Uniqueness theorems for right LCM semigroup C^*-algebras

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based on joint work with
N. Brownlowe, N. Stammeier, and B. Kwaśniewski

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Yes, for positive cones of ordered subgroups of $\mathbb{R}$ (Douglas), and for positive cones in totally ordered groups (Murphy).

C^*-algebras of quasi-lattice ordered groups

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with $r = p \vee q$ their least common upper bound.

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Examples:

- (G, P), G totally ordered abelian with positive cone P (Douglas, Murphy);
- $(\mathbb{F}_n, \mathbb{F}_n^+)$ (Nica):
- right-angled and finite type Artin groups (Crisp-Laca);
- $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ (Laca-Raeburn).

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$C^*(G, P)$: generated by a universal Nica cov. repres. v of P.

A uniqueness result for $C^*(G, P)$

Theorem (Laca-Raeburn 1996): Let (G, P) be qlo and $C^*(G, P) = \mathcal{D} \rtimes P$. If the canonical conditional expectation $\Phi : C^*(G, P) \to \mathcal{D}$,

$$\Phi\big(\sum_{p,q\in F} a_{p,q} v_p v_q^*\big) = \sum_{p\in F} a_{p,p} v_p v_p^*$$

for $a_{p,q} \in \mathbb{C}$ is faithful, then a representation $\pi_W \times W$ of $C^*(G, P)$ obtained from a Nica covariant isometric representation W of P is faithful iff

$$(\#) \quad \prod_{p\in F}(I - W_p W_p^*) \neq 0$$

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Question: is injectivity of $*$-homomorphisms on $C^*(S)$ for a larger class of semigroups still characterised by means of a condition expressed in $\mathcal{D}$, similar to $(\#)$?

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However, there is a large class of left cancellative monoids for which a condition analogous to (1) still holds (essentially, due to structure of principal right ideals of the semigroup). Thus $C^*(P)$ has a familiar spanning set. This class is adequate to consider.

A good class of monoids: right LCM

Definition (Brownlowe-Ramagge-Robertson-Whittaker, Lawson, Norling)

A left cancellative monoid S is **right LCM** *(or is said to satisfy Clifford's condition) if*

$$pS \cap qS \neq \emptyset \Rightarrow pS \cap qS = rS;$$

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[BRRW] studied $C^*(S)$ for $S = U \bowtie A$ both as a C^*-algebra with generators and relations and as a semigroup C^*-algebra.

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$\forall x \in S^* \setminus \{e\}\ \exists r \in tS : rS \not\subseteq X(F),\ x(s^{-1}rS) \cap (s^{-1}rS) = \emptyset.$

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Theorem (Brownlowe-L-Stammeier):
Let S be unital cancellative right LCM s. t. $\Phi : C^*(S) \to \mathcal{D}$, $\Phi(\sum_{p,q\in F} a_{p,q} v_p v_q^*) = \sum_{p\in F} a_{p,p} v_p v_p^*$, for $a_{p,q} \in \mathbb{C}$, gives a faithful conditional expectation. Assume condition (TC). Then a $*$-homomorphism $\pi : C^*(S) \to B$ is injective iff

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Ex: $S = G \rtimes_\theta P$ with both finite and infinite index for $\theta_p(G)$'s.

Algebraic dynamical systems

Motivating class of examples (Brownlowe-L-Stammeier)

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Definition (BLS)

An algebraic dynamical system is a triple (G, P, θ) consisting of a countable, discrete group G, a countable right LCM monoid P, and an action θ of P by injective endomorphisms of G that is order-preserving, i.e. s.t. for all $p, q \in P$

$$\theta_p(G) \cap \theta_q(G) = \theta_r(G) \text{ if } r \in P \text{ satisfies } pP \cap qP = rP.$$

Proposition (BLS)

The monoid $\mathscr{P} = G \rtimes_\theta P$ with operation

$$(g, p)(h, q) = (g\theta_p(h), pq) \text{ for } g, h \in G, p, q, \in P$$

is right LCM.

Algebraic dynamical systems: constructible ideals

Proposition

Let $X_{(g,p)}$ and $X_{(h,q)}$ be principal right ideals of $\mathscr{P} = G \rtimes_\theta P$, for $g, h \in G$ and $p, q \in P$. Then

$$X_{(g,p)} \cap X_{(h,q)} = \begin{cases} X_{(g\theta_p(k),r)} & \text{if } pP \cap qP = rP, g\theta_p(k) \in h\theta_q(G) \\ \emptyset & \text{otherwise.} \end{cases}$$

for some $r \in P$ and $k \in G$.

Example satisfying (TC)

Let $G = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and P be the unital subsemigroup of $\mathbb{N}^\times$ generated by 2 and 3. Define an action θ of P by injective endomorphisms of G as follows: for $g = (g_n)_{n\in\mathbb{N}} \in G$, let

$$\theta_2(g) = 2g, \ (\theta_3(g))_0 = 3g_0 \text{ and } (\theta_3(g))_n = g_n \text{ for all } n \geq 1.$$

Fact: θ is order-preserving, therefore $\mathscr{P} = G \rtimes_\theta P$ is right LCM.

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Fact: θ is order-preserving, therefore $\mathscr{P} = G \rtimes_\theta P$ is right LCM.
We have $[G : \theta_2(G)] = \infty$ and $[G : \theta_3(G)] = 3$. In fact, $[G : \theta_{2^k}(G)] = \infty$ for all $k \geq 1$, which gives the flexibility required for establishing (TC).

Another perspective on uniqueness

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This requires explaining some notions: Hilbert module and C^*-correspondence over a C^*-algebra, a product system of C^*-correspondences over a semigroup, representations in this context, associated C^*-algebras...

Hilbert modules and C^*-correspondences

Example (Pimsner): given a dynamical system $(A, \mathbb{Z}, \alpha)$, give $X = A$ a right module structure by $x \cdot a = xa$ for $x, a \in A$ and pre-inner product $\langle x, y \rangle = x^* y$, $x, y \in A$. Complete to get a Hilbert A-module. Obtain a C^*-correspondence via a left action of A as adjointable operators on X, i.e. $\phi : A \to \mathcal{L}(X)$ homomorphism

$$a \cdot x = \phi(a)x = \alpha(a)x.$$

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Definition (Pimsner, Fowler-Raeburn)

Given a C^*-correspondence X over A, a Toeplitz representation of X in a C^*-algebra is a pair (ψ, π) with $\pi : A \to B$ homomorphism and $\psi : X \to B$ linear s.t. for $a \in A$, $x, y \in X$

$$\pi(\langle x, y\rangle) = \psi(x)^*\psi(y)$$
$$\psi(x \cdot a) = \psi(x)\pi(a)$$
$$\psi(\phi(a)x) = \pi(a)\psi(x).$$

Product systems of C^*-correspondences

The Toeplitz algebra $\mathcal{T}_X$ of a C^*-correspondence X over A is defined as the universal C^*-algebra for representations of X. It is generally tractable, for example there are powerful uniqueness theorems due to Fowler-Raeburn.

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Idea: a product system formalises a multiplicative collection of C^*-correspondences over a fixed C^*-algebra. Motivation comes from similar construction for Hilbert spaces due to Arveson (continuous semigroups) and Dinh (discrete semigroups).

Product systems of C^*-correspondences I

Setup (Fowler): $\mathscr{P}$ left cancellative monoid, semigroup

$$X = \bigsqcup_{p \in \mathscr{P}} X_p \; s.t.$$

1. X_p is a C^*-correspondence over (fixed) A, $\forall p \in \mathscr{P}$;
2. $X_p \otimes_A X_q \cong X_{pq}$, $x \otimes_A y \mapsto xy$, $\forall p, q \in \mathscr{P}, p \neq e$;
3. $X_e = {}_A A_A$, the standard C^*-correspondence;
4. $X_e \times X_p \to X_p$ and $X_p \times X_e \to X_p$, $p \in \mathscr{P}$, are the module actions.

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Toeplitz algebra $\mathcal{T}_X$, universal for representations of X. Generally unmanageable. Throw in more structure on $\mathscr{P}$ and look for quotients of $\mathcal{T}_X$.

Product systems of C^*-correspondences II

For $p, q \in \mathscr{P}$, there is a homomorphism

$$\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq}), \iota_p^{pq}(T)(xy) = (Tx)y \qquad (2)$$

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Definition (Fowler (qlo), Brownlowe-L-Stammeier (rLCM))

(a) X is compactly aligned if for all $p, q, r \in \mathscr{P}$ such that $p\mathscr{P} \cap q\mathscr{P} = r\mathscr{P}$ and all $T_p \in \mathcal{K}(X_p)$, $T_q \in \mathcal{K}(X_q)$ we have

$$\iota_p^r(T_p)\iota_q^r(T_q) \in \mathcal{K}(X_r).$$

Product systems of C^*-correspondences II

For $p, q \in \mathscr{P}$, there is a homomorphism

$$\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq}), \iota_p^{pq}(T)(xy) = (Tx)y \tag{2}$$

for all $x \in X_p, y \in X_q$.

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(b) A representation ψ of X in B is Nica covariant if

$$\psi^{(p)}(T_p)\psi^{(q)}(T_q) = \begin{cases} \psi^{(r)}\big(\iota_p^r(T_p)\iota_q^r(T_q)\big) & \text{if } p\mathscr{P} \cap q\mathscr{P} = r\mathscr{P} \\ 0 & \text{otherwise,} \end{cases}$$

where $p, q \in \mathscr{P}$ are arbitrary and r is in $\mathscr{P}$.

The full Nica-Toeplitz algebra of X

Following Fowler (Sims-Yeend, Carlsen-L-Sims-Vittadello, Brownlowe-L-Stammeier): *the Nica Toeplitz algebra* $\mathcal{N}\mathcal{T}(X)$ of X is the universal C^*-algebra for Nica covariant representations of X,

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The Fock representation $\mathbb{L}$ acts in the Hilbert A-module

$$\mathcal{F}(X) = \Big\{(x_p)_{p\in\mathscr{P}} \mid x_p \in X_p, \sum_{p\in\mathscr{P}} \|x_p\|_p^2 < \infty\Big\}$$

equipped with the inner product

$\langle (x_p)_{p\in\mathscr{P}}, (y_p)_{p\in\mathscr{P}}\rangle = \sum_{p\in\mathscr{P}} \langle x_p, y_p\rangle_p$ and obvious actions;

$$\mathbb{L}(x)(y_q)_{q\in\mathscr{P}} = (\chi_{p\mathscr{P}}(q) \cdot x y_{p^{-1}q})_{q\in\mathscr{P}} \text{ for } x \in X_p.$$

A product system for algebraic dynamical systems

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$$\alpha : P \curvearrowright C^*(G), \alpha_p(\delta_g) = \delta_{\theta_p(g)}$$

by endomorphisms of $C^*(G)$, and

$$L : P^{\mathrm{op}} \curvearrowright C^*(G), L_p(\delta_g) = \chi_{\theta_p(G)}(g)\delta_{\theta_p^{-1}(g)}$$

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by unital, positive, linear maps $L_p : C^*(G) \to C^*(G)$. Turn $C^*(G)$ into a C^*-correspondence M_p with right action and pre-inner product

$$a \cdot b = a\alpha_p(b), \; \langle a, b\rangle_p = L_p(a^*b).$$

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$$a \cdot b = a\alpha_p(b), \ \langle a, b\rangle_p = L_p(a^*b).$$

The left action ϕ_p on M_p is given by left multiplication. Let $\pi_p(\delta_g)$ denote the image of δ_g in M_p for $p \in P$, $g \in G$.

Algebraic dynamical systems and their C^*-algebras

Form the semigroup $M = \bigsqcup_{p \in P} M_p$, a product system over P of Exel-type correspondences with operation

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Hence a representation of $C^*(\mathscr{P})$ arises as a representation ψ_* for a Nica covariant representation ψ of M. Through this perspective we investigate the injectivity of a representation of $C^*(\mathscr{P})$.

Nica covariant representations

Proposition (Kwaśniewski-L)

Let (G, P, θ) be an algebraic dynamical system. Let $A = C^*(G)$ and recall the action L of P^{op} by transfer operators of A. There is a 1-1 correspondence $\psi \longleftrightarrow (\pi, W)$ where $\psi : M \to B(H)$ is Nica covariant, $\pi : A \to B(H)$ is a nondeg. repres. and $W : P \to B(H)$ a homomorphism s.t. (π, W) is Nica covariant ($\sharp$) for (A, P^{op}, L). Specifically,

$$\pi(a)W_p = \psi_p(a) \text{ for } p \in P, a \in M_p.$$

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$$\pi(a)W_p = \psi_p(a) \text{ for } p \in P, a \in M_p.$$

(♯) means preserves redundancies: if $a \in \overline{\pi(A)W_p\pi(A)W_q^*\pi(A)}$, $b \in \overline{\pi(A)W_s\pi(A)W_t^*\pi(A)}$, $k \in \overline{\pi(A)W_{pm}\pi(A)W_{tn}^*\pi(A)}$ for $qP \cap sP = rP$ and $qm = r = sn$ such that

$$ab\pi(c)W_{tn} = k\pi(c)W_{tn}$$

for $c \in A$, then $ab = k$.

Some further preparation

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Fact/Def: A group $\{\alpha_h\}_{h\in H}$ of automorphisms of a C^*-algebra C is aperiodic if for every $h \in H \setminus \{e\}$ and every non-zero hereditary subalgebra D of C we have

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Fact/Def: A group $\{\alpha_h\}_{h\in H}$ of automorphisms of a C^*-algebra C is aperiodic if for every $h \in H \setminus \{e\}$ and every non-zero hereditary subalgebra D of C we have

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In the context of product systems, there are natural equivalent characterisations, e.g. as aperiodicity of a certain Fell bundle.

Uniqueness theorem for left semidirect products

Theorem (Kwaśniewski-L)

Let $\mathscr{P} = G \rtimes_\theta P$ where (G, P, θ) is an algebraic dynamical system. Suppose that either $P^* = \{e\}$ or that the action of $\{\alpha_h\}_{h\in P^*}$ on $A = C^*(G)$ is aperiodic. Assume that M is amenable. Let (π, W) be a Nica covariant representation of (A, P, L) and let Q_p be the projection onto the space $\overline{\pi(A)W_pH}$, $p \in P$. Then there is a surjective homomorphism

$$C^*(\mathscr{P}) \mapsto \overline{\text{span}}\{\pi(a)W_pW_q^*\pi(b) : a, b \in A\}$$

with $p, q \in P$, which is an isomorphism if for every finite family $q_1, \ldots, q_n$ in $P \setminus P^*$, the representation $a \mapsto \pi(a)\Pi_{i=1}^n(1 - Q_{q_i})$ of $C^*(G)$ is faithful. If in addition $G/\theta_p(G)$ is finite for every P, then the converse holds.

Uniqueness theorem for right semidirect products

Form the *(right) semidirect product* $\mathscr{P} = P_\vartheta \ltimes G$ where ϑ is a right action of a right LCM semigroup P on a group G.

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Generally, given monoids T, P and a right action $T \overset{\vartheta}{\curvearrowleft} P$, the right semidirect product $P_\vartheta \ltimes T$, is the semigroup $P \times T$ with composition

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Lemma

If ϑ is a right action of a right LCM semigroup P on a group G, then $\mathscr{P} = P_\vartheta \ltimes G$ is right LCM. Its constructible right ideals satisfy

$$\mathcal{J}(P) \cong \mathcal{J}(P_\vartheta \ltimes G) \quad \textit{and} \quad (P_\vartheta \ltimes G)^* = {P^*}_\vartheta \ltimes G.$$

Moreover, $P_\vartheta \ltimes G$ is cancellative if and only if P is cancellative and every ϑ_p is injective, $p \in P$.

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Let $\mathscr{P} = P_\vartheta \ltimes G$ where ϑ is a right action of a right LCM semigroup P on a group G. Suppose that either $P^* = \{e\}$ or that the action of $\{\alpha_h\}_{h \in P^{*op}}$ on $C^*(G)$ is aperiodic. Assume that the system $(C^*(G), P, \alpha)$ is amenable. For a Nica covariant representation (π, W) of $(C^*(G), P, \alpha)$ there is a canonical surjective homomorphism

$$C^*(\mathscr{P}) \mapsto \overline{\text{span}}\{W_p \pi(a) W_q^* : a \in \mathcal{A}_{p,q}, p, q \in P\}$$

with $\mathcal{A}_{p,q} = \alpha_p(C^*(G))C^*(G)\alpha_q(C^*(G))$.
This map is an isomorphism if and only if for every finite family $q_1, \ldots, q_n$ in $P \setminus P^*$, the representation $a \mapsto \pi(a)\Pi_{i=1}^n(1 - W_{q_i} W_{q_i}^*)$ of $C^*(G)$ is faithful.

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Here Nica covariance of (π, W) is determined as Nica covariance of W.

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with the action by left shifts $\vartheta_p\big((\gamma_r)_{r\in P}\big) := (\gamma_{rp})_{r\in P}$ for $p \in P$ and $(\gamma_r)_{r\in P} \in G = \bigoplus_{p\in P} \Gamma$. Here, $\vartheta_p \circ \vartheta_q = \vartheta_{qp}$ for all $p, q \in P$. Denote $a\delta_q$ the element in $C^*(G)$ with a in the q-th component and zero elsewhere. Define $\alpha_p(a\delta_q) = a\delta_r$ where $rp = q$.

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Suppose W is a Nica covariant isometric representation of P on a Hilbert space H and π a nondeg. representation of $C^*(G)$ on H s.t.

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Then the repr of $C^*(\mathscr{P})$ is injective iff for every $q_1, \ldots, q_n \in P \setminus P^*$ and $q \in P$, the representation $C^*(\Gamma) \ni a \mapsto \pi(a\delta_q)\Pi_{i=1}^n(1 - W_{q_i}W_{q_i}^*)$ is faithful.