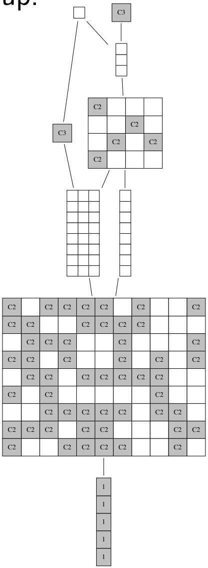

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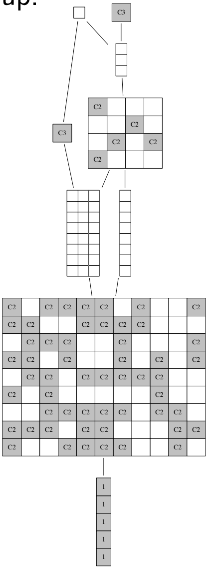

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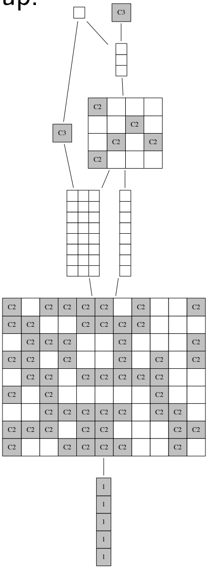

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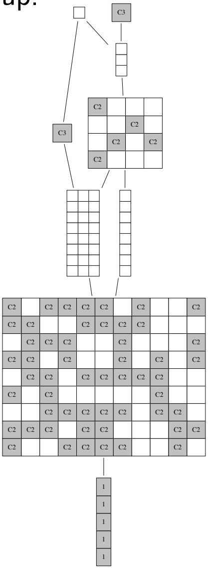

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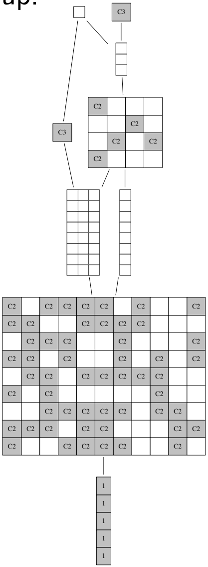

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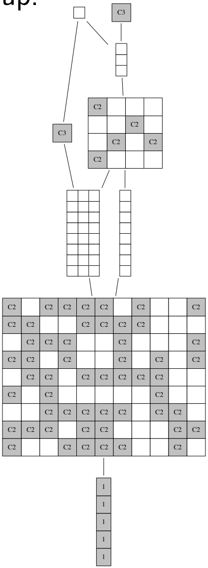
- Egg-box diagrams tell us a lot about the structure of a semigroup. But not everything.



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- Caution:
  - Green's relations on subsemigroups are not necessarily inherited.

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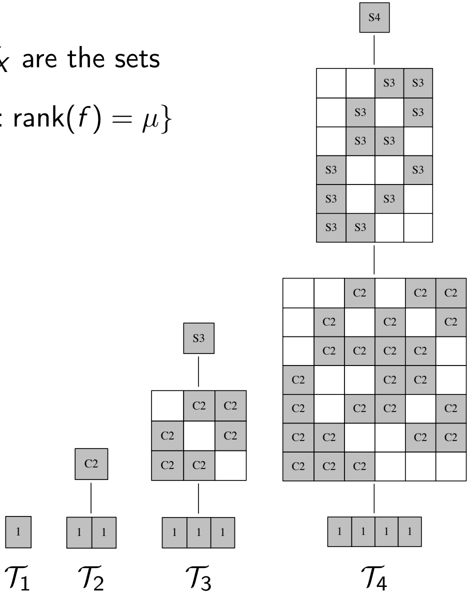

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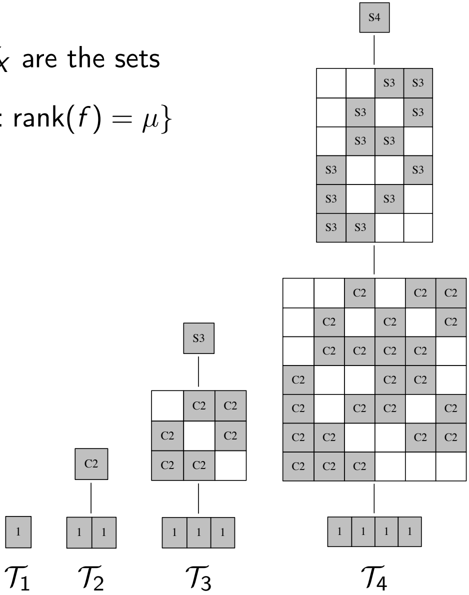

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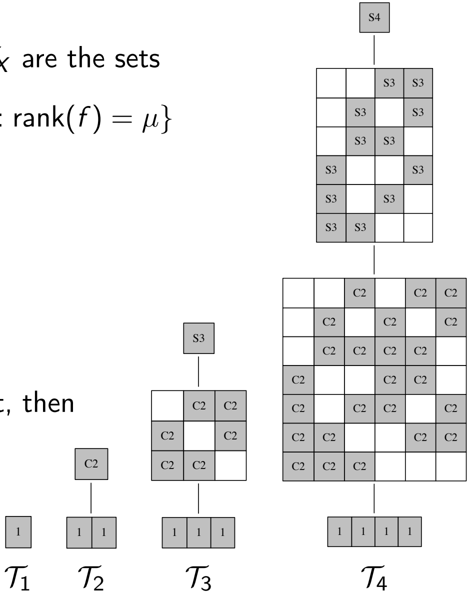

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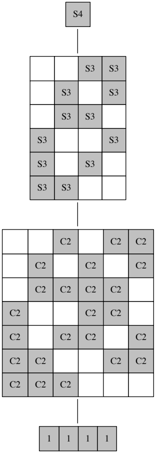

$\mathcal{T}_4$

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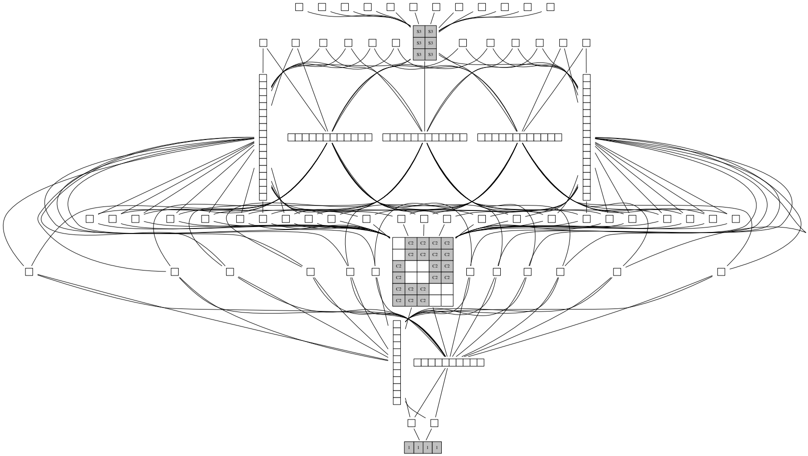

$\mathcal{T}_4^a,\ a = [1, 2, 3, 3]$

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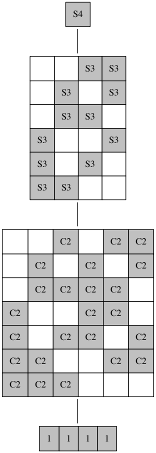

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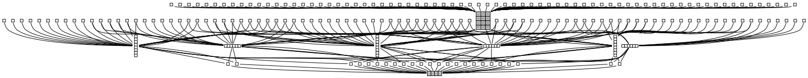
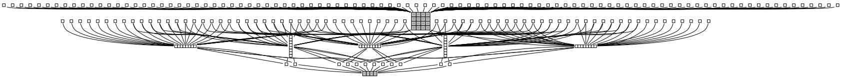


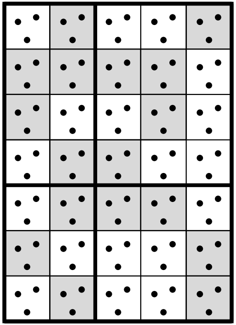
$\mathcal{T}_4^b,\ b = [1, 2, 2, 2]$

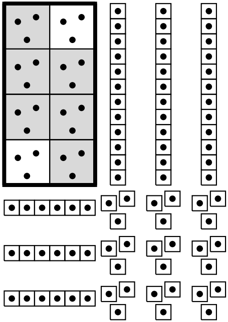


$\mathcal{T}_4^c,\ c = [1, 1, 2, 2]$

“High energy semigroup theory” — Attila Egri-Nagy

$\mathscr{D}$-class of $\mathcal{T}_X$ $\longrightarrow$ $\mathscr{D}$-classes of $\mathcal{T}_X^a$





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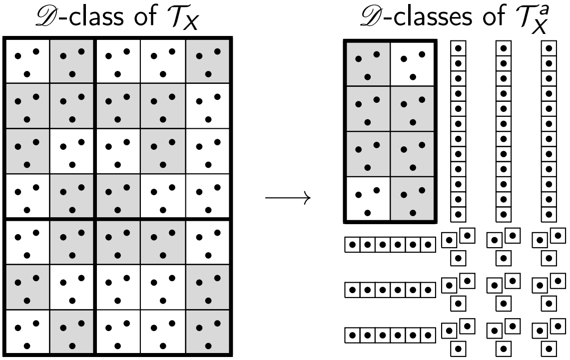

- A (regular) $\mathscr{D}$-class of $\mathcal{T}_X$ yields:
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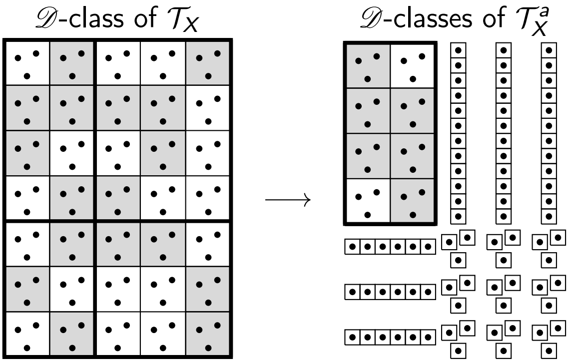
“High energy semigroup theory” — Attila Egri-Nagy



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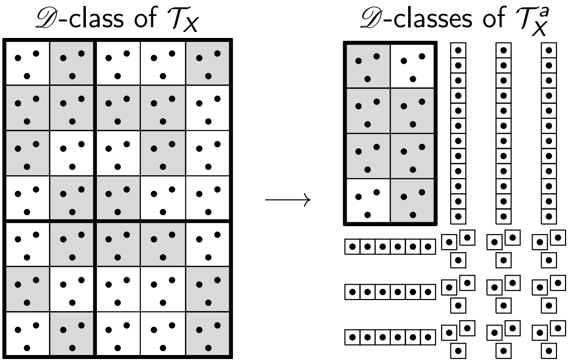

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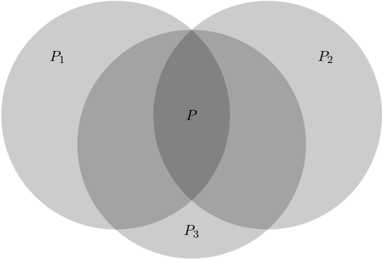

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Theorem: For any $x \in S$,

- $R_x^a = \begin{cases} R_x \cap P_1 & \text{if } x \in P_1 \\ \{x\} & \text{if } x \in S \setminus P_1, \end{cases}$
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- $H_x^a = \begin{cases} H_x & \text{if } x \in P \\ \{x\} & \text{if } x \in S \setminus P, \end{cases}$
- $D_x^a = \begin{cases} D_x \cap P & \text{if } x \in P \\ L_x^a & \text{if } x \in P_2 \setminus P_1 \\ R_x^a & \text{if } x \in P_1 \setminus P_2 \\ \{x\} & \text{if } x \in S \setminus (P_1 \cup P_2), \end{cases}$
- $J_x^a = \begin{cases} J_x \cap P_3 & \text{if } x \in P_3 \\ D_x^a & \text{if } x \in S \setminus P_3. \end{cases}$

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Theorem

- $\text{Reg}(S^a) = \text{Reg}(S) \cap P$.
- If S is regular, then $\text{Reg}(S^a) = P$ is a subsemigroup of S^a.

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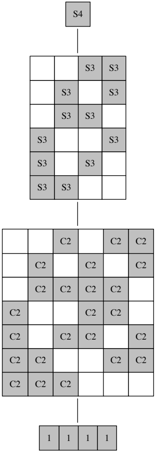

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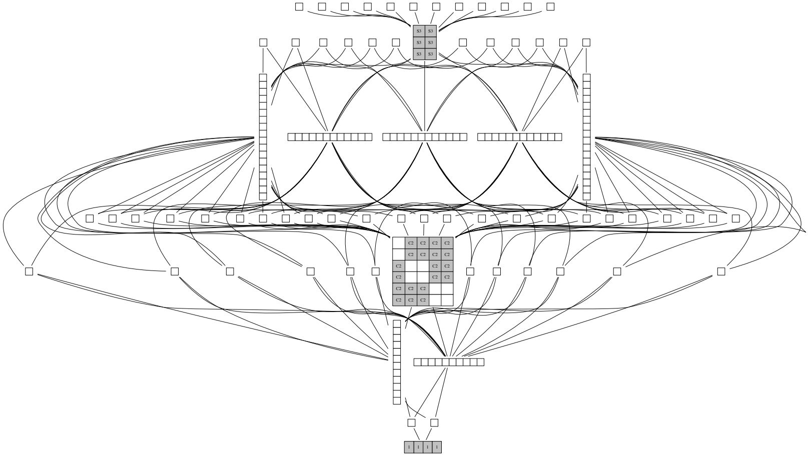

$\mathcal{T}_4^a$, $a = [1, 2, 3, 3]$

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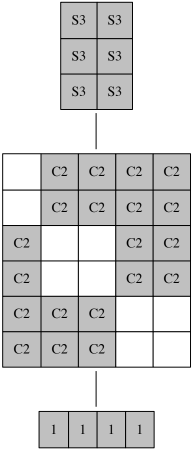

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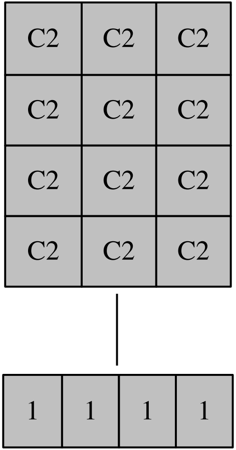

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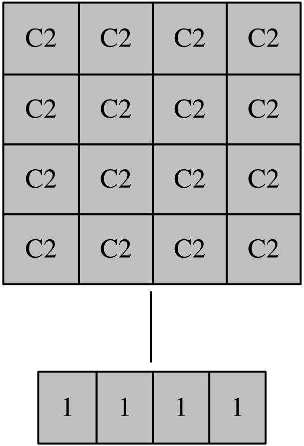

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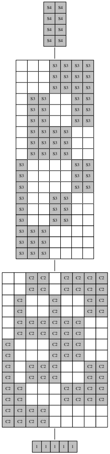

$\text{Reg}(\mathcal{T}_5^a)$, $a = [1, 2, 3, 4, 4]$

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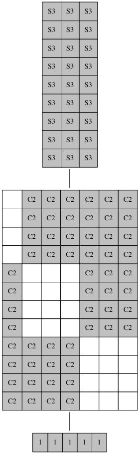

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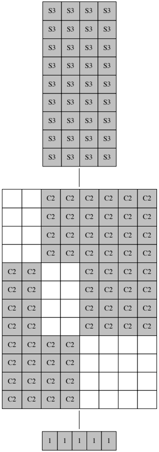

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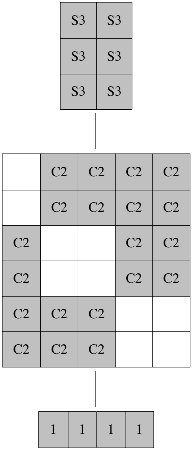
Look familiar?

$\mathrm{Reg}(\mathcal{T}_4^a)$, $\mathrm{rank}(a) = 3$



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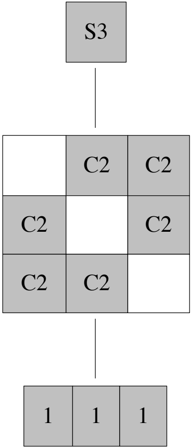

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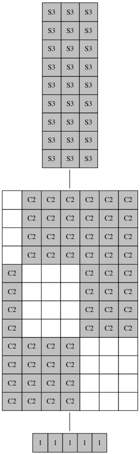

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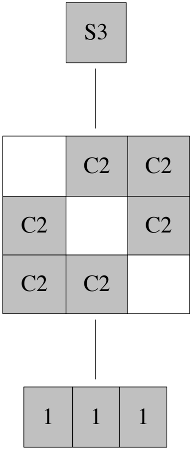

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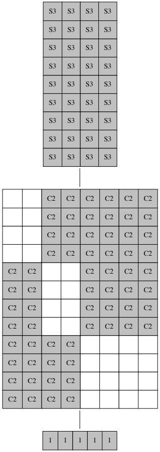

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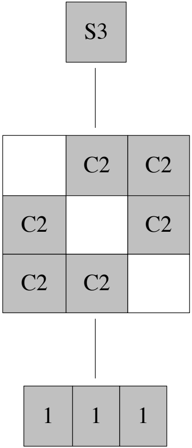
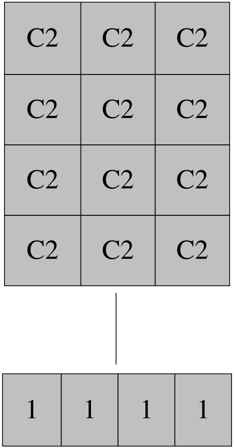
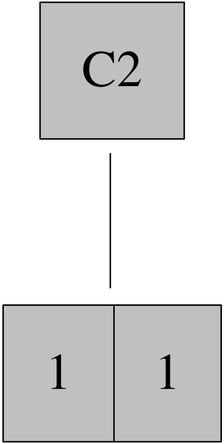


Look familiar?

Reg($\mathcal{T}_4^a$), rank(a) = 2



$\mathcal{T}_2$



Look familiar?

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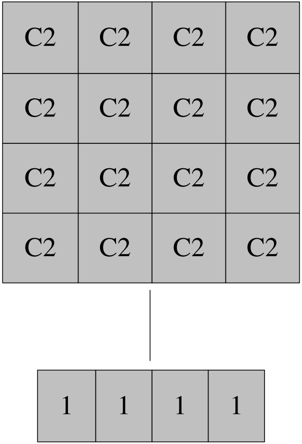

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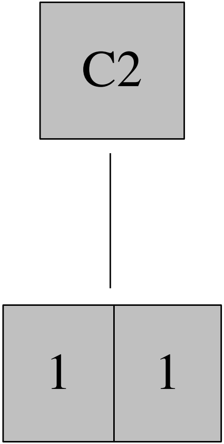

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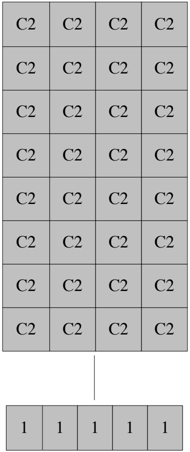

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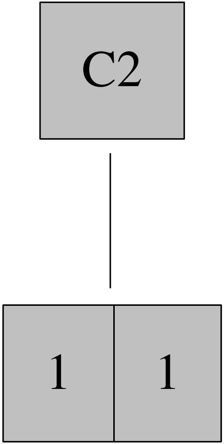

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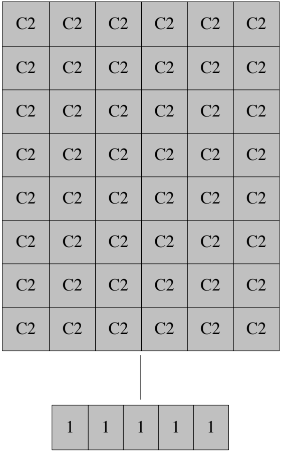

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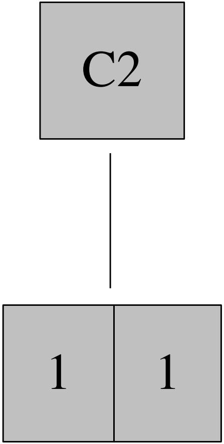
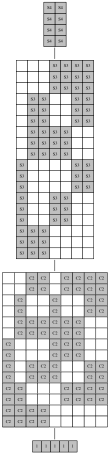


Look familiar?

$\mathrm{Reg}(\mathcal{T}_5^a)$, $\mathrm{rank}(a) = 4$



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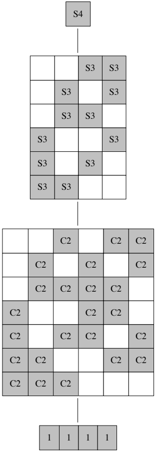

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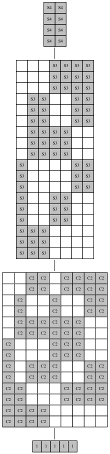

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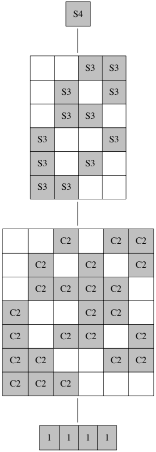

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  - Something weaker than regularity is sufficient, but we'll KIS(S).

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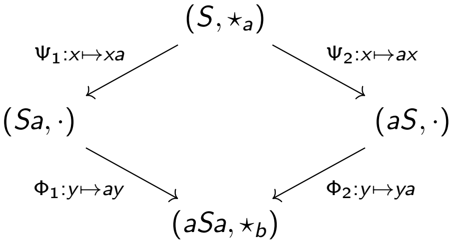

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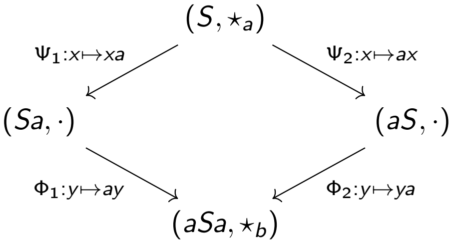

- Key fact: The above extends to the regular subsemigroups...

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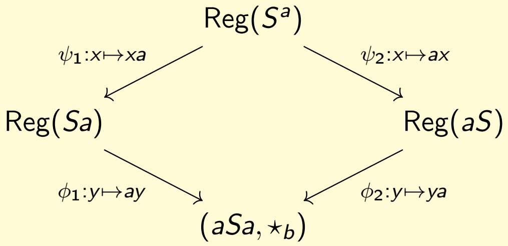

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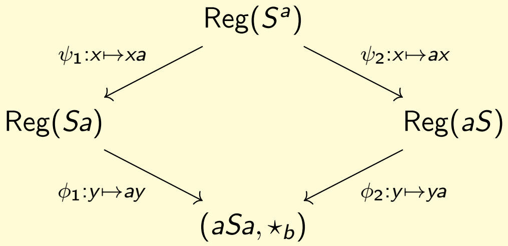

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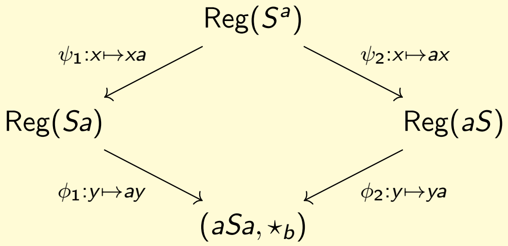

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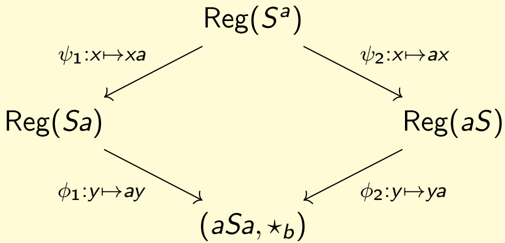

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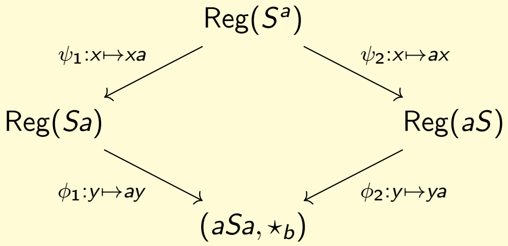

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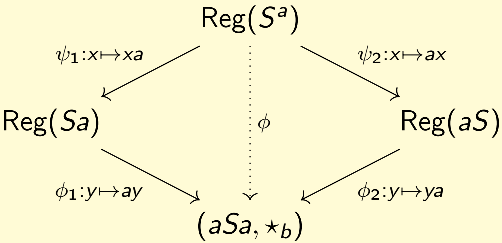

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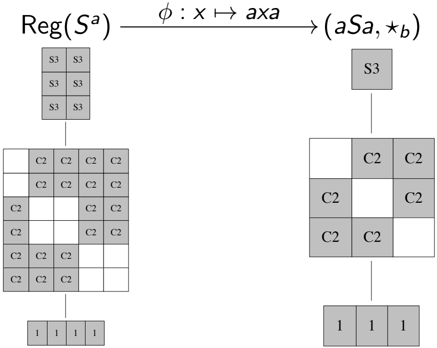

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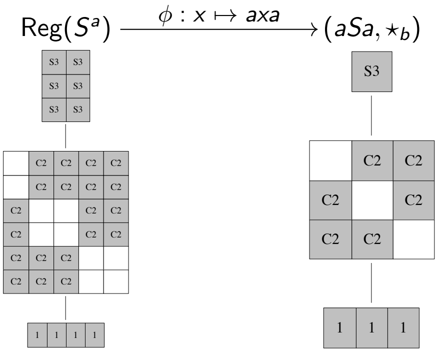

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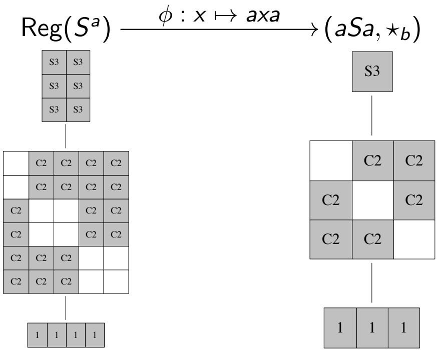

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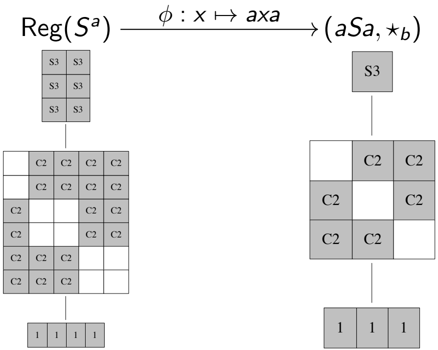

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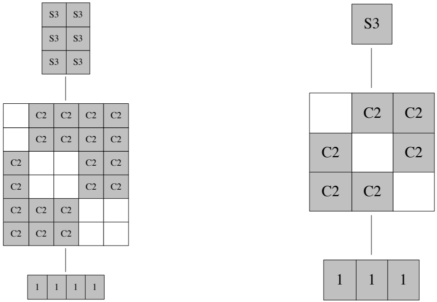

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- Theorem: $|\,\text{Reg}(\mathcal{T}^a_{XY})| = \displaystyle\sum_{\mu=1}^{\alpha} \mu!\mu^\beta S(\alpha, \mu) \sum_{\substack{J \subseteq I \\ |J| = \mu}} \Lambda_J$.

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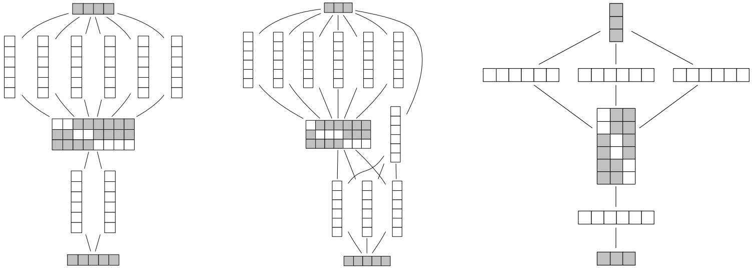

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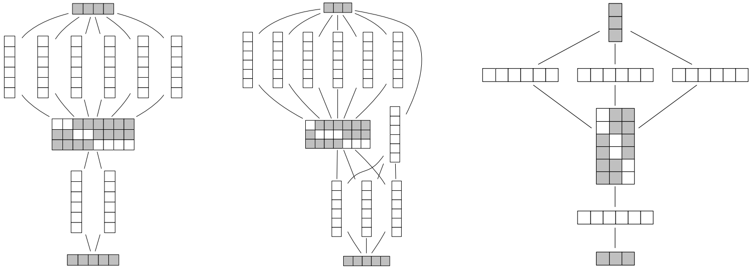

- We obtain new (and known) results on these as corollaries.

Thanks for having me in Newcastle!

- Variants of finite full transformation semigroups — IJAC, 2015
- Semigroups of matrices under a sandwich operation — SF, 2017?
- Sandwich semigroups in locally small categories — coming soon!