# Nica-Toeplitz algebras for product systems of C*-correspondences over right LCM semigroups

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based on joint work with B. Kwaśniewski

# Product systems of $C^*$-correspondences

Setup (Fowler): $\mathscr{P}$ left cancellative monoid, semigroup

$$X = \bigsqcup_{p \in \mathscr{P}} X_p \; s.t.$$

1. $X_p$ is a $C^*$-correspondence over (fixed) $A$, $\forall p \in \mathscr{P}$;
2. $X_p \otimes_A X_q \cong X_{pq}$, $x \otimes_A y \mapsto xy$, $\forall p, q \in \mathscr{P}, p \neq e$;
3. $X_e = {}_A A_A$, the standard $C^*$-correspondence;
4. $X_e \times X_p \to X_p$ and $X_p \times X_e \to X_p$, $p \in \mathscr{P}$, are the module actions.

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A representation $\psi : X \to B$ is given by Toeplitz representations $\psi_p$ of $X_p$ in $B$, for all $p \in \mathscr{P}$ s.t.

$$\psi(xy) = \psi(x)\psi(y) \text{ for all } x \in X_p, y \in X_q.$$

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Toeplitz algebra $\mathcal{T}_X$, universal for representations of $X$. We are interested in $\mathcal{NT}(X)$, a quotient of $\mathcal{T}_X$.

# Product systems: when $\mathscr{P}$ has units

Suppose $h \in \mathscr{P}^*$. Then there are isomorphisms

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In particular, $X_h$ and $X_{h^{-1}}$ are mutually adjoint bimodules, i.e. there is an antilinear isometric bijection $\flat_h : X_h \to X_{h^{-1}}$ s.t.

$$\flat_h(ab) = \flat_h(b)a \text{ and } \flat_h(ba) = a\flat_h(b)$$

for all $a \in A$ and $b \in X_h$ (cf. Clare-Crisp-Higson).

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*Consequence*: recall $\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$, where $\iota_p^{pq}(T)(xy) = (Tx)y$. Then for $h \in \mathscr{P}^*$,

$$\iota_p^{ph} : \mathcal{L}(X_p) \to \mathcal{L}(X_{ph})$$

is an isomorphism mapping $\mathcal{K}(X_p)$ onto $\mathcal{K}(X_{ph})$.

# Product systems of $C^*$-correspondences

## Corollary

*For all* $h \in \mathscr{P}^*$, $p, q, r \in \mathscr{P}$, $T_p \in \mathcal{K}(X_p)$, $T_q \in \mathcal{K}(X_q)$,

$$\iota_p^r(T_p)\iota_q^r(T_q) \in \mathcal{K}(X_r) \Leftrightarrow \iota_p^{rh}(T_p)\iota_q^{rh}(T_q) \in \mathcal{K}(X_{rh}).$$

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**Definition (Fowler (qlo), Brownlowe-L-Stammeier (rLCM))**

*(a) $X$ is compactly aligned if for all $p, q, r \in \mathscr{P}$ such that $p\mathscr{P} \cap q\mathscr{P} = r\mathscr{P}$ and all $T_p \in \mathcal{K}(X_p), T_q \in \mathcal{K}(X_q)$ we have*

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*(b) A representation $\psi$ of $X$ in $B$ is Nica covariant if*

$$\psi^{(p)}(T_p)\psi^{(q)}(T_q) = \begin{cases} \psi^{(r)}\big(\iota_p^r(T_p)\iota_q^r(T_q)\big) & \text{if } p\mathscr{P} \cap q\mathscr{P} = r\mathscr{P} \\ 0 & \text{otherwise.} \end{cases}$$

# The full Nica-Toeplitz algebra of $X$

Following Fowler (Sims-Yeend, Carlsen-L-Sims-Vittadello, Brownlowe-L-Stammeier): *the Nica Toeplitz algebra* $\mathcal{NT}(X)$ of $X$ is the universal $C^*$-algebra for Nica covariant representations of $X$,

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The Fock representation $\mathbb{L}$ acts in the Hilbert $A$-module

$$\mathcal{F}(X) = \{(x_p)_{p\in\mathscr{P}} \mid x_p \in X_p, \sum_{p\in\mathscr{P}} \|x_p\|_p^2 < \infty\}$$

equipped with the inner product
$\langle (x_p)_{p\in\mathscr{P}}, (y_p)_{p\in\mathscr{P}}\rangle = \sum_{p\in\mathscr{P}} \langle x_p, y_p\rangle_p$ and obvious actions;

$$\mathbb{L}(x)(y_q)_{q\in\mathscr{P}} = (\chi_{p\mathscr{P}}(q) \cdot xy_{p^{-1}q})_{q\in\mathscr{P}} \text{ for } x \in X_p.$$

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$$\pi(\beta_p(a)) = V_p \pi(a) V_p^*$$

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The Toeplitz crossed product $\mathcal{T}(A \rtimes_\beta \mathscr{P})$ is $\mathcal{T}_X$ for the product system with $X_p := \{p\} \times \beta_p(1)A$, actions

$$(p, x) \cdot a = (p, xa), a \cdot (p, x) = (p, \beta_p(a))$$

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and inner product $\langle (p, x), (p, y) \rangle = x^* y$. Unfortunately not much known about $\mathcal{T}(A \rtimes_\beta \mathscr{P})$. Possible $\mathcal{T}(A \rtimes_{\beta, r} \mathscr{P})$.

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(i) The obvious (meaningful) regular representation of a universally defined $C^*$-algebra is injective.

(ii) A "canonical" conditional expectation $\Phi$ on a $C^*$-algebra $C$ to a subalgebra $D$, i.e. a linear map with module-type properties with respect to $D \subseteq C$, is faithful on positive elements. This means $\Phi(c^*c) = 0$ in $D$ implies $c = 0$ in $C$.

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Given $(\mathscr{G}, \mathscr{P})$ qlo, let $B_{\mathscr{P}}$ be the subalgebra of $\ell^\infty(\mathscr{P})$ generated by (even spanned by) $\mathbb{1}_p$, the characteristic function of $p\mathscr{P}$. Let $\tau$ be the translation action of $\mathscr{P}$ on $B_{\mathscr{P}}$.

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Given a product system $X$ over $\mathscr{P}$ of $C^*$-correspondences over $A$, it is possible to define a crossed product for $(B_{\mathscr{P}}, \mathscr{P}, \tau, X)$ by forming 'covariant pairs' $(L^\psi, \psi)$ with $\psi$ repres. of $X$.

# Fowler's uniqueness theorem

### Theorem (Fowler '02, Theorem 7.2)

*Let $X$ be compactly aligned over $(\mathscr{G}, \mathscr{P})$ qlo with every $X_p$, $p \in \mathscr{P}$ essential such that the dynamical system twisted by a product system $(B_{\mathscr{P}}, \mathscr{P}, \tau, X)$ is amenable. Let $\psi$ be a Nica covariant representation of $X$ on a Hilbert space $H$, denote $\mathbb{1}_p$ the canonical projections that span $B_{\mathscr{P}}$ and let $L^{\psi}$ be the representation of $B_{\mathscr{P}}$ which yields a representation $L^{\psi} \times \psi$ of $B_{\mathscr{P}} \rtimes_{\tau,X} \mathscr{P}$. Then $L^{\psi} \times \psi$ is faithful iff*

$$\forall n \geq 1, \forall p_1, \ldots, p_n \in \mathscr{P} \setminus \{e\},$$
$$a \mapsto \psi_e(a) \prod_{k=1}^{n} (1 - L^{\psi}(\mathbb{1}_{p_k})$$

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Mystery: Where is $\mathcal{NT}(X)$? How about its representations?

# Fowler's uniqueness theorem: some explanation

Fact: $B_{\mathscr{P}} \rtimes_{\tau,X} \mathscr{P}$ is generated by a canonical pair formed of $\rho$, a representation of $B_{\mathscr{P}}$ and $i_X$, a Nica covariant representation of $X$.

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For compactly aligned $X$ over a qlo pair, we have

$$\mathcal{NT}(X) = \overline{\mathrm{span}}\{i_X(x)i_X(y)^* \mid x, y \in X\}$$

$$B_{\mathscr{P}} \rtimes_{\tau,X} \mathscr{P} = \overline{\mathrm{span}}\{i_X(x)\rho(\mathbb{1}_p)i_X(y)^* \mid x, y \in X, p \in \mathscr{P}\}.$$

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Fact (Fowler): $B_{\mathscr{P}} \rtimes_{\tau,X} \mathscr{P} = \mathcal{NT}(X)$ when all left actions of $A$ in $X_p$ for $p \in \mathscr{P}$ are by generalised compacts.

# $C^*$-precategories from product systems

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$$\mathcal{L}_X(p,q) := \begin{cases} \mathcal{L}(X_q, X_p), & \text{if } p,q \in \mathscr{P} \setminus \{e\}, \\ \mathcal{K}(X_q, X_p), & \text{otherwise.} \end{cases}$$

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Further

1. $\mathcal{L}_X(p,e) = \mathcal{K}(X_e, X_p) \cong X_p$ via $x \to t_x$ for $t_x(a) = x \cdot a$;
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Well-defined product of Banach spaces for $p, q, t \in \mathscr{P}$

$$\mathcal{L}_X(p,q) \times \mathcal{L}_X(q,t) \to \mathcal{L}_X(p,t) \text{ s.t.}$$

$$\|ab\| \leq \|a\| \cdot \|b\| \text{ for } a \in \mathcal{L}_X(p,q), b \in \mathcal{L}_X(q,t).$$

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Adjoint $^*: \mathcal{L}_X(p,q) \to \mathcal{L}_X(q,p)$ for all $p,q$; every $\mathcal{L}_X(p,p)$ is a $C^*$-algebra. Then $\mathcal{L} = \{\mathcal{L}_X(p,q)\}_{p,q\in\mathscr{P}}$ is a $C^*$-precategory.

# Another $C^*$-precategory from product systems

$X$: product system over $\mathscr{P}$ with coefficients in $A$. Recall

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if $p, q \in \mathscr{P} \setminus \{e\}$, $\mathcal{L}_X(p,e) = \mathcal{K}(X_e, X_p) \cong X_p$, $\mathcal{L}_X(e,q) \cong \widetilde{X}_q$.

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New $C^*$-precategory $\mathcal{K} = \{\mathcal{K}_X(p,q)\}_{p,q\in\mathscr{P}}$ where

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Say that $\mathcal{K}$ is an ideal in $\mathcal{L}$, even *essential*, i.e. $\mathcal{K}_X(p,p)$ essential ideal in $\mathcal{L}_X(p,p)$ for all $p$.

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New $C^*$-precategory $\mathcal{K} = \{\mathcal{K}_X(p,q)\}_{p,q\in\mathscr{P}}$ where

$$\mathcal{K}_X(p,q) := \mathcal{K}(X_q, X_p) \text{ for all } p,q \in \mathscr{P}.$$

Say that $\mathcal{K}$ is an ideal in $\mathcal{L}$, even *essential*, i.e. $\mathcal{K}_X(p,p)$ essential ideal in $\mathcal{L}_X(p,p)$ for all $p$.

Aim: for $\mathscr{P}$ right LCM, define Nica-Toeplitz $C^*$-algebras for $\mathcal{L}$ and $\mathcal{K}$ and compare with $\mathcal{NT}(X)$.

# Another $C^*$-precategory from product systems

$X$: product system over $\mathscr{P}$ with coefficients in $A$. Recall

$$\mathcal{L} = \{\mathcal{L}_X(p,q)\}_{p,q\in\mathscr{P}} \text{ for } \mathcal{L}_X(p,q) = \mathcal{L}(X_q, X_p)$$

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Aim: for $\mathscr{P}$ right LCM, define Nica-Toeplitz $C^*$-algebras for $\mathcal{L}$ and $\mathcal{K}$ and compare with $\mathcal{NT}(X)$. Worthwhile doing both at once to get uniqueness theorems.

# $C^*$-precategories

## Definition

*(Ghez-Lima-Roberts, Kwaśniewski) A $C^*$-precategory is a collection of Banach spaces $\{\mathcal{L}(p,q)\}_{p,q\in P}$, viewed as morphisms, equipped with bilinear maps, viewed as composition of morphisms,*

$$\mathcal{L}(p,q) \times \mathcal{L}(q,r) \ni (a,b) \to ab \in \mathcal{L}(p,r), p,q,r \in P,$$

*satisfying $\|ab\| \leq \|a\| \cdot \|b\|$, and an antilinear involutive contravariant map $^* : \mathcal{L} \to \mathcal{L}$ s.t.*

$$a \in \mathcal{L}(p,q) \Rightarrow a^* \in \mathcal{L}(q,p)$$

*and $\|a^*a\| = \|a\|^2$ holds. In particular, $\mathcal{L}(p,p)$ is a $C^*$-algebra, and we require that for every $a \in \mathcal{L}(q,p)$ the element $a^*a$ is positive in the $C^*$-algebra $\mathcal{L}(p,p)$.*
*We say $\mathcal{L}$ is $C^*$-category if $\mathcal{L}$ is a category.*

# Right tensoring on $C^*$-precategories

Recall homomorphisms $\iota_p^{pq} : \mathcal{L}(X_p, X_p) \to \mathcal{L}(X_{pq}, X_{pq})$,

$$\iota_p^{pq}(T)(xy) = (Tx)y. \tag{1}$$

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For $q \neq e$ define $\iota_{p,q}^{pr,qr} : \mathcal{L}(X_q, X_p) \to \mathcal{L}(X_{qr}, X_{pr})$ by

$$\iota_{p,q}^{pr,qr}(T)(xy) := (Tx)y \tag{2}$$

for $x \in X_q, y \in X_r$ and $T \in \mathcal{L}(X_q, X_p)$.

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For $q = e$ let $\iota_{p,e}^{pr,r} : \mathcal{K}(X_e, X_p) \to \mathcal{L}(X_r, X_{pr})$ by

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Write $T \otimes 1_r := \iota_{p,q}^{pr,qr}(T)$ and $t_x \otimes 1_r := \iota_{p,e}^{pr,r}$ using $X_q \otimes_A X_r \cong X_{qr}$ and $X_e \otimes_A X_r \to X_r$ etc.
Say $\{\otimes 1_r\}_{r \in \mathscr{P}}$ is a right tensoring on $\mathcal{L}$.

# Properties of right tensoring on $C^*$-precategories

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In general, a right tensoring on $\mathcal{L} = \{\mathcal{L}(p,q)\}_{p,q\in\mathscr{P}}$ satisfies

1. $(a \otimes 1_r) \otimes 1_s = a \otimes 1_{rs}$, $p, r \in \mathscr{P}$, $a \in \mathcal{L}(p,q)$,
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Now introduce representations of $\mathcal{L}$ and of $\mathcal{K}$.

# Representations of $C^*$-precategories I

Given $\mathcal{L} = \{\mathcal{L}(p,q)\}_{p,q\in\mathscr{P}}$ and a $C^*$-algebra $B$. A *representation* $\Phi : \mathcal{L} \to B$ is a family $\Phi = \{\Phi_{p,q}\}_{p,q\in\mathscr{P}}$ of linear maps $\Phi_{p,q} : \mathcal{L}(p,q) \to B$ s. t.

$$\Phi_{p,q}(a)^* = \Phi_{q,p}(a^*) \text{ and } \Phi_{p,r}(ab) = \Phi_{p,q}(a)\Phi_{q,r}(b),$$

for all $a \in \mathcal{L}(p,q)$, $b \in \mathcal{L}(q,r)$.

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for all $a \in \mathcal{L}(p,q)$, $b \in \mathcal{L}(q,r)$. Properties and notation:

1. $\Phi_{p,q}$, $p,q \in \mathscr{P}$, are contractions;
2. $\Phi_{p,p}$, $p \in \mathscr{P}$, are isometric iff injective;
3. $C^*(\Phi(\mathcal{L}))$ is the $C^*$-algebra generated by $\Phi(\mathcal{L}(p,q))$, $p,q \in \mathscr{P}$.
4. If $\mathcal{K}$ is an ideal in $\mathcal{L}$ and $\Psi$ is a representation of $\mathcal{K}$ on a Hilbert space $H$, there is a unique extension $\overline{\Psi}$ of $\Psi$ to a representation of $\mathcal{L}$ s.t. the essential subspace of $\overline{\Psi}_{p,q}$ is contained in the essential subspace of $\Psi_{p,q}$, for every $p,q \in \mathscr{P}$.

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Suppose $\{\otimes 1_r\}_{r\in\mathscr{P}}$ is a right tensoring on $\mathcal{L}$ and $\mathcal{K}$ is a well-aligned ideal in $\mathcal{L}$, i.e.

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## Definition

*A representation $\Psi : \mathcal{K} \to B$ is Nica covariant if for all $a \in \mathcal{K}(p,q)$, $b \in \mathcal{K}(s,t)$ we have*

$$\Psi(a)\Psi(b) = \begin{cases} \Psi\left((a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r})\right) & \text{if } qP \cap sP = rP \\ 0 & \text{otherwise,} \end{cases}$$

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Are there any Nica covariant reps?

# The full and reduced Nica-Toeplitz algebras

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$\mathbb{L}$ acts in the Fock module $\mathcal{F}_{\mathcal{K}} := \bigoplus_{p\in\mathscr{P}} X_p$ for $X_p := \mathcal{K}(p,e)$

$$\mathbb{L}_{p,q}(a)x = \begin{cases}(a\otimes 1_{q^{-1}s})x & \text{if } s\in qP,\\ 0 & \text{otherwise,}\end{cases}$$

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The *reduced Nica-Toeplitz algebras* of $\mathcal{K}$ and $\mathcal{L}$ are

$$\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) := C^*(\mathbb{L}(\mathcal{K})) \text{ and } \mathcal{NT}^r(\mathcal{L}) := C^*(\overline{\mathbb{L}}(\mathcal{L})).$$

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$$(\mathcal{NT}_{\mathcal{L}}(\mathcal{K}), i_{\mathcal{K}}) \text{ and } (\mathcal{NT}(\mathcal{L}), i_{\mathcal{L}})$$

generated by an injective Nica covariant representation $i_{\mathcal{K}}$ and $i_{\mathcal{L}}$, respectively, with a universal property.

# The $C^*$-precategory from a product system

Let $X$ be a product system over a right LCM monoid $\mathscr{P}$, let

$$\mathcal{L}_X = \{\mathcal{L}_X(p,q)\}_{p,q\in\mathscr{P}} \text{ for } \mathcal{L}_X(p,q) = \mathcal{L}(X_q, X_p)$$

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### Lemma

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*There is a 1-1 correspondence given by $\Psi_{p,q}(\Theta_{x,y}) = \psi_p(x)\psi_q(y)^*$ between repres. $\Psi$ of $\mathcal{K}_X$ and representations $\psi$ of $X$. This preserves Nica covariance.*

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## Corollary

*If $X$ is compactly aligned we have $\mathcal{NT}_{\mathcal{L}_X}(\mathcal{K}_X) \cong \mathcal{NT}(X)$.*

# New $C^*$-algebras associated to $X$

## Definition

*The reduced Nica-Toeplitz algebra of $X$ is* $\mathcal{N}\mathcal{T}^r(X) := \mathcal{N}\mathcal{T}^r_{\mathcal{L}_X}(\mathcal{K}_X)$.

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## Lemma (Doplicher-Roberts type algebra of $\mathcal{NT}(X)$)

*There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{NT}(X) & \xrightarrow{\hookrightarrow} & \mathcal{NT}(\mathcal{L}_X) \\ \Lambda \downarrow & & \downarrow \overline{\Lambda} \\ \mathcal{NT}^r(X) & \xrightarrow{\hookrightarrow} & \mathcal{NT}^r(\mathcal{L}_X) \end{array}$$

*in which the map $\overline{\Lambda}$ is injective on the core subalgebra $\overline{\operatorname{span}}\{i_{\mathcal{L}_X}(a) : a \in \mathcal{L}_X(p,p), p \in \mathscr{P}\}$ of $\mathcal{NT}^r(\mathcal{L}_X)$ and $\Lambda$ is injective on $\overline{\operatorname{span}}\{i_X(x)i_X(y)^* : x, y \in X\}$.*

# Towards uniqueness theorems: useful projections

Fix $X$ compactly-aligned over $\mathscr{P}$ rLCM monoid.

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is injective. Similar for $\mathcal{K}_X$.
If $\psi : X \to B(H)$ is a representation of $X$, let $Q_p^\psi$ be the projection in $B(H)$ s.t.

$$Q_p^\psi H = \begin{cases} \overline{\psi^{(p)}(\mathcal{K}(X_p))H} & \text{if } p \in P \setminus \{e\}, \\ \overline{\psi(X_e)H} & \text{if } p = e. \end{cases}$$

# Uniqueness theorems for $C^*$-algebras from $X$ I

## Theorem (Kwaśniewski-L)

*Assume $\mathcal{K}_X$ is amenable and $\mathscr{P}$ has no nontrivial units. Given a Nica covariant representation $\psi : X \to B(H)$, consider*

1. *$\psi$ satisfies condition (C), i.e. $\forall q_1, \ldots, q_n \in \mathscr{P} \setminus \{e\}$,*

$$A \ni a \longmapsto \psi_e(a) \prod_{i=1}^{n} (1 - Q_{q_i}^{\psi}) \text{ is injective.}$$

2. *$\psi \times P$ is injective on $\mathcal{NT}(X)$.*
3. *$\psi$ is inj. and Toeplitz covariant, i.e. $\forall q_1, \ldots, q_n \in \mathscr{P} \setminus \{e\}$,*

$$\psi_e(A) \cap \overline{\operatorname{span}}\{\psi^{(q_i)}(\mathcal{K}(X_{q_i})) : i = 1, \ldots, n\} = \{0\}.$$

*Then (1)⇒(2)⇒(3), with equivalence if $A$ acts by compacts.*

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2. *$\psi \times P$ is injective on $\mathcal{NT}(X)$.*
3. *$\psi$ is inj. and Toeplitz covariant, i.e. $\forall q_1, \dots, q_n \in \mathscr{P} \setminus \{e\}$,*

$$\psi_e(A) \cap \overline{\text{span}}\{\psi^{(q_i)}(\mathcal{K}(X_{q_i})) : i = 1, \dots, n\} = \{0\}.$$

*Then (1)⇒(2)⇒(3), with equivalence if $A$ acts by compacts.*

Remark: Thm holds if $\{X_h \mid h \in \mathscr{P}^*\}$ periodic Fell bundle.

# Uniqueness theorems for $C^*$-algebras from $X$ II

## Theorem (Kwaśniewski-L)

*Assume $\mathcal{L}_X$ is amenable and $\mathscr{P}$ has no nontrivial units. Given a Nica covariant representation $\psi : X \to B(H)$, TFA*

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2. *$\overline{\psi \rtimes P}$ is an isomorphism from $\mathcal{NT}(\mathcal{L}_X)$ onto the closed linear span of operators $T$ s.t. $T \in \psi(X_e) \cup \psi(X_e)^*$ or*

$$T \in Q^{\psi}_p B(H) Q^{\psi}_q, T\psi(X_q) \subseteq \psi(X_p), T^*\psi(X_p) \subseteq \psi(X_q)$$

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*The isomorphism in (2) restricts to an isomorphism of $\mathcal{NT}(X)$ onto $\overline{\text{span}}\{\psi(x)\psi(y)^* \mid x, y \in X\}$.*

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Define $\mathcal{FR}(X) = C^*(B_{\mathscr{P}} \cdot \mathcal{NT}(X)) \subseteq \mathcal{NT}(\mathcal{L}_X)$. Then $\mathcal{FR}(X) = \mathcal{NT}(X)$ precisely when $A$ acts by gen. compacts.

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2. $\mathcal{FR}(X) \cong \overline{\text{span}}\{\psi(x) Q^{\psi}_p \psi(y) \mid x, y \in X\}$.