The crux of generalised scales

(partly based on joint work with Afsar–Brownlowe–Larsen)

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Interactions Between Semigroups and Operator Algebras

1. Examples of right LCM monoids

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2. A dabbler's working definition

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OK. — So what is a generalised scale and what are the issues?

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Definition (Afsar–Brownlowe–Larsen–S.)

A nontrivial homomorphism $N \colon S \to \mathbb{N}^\times$ is a generalised scale if for all $n \in N(S)$, every transversal for $N^{-1}(n)/_\sim$ is an accurate foundation set of size n.

Proposition (Afsar–Brownlowe–Larsen–S.)

Let $N\colon S \to \mathbb{N}^{\times}$ *be a generalised scale on a right LCM semigroup* S*.*

(i) *The kernel of* N *is* S_c*. In particular,* $S_c \neq S$*.*

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Let $\mathbb{F}_n^+ := \langle \{a_i \mid 1 \leq i \leq n\} \rangle$ be the free monoid in $2 \leq n \leq \infty$ generators. Then $\mathbb{F}_n^+$ admits a unique generalised scale given by $a_i \mapsto n$.

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Sketch of proof: The core of $\mathbb{F}_n^+$ is trivial. The generators a_i are mutually orthogonal and irreducible. Thus they need to be mapped to the same value, which has to be n to meet the size constraint.

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With the following tool, we gain insights into more examples:

Proposition (Afsar–Brownlowe–Larsen–S.)

Suppose S is a Zappa-Szép product $S = U \bowtie A$ of two right LCM monoids U and A such that for all $(u,a),(v,b) \in S$

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for some $c \in A$. Then the restriction $N \mapsto N|_U$ defines a one-to-one correspondence between generalised scales N on S and generalised scales M on U with $M_{a(u)} = M_u$ for all $a \in A, u \in U$.

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The Baumslag–Solitar monoid $BS(c,d)^+ = \langle a, b \mid ab^c = b^d a \rangle$ for $c, d \in \mathbb{N}^\times$ admits a generalised scale if and only if $d > 1$. For $d > 1$, the unique generalised scale N is determined by $N_a = d$ and $N_b = 1$.

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Example

For every self-similar action (G, X) (with the standing assumption $2 \leq |X| < \infty$), the right LCM monoid $X^* \bowtie G$ admits a unique generalised scale given by $(w, g) \mapsto |X|^{\ell(w)}$, where $\ell \colon X^* \to \mathbb{N}$ denotes the length function for the generating set X of X^*.

Let us now consider countable, undirected graphs $\Gamma = (V, E)$ without loops or multiple edges. For every such graph, we can construct its right-angled Artin monoid $A_\Gamma^+ := \langle (a_v)_{v \in V} \mid (v, w) \in E \Rightarrow a_v a_w = a_w a_v \rangle$.

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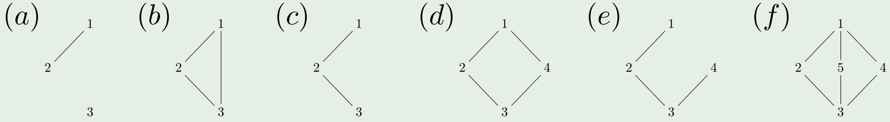

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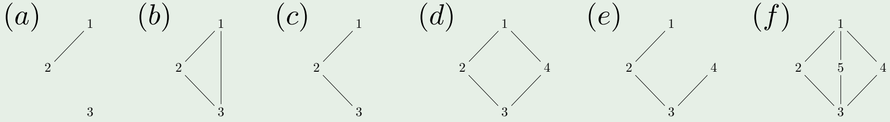

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$$\Gamma^{\mathsf{opp}} := \big(V, V \times V \setminus (E \cup \{(v, v) \mid v \in V\})\big)$$

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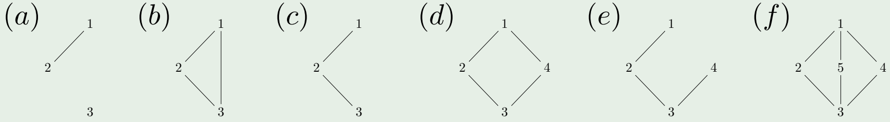

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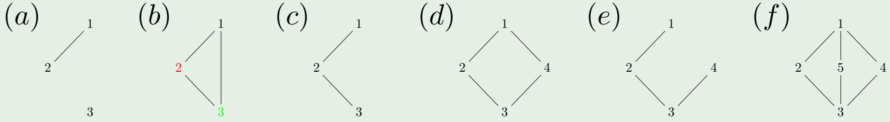

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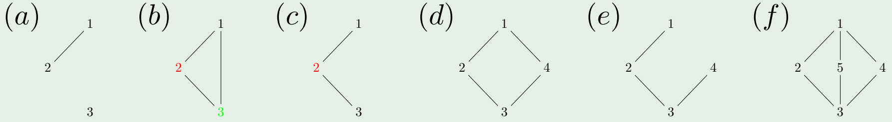

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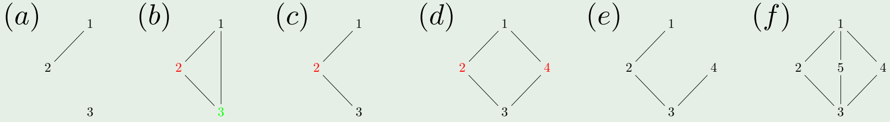

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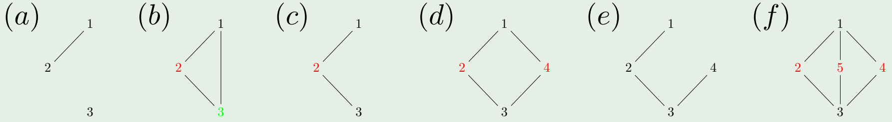

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Theorem (S.)

The right-angled Artin monoid A_Γ^+ admits a generalised scale N if and only if

(i) Γ *is not the complete graph on* V*;*

(ii) all coconnected components Γ_i *are finite and edge-free; and*

(iii) $\bigoplus_{i \in I_2} |V_i|$ *is rationally independent, where* $I_2 \subset I$ *contains all indices whose coconnected components have at least two vertices.*

In this case, N *is unique.*

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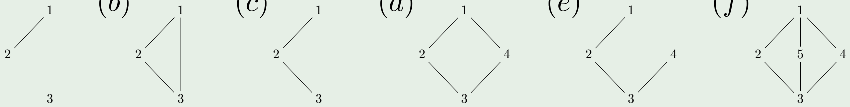
(ii) all coconnected components Γ_i *are finite and edge-free; and*

(iii) $\bigoplus_{i \in I_2} |V_i|$ *is rationally independent, where* $I_2 \subset I$ *contains all indices whose coconnected components have at least two vertices.*

In this case, N *is unique.*

Examples



(a) $\mathbb{N}^2 * \mathbb{N}$

(b) $\mathbb{N}^3$

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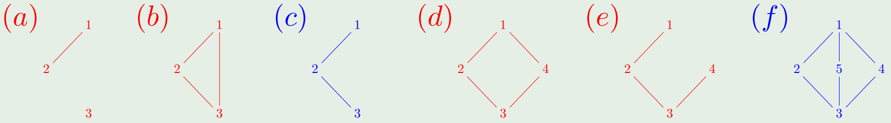

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The central reason behind this rigidity is the absence of property (AR) for the vast majority of right-angled Artin monoids:

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For a graph Γ, the right-angled Artin monoid A_Γ^+ has property (AR) if and only if every finite coconnected component Γ_i of Γ is edge-free, that is, every finitely generated direct summand of A_Γ^+ is free.

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Proposition (S.)

Suppose $m \in \bigoplus_{j \in J}\{k \in \mathbb{N} \mid 2 \leq k < \infty\}$ and M is a free abelian monoid. Then $S := M \oplus \bigoplus_{j \in J} \mathbb{F}_{m_j}^+$ admits a generalised scale $N \colon S \to \mathbb{N}^\times$ if and only if m is rationally independent. In this case, N restricts to the unique generalised scale on $\mathbb{F}_{m_j}^+$, and is therefore unique.

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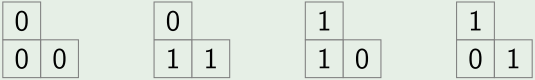
Example (Ledrappier's shift)

Consider the two-sided subshift of finite type

$$X_L = \{x \in \{0,1\}^{\mathbb{Z}^2} \mid x_{m,n} + x_{m+1,n} = x_{m,n+1} (\text{mod } 2)\}$$

equipped with the shift action of $\mathbb{Z}^2$. The elements in X_L are describable as pavings of the plane using the four tiles



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0	0	1	1	1	0	0	1

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Wait! — Where is the semigroup?

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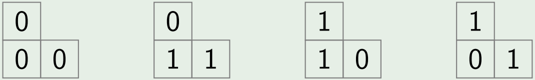

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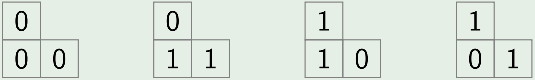

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NB: This simple observation sheds light on the existence and uniqueness problem of generalised scales for right LCM monoid of the form $G \rtimes_\theta P$ built from algebraic dynamical systems (G,P,θ).

Facets of Irreversibility: Inverse Semigroups, Groupoids, and Operator Algebras

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